Section 3.6
Optimization
MATH 1190

1. Of all the numbers whose sum is 70, find the two that have the maximum product.

   - You want to find two numbers, say x and y.
   - Need maximum product, so we need to maximize $Q = xy$.
   - You have a function with 2 variables. From the previous sections, you know how to find the maximum and minimum of a function with one variable (find the critical numbers and determine whether it is a maximum or minimum). Therefore, you need an additional equation to relate the two numbers x and y. It is given that the sum of the numbers needs to be 70, i.e.

     $$x + y = 70$$

   - Solve the above equation for one of the variables and substitute into $Q = xy$. So, solving for y, we have

     $$y = 70 - x.$$

     Thus,

     $$Q = x(70 - x) = 70x - x^2.$$

   - To find the maximum of Q, find the critical numbers.

     $$Q'(x) = 70 - 2x$$

     Setting $Q'(x) = 0$, we have $x = 35$ is the only critical number.
   - Need to make sure the critical number gives a maximum. Use the second derivative test:

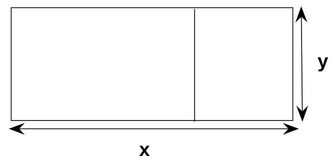
     $$Q''(x) = -2$$

     so

     $$Q''(35) = -2 < 0 \text{ (concave down - so maximum)}$$

   - Look back at the question and make sure what it is asking for: two numbers whose sum is 70 and has a maximum product. We have $x = 35$. To find y, plug $x = 35$ into the equation for y: $y = 70 - x = 70 - 35 = 35$. So,the answer is $x = 35$ and $y = 35$.

2. Melissa has 400 feet of fencing with which to enclose two adjacent lots. Determine the dimensions x and y that maximize the total area. What is the maximum area.



- Want to maximize $A = xy$
- Need an additional equation. You have 400 feet of fencing, so

$$2x + 3y = 400$$

Solving for y, you get

$$y = \frac{400}{3} - \frac{2}{3}x$$

- Substitute the equation for y into $A = xy$ to get

$$A = x\left(\frac{400}{3} - \frac{2}{3}x\right) = \frac{400}{3}x - \frac{2}{3}x^2$$

- To find the maximum, we need to first find the critical numbers.

$$A' = \frac{400}{3} - \frac{4}{3}x$$

Setting this equal to 0, we get $x = 100$.

- To make sure $x = 100$ gives a maximum, we need to use the second derivative test.

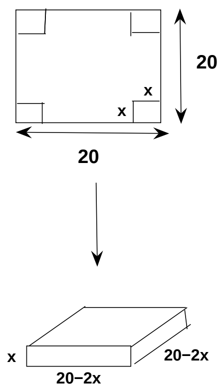
$$A'' = -\frac{4}{3}$$

so

$$A''(100) = -\frac{4}{3} < 0 \text{ (concave down - so maximum)}$$

- Now look to see what is asked for - the dimensions x and y and the maximum area. $x = 100$, so $y = \frac{400}{3} - \frac{2}{3}(100) = \frac{200}{3}$. Maximum area is $A = 100(\frac{200}{3}) = \frac{20000}{3}$ square feet.

3. From a thin piece of cadboard 20 in. by 20 in., square corners are cut out so that the sides can be folded up to make a box. What dimensions yield a box of maximum volume? What is the maximum volume?



- You are looking for the maximum volume, so you want to find

$$V = lwh$$

Looking at the picture, you can see the length and width are both $20 - 2x$ and width is x, so

$$V = (20 - 2x)^2 x$$

- Finding the critical numbers, you have

$$V' = 2(20 - 2x)(-2)x + (20 - 2x)^2 = (20 - 2x)(-4x + (20 - 2x)) = (20 - 2x)(20 - 6x)$$

So, the critical numbers are $x = 10$ and $x = \frac{10}{3}$.
You disregard $x = 10$, because if $x = 10$, then the length and width are 0 which is impossible. So, the only critical number possible is $x = \frac{10}{3}$.

- To check that $x = \frac{10}{3}$ gives a maximum, use the second derivative test.

$$V'' = -6(20 - 2x) - 2(20 - 6x)$$

so

$$V''(\frac{10}{3}) < 0 \text{ (concave down - so maximum)}$$

- The problem asks for the dimensions of the box. The height is x, so the height is $\frac{10}{3}$ in. The width is $20 - 2x = \frac{40}{3}$ in. The length is the same, $\frac{40}{3}$ in. The problem also asks for the volume, which is l*w*h or

$$V = (\frac{10}{3})(\frac{40}{3})(\frac{40}{3}) = 592.6 \text{ cubic inches}$$