Overview Sections 1.3 and 2.1

Rates of Change and Tangent Lines

MATH 1190

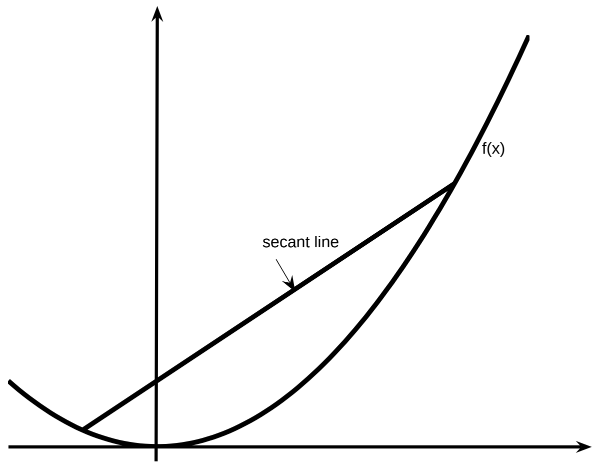
- Rate of change describes how fast or slow something changes. For example, how fast or slow a car moves. Average velocity is described as

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed or change in time}}$$

This is like the slope of a line. If $s(t)$ represents the distance traveled, then average velocity is given by

$$\text{average velocity} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} \left(\frac{y_2 - y_1}{x_2 - x_1} \right)$$

- Average rate of change is described as the slope of a secant line (a line joining two points on a graph); see graph below.



- So, average rate of change describes how fast or slow something moves across a period of time. For instance, if the average velocity from a 30 minute trip is 45 m.p.h., that does not necessarily mean that the car was always going 45 m.p.h. It may have gone slower or faster at different points.
- Instantaneous rate of change describes how fast or slow something is changing at a specific instance in time. This can be found by taking the limit of the average rate of change as the time elapsed decreases to 0, i.e.

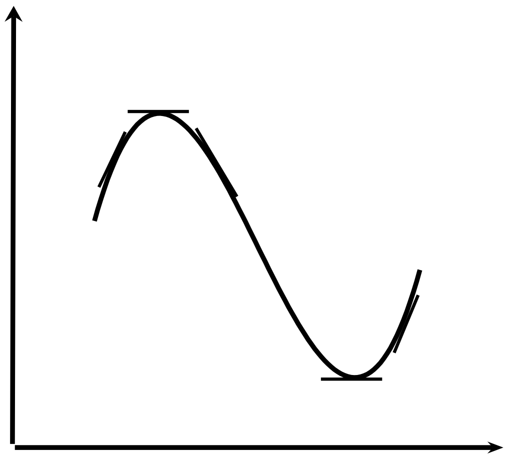
$$\text{Instantaneous rate of change} = \lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

or if we let $t_2 - t_1 = h$, then $t_2 = t_1 + h$ and we can write

$$\text{Instantaneous rate of change} = \lim_{h \to 0} \frac{s(t_1 + h) - s(t_1)}{h}$$

- The instantaneous rate of change is described as the slope of a tangent line to a curve. A tangent line to a curve at a point P matches the "steepness" of the curve at the point and touches the curve at only point P.
- So, the instantaneous rate of change and the slope of the tangent coincide. The slope of the tangent line for a curve $y = f(x)$ at the point (x_1, y_1) is given by

$$m_{tan} = \lim_{h \to 0} \frac{(f(x_1 + h) - f(x_1)}{h}$$



- Example: Find the equation for the tangent line to the curve $y = x^2 + 1$ at the point $(1, 2)$.

$$\begin{aligned} m_{tan} &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{((1+h)^2+1)-(1^2+1)}{h} \\ &= \lim_{h\to 0} \frac{(1+2h+h^2+1)-2}{h} \\ &= \lim_{h\to 0} \frac{2h+h^2+2-2}{h} \\ &= \lim_{h\to 0} \frac{h(2+h)}{h} \\ &= \lim_{h\to 0} 2+h \\ &= 2+0 \\ &= 2 \end{aligned}$$