Linear Transformations

Linear Algebra

MATH 2010

- **Functions in College Algebra:** Recall in college algebra, functions are denoted by

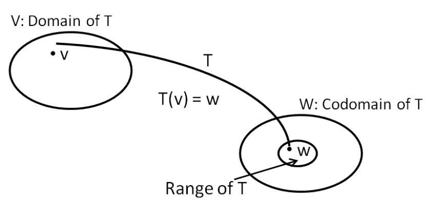
$$f(x) = y$$

where $f : dom(f) \rightarrow range(f)$.

- **Mappings:** In Linear Algebra, we have a similar notion, called a *map*:

$$T : V \rightarrow W$$

where V is the domain of T and W is the codomain of T where both V and W are vector spaces.



- **Terminology:** If

$$T(v) = w$$

then

  - w is called the *image* of v under the mapping T
  - v is caled the *preimage* of w
  - the set of all images of vectors in V is called the *range of* T

- **Example:** Let

$$T([v_1, v_2]) = [2v_2 - v_1, v_1, v_2]$$

then $T : \Re^2 \rightarrow \Re^3$.

  - Find the image of $v = [0, 6]$.

$$T([0, 6]) = [2(6) - 0, 0, 6] = [12, 0, 6]$$

  - Findthe preimage of $w = [3, 1, 2]$.

$$[3, 1, 2] = [2v_1 - v_1, v_1, v_2]$$

which means

$$\begin{aligned} 2v_2 - v_1 &= 3 \\ v_1 &= 1 \\ v_2 &= 2 \end{aligned}$$

So, $v = [1, 2]$.

- **Example:** Let
$$T([v_1, v_2, v_3]) = [2v_1 + v_2, v_1 - v_2]$$
Then $T : \Re^3 \to \Re^2$.
  - Find the image of $v = [2, 1, 4]$:
  $$T([2, 1, 4]) = [2(2) + 1, 2 - 1] = [5, 1]$$
  - Find the preimage of $w = [-1, 2]$
  $$[-1, 2] = [2v_1 + v_2, v_1 - v_2]$$
  This leads to
  $$\begin{array}{rcrcr} 2v_1 & + & v_2 & = & -1 \\ v_1 & - & v_2 & = & 2 \end{array}$$
  Recall that you are looking for $v = [v_1, v_2, v_3]$. So, there are really 3 unknowns in the system:
  $$\begin{array}{rcrcrcr} 2v_1 & + & v_2 & + & 0v_3 & = & -1 \\ v_1 & - & v_2 & + & 0v_3 & = & 2 \end{array}$$
  This leads to the solution
  $$v = [\frac{1}{3}, -\frac{5}{3}, k]$$
  where k is an real number.
- **Definition:** Let V and W be vector spaces. The function $T : V \to W$ is called a *linear transformation* of V into W if the following 2 properties are true for all u and v in V and for any scalar c:
  1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
  2. $T(c\mathbf{u}) = cT(\mathbf{u})$
- **Example:** Determine whether $T : \Re^3 \to \Re^3$ defined by
$$T([x, y, z]) = [x + y, x - y, z]$$
is a linear transformation.
  1. Let $u = [x_1, y_1, z_1]$ and $v = [x_2, y_2, z_2]$. Then we want to prove $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
  $$\begin{array}{rcl} T(\mathbf{u} + \mathbf{v}) & = & T([x_1, y_1, z_1] + [x_2, y_2, z_2]) \\ & = & T([x_1 + x_2, y_1 + y_2, z_1 + z_2]) \\ & = & [x_1 + x_2 + y_1 + y_2, x_1 + x_2 - (y_1 + y_2), z_1 + z_2] \end{array}$$
  and
  $$\begin{array}{rcl} T(\mathbf{u}) + T(\mathbf{v}) & = & T([x_1, y_1, z_1]) + T([x_2, y_2, z_2]) \\ & = & [x_1 + y_1, x_1 - y_1, z_1] + [x_2 + y_2, x_2 - y_2, z_2] \\ & = & [x_1 + y_1 + x_2 + y_2, x_1 - y_1 + x_2 - y_2, z_1 + z_2] \\ & = & [x_1 + x_2 + y_1 + y_2, x_1 + x_2 - (y_1 + y_2), z_1 + z_2] \end{array}$$
  Therefore, $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
  2. We want to prove $T(c\mathbf{u}) = cT(\mathbf{u})$.
  $$\begin{array}{rcl} T(c\mathbf{u}) & = & T(c[x_1, y_1, z_1]) \\ & = & T([cx_1, cy_1, cz_1]) \\ & = & [cx_1 + cy_1, cx_1 - cy_1, cz_1] \end{array}$$
  and
  $$\begin{array}{rcl} cT(\mathbf{u}) & = & cT([x_1, y_1, z_1]) \\ & = & c[x_1 + y_1, x_1 - y_1, z_1] \\ & = & [c(x_1 + y_1), c(x_1 - y_1), cz_1] \\ & = & [cx_1 + cy_1, cx_1 - cy_1, cz_1] \end{array}$$
  So, $T(c\mathbf{u}) = cT(\mathbf{u})$.

Therefore, T is a linear transformation.

- **Example:** Determine whether $T : \Re^2 \to \Re^2$ defined by

$$T([x, y]) = [x^2, y]$$

is a linear transformation.

1. Let $u = [x_1, y_1]$ and $v = [x_2, y_2]$. Then we want to prove $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T([x_1, y_1] + [x_2, y_2]) \\ &= T([x_1 + x_2, y_1 + y_2]) \\ &= [(x_1 + x_2)^2, y_1 + y_2] \\ &= [x_1^2 + 2x_1x_2 + x_2^2, y_1 + y_2] \end{aligned}$$

and

$$\begin{aligned} T(\mathbf{u}) + T(\mathbf{v}) &= T([x_1, y_1]) + T([x_2, y_2]) \\ &= [x_1^2, y_1] + [x_2^2, y_2] \\ &= [x_1^2 + x_2^2, y_1 + y_2] \end{aligned}$$

Since, $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$, T is not a linear transformation. There is no need to test the second criteria. However, you could have proved the same thing using the second criteria:

2. We would want to prove $T(c\mathbf{u}) = cT(\mathbf{u})$.

$$\begin{aligned} T(c\mathbf{u}) &= T(c[x_1, y_1]) \\ &= T([cx_1, cy_1]) \\ &= [(cx_1)^2, cy_1] \\ &= [c^2x_1^2, cy_1] \end{aligned}$$

and

$$\begin{aligned} cT(\mathbf{u}) &= cT([x_1, y_1]) \\ &= c[x_1^2, y_1] \\ &= [cx_1^2, cy_1] \end{aligned}$$

So, $T(c\mathbf{u}) \neq cT(\mathbf{u})$ either. Thus, again, we would have showed, T was not a linear transformation.

- **Two Simple Linear Transformations:**
  - Zero Transformation: $T : V \to W$ such that $T(v) = 0$ for all v in V
  - Identity Transformation: $T : V \to V$ such that $T(v) = v$ for all v in V

- **Theorem:** Let T be a linear transformation from V into W, where u and v are in V. Then

1. $T(0) = 0$
2. $T(-v) = -T(v)$
3. $T(u - v) = T(u) - T(v)$
4. If

$$v = c_1v_1 + c_2v_2 + ... + c_nv_n$$

then

$$T(v) = c_1T(v_1) + c_2T(v_2) + ... + c_nT(v_n)$$

- **Example:** Let $T : \Re^3 \to \Re^3$ such that

$$T([1, 0, 0]) = [2, 4, -1] \qquad T([0, 1, 0]) = [1, 3, -2] \qquad T([0, 0, 1]) = [0, -2, 2]$$

Find $T([-2, 4, -1])$. Since

$$[-2, 4, -1] = -2[1, 0, 0] + 4[0, 1, 0] - 1[0, 0, 1]$$

we can say

$$T([-2, 4, -1]) = -2T([1, 0, 0]) + 4T([0, 1, 0]) - 1T([0, 0, 1]) = -2[2, 4, -1] + 4[1, 3, -2] - [0, -2, 2] = [0, 6, -8]$$

- **Theorem:** Let A be a mxn matrix. The function T defined by

$$T(v) = Av$$

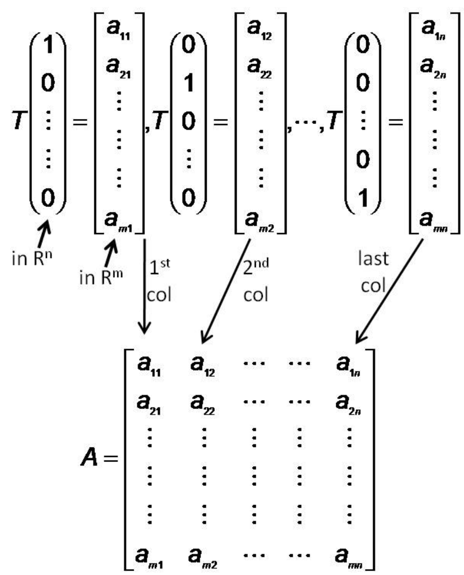
is a linear transformation from $\Re^n \to \Re^m$.

- **Examples:**
  - If $T(v) = Av$ where
$$A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}$$
    then $T : \Re^2 \to \Re^3$.
  - If $T(v) = Av$ where
$$A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$$
    then $T : \Re^5 \to \Re^2$.

- **Standard Matrix:** Every linear transformation $T : \Re^n \to \Re^m$ has a mxn standard matrix A associated with it where

$$T(v) = Av$$

To find the standard matrix, apply T to the basis elements in $\Re^n$. This produces vectors in $\Re^m$ which become the *columns* of A:



For example, let

$$T([x_1, x_2, x_3]) = [2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3]$$

Then

$$T([1,0,0]) = [2,-1,0] \qquad T([0,1,0]) = [1,3,3] \qquad T([0,0,1]) = [-1,-2,4]$$

these vectors become the *columns* of A:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix}$$

- **Shortcut Method for Finding the Standard Matrix:** Two examples:

  1. Let T be the linear transformation from above, i.e.,

  $$T([x_1, x_2, x_3]) = [2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3]$$

  Then the first, second and third components of the resulting vector w, can be written respectively as

  $$\begin{array}{rcrcrcr} w_1 &=& 2x_1 &+& x_2 &-& x_3 \\ w_2 &=& -x_1 &+& 3x_2 &-& 2x_3 \\ w_3 &=& && 3x_2 &+& 4x_3 \end{array}$$

  Then the standard matrix A is given by the coefficient matrix or the right hand side:

  $$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix}$$

  So,

  $$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

  2. **Example:** Let

  $$T([x, y, z]) = [x - 2y, 2x + y]$$

  Since $T : \Re^3 \to \Re^2$, A is a 3x2 matrix:

  $$\begin{array}{rcrcrcr} w_1 &=& x &-& 2y &+& 0z \\ w_2 &=& 2x &+& y &+& 0z \end{array}$$

  So,

  $$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- **Geometric Operators:**

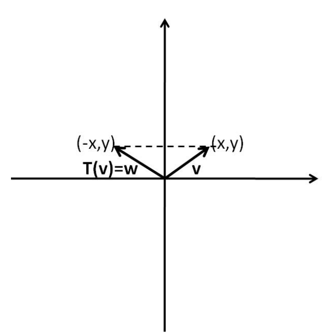
  - **Reflection Operators:**

    * *Reflection about the y-axis:* The schematic of reflection about the y-axis is given below. The transformation is given by

    $$\begin{array}{rcr} w_1 &=& -x \\ w_2 &=& y \end{array}$$

    with standard matrix

    $$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

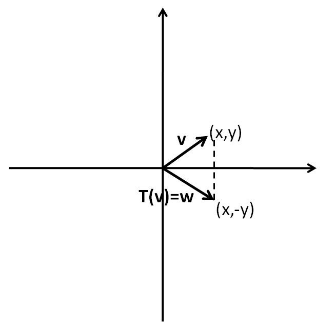


* *Reflection about the x-axis:* The schematic of reflection about the x-axis is given below. The transformation is given by

$$\begin{aligned} w_1 &= x \\ w_2 &= -y \end{aligned}$$

with standard matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

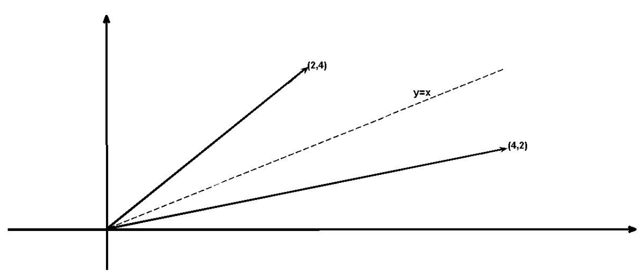


* *Reflection about the line $y = x$:* The schematic of reflection about the line $y = x$ is given below. The transformation is given by

$$\begin{aligned} w_1 &= y \\ w_2 &= x \end{aligned}$$

with standard matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



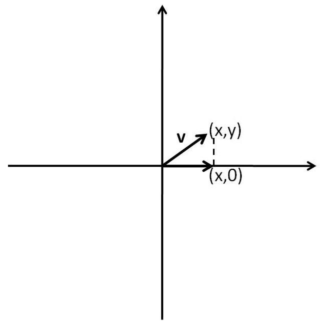
– **Projection Operators:**

* *Projected onto x-axis:* The schematic of projection onto the x-axis is given below. The transformation is given by

$$\begin{aligned} w_1 &= x \\ w_2 &= 0 \end{aligned}$$

with standard matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

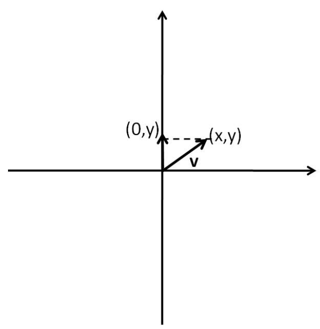


* *Projected onto y-axis:* The schematic of projection onto the y-axis is given below. The transformation is given by

$$\begin{aligned} w_1 &= 0 \\ w_2 &= y \end{aligned}$$

with standard matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



* In $\Re^3$, you can project onto a plane. The standard matrices for the projection is given below.
  · *Projection onto xy-plane:*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

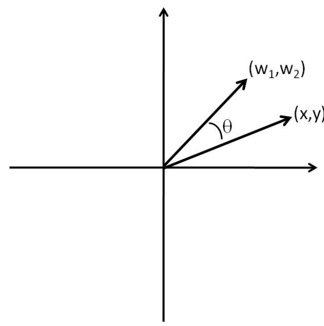
  · *Projection onto xz-plane:*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

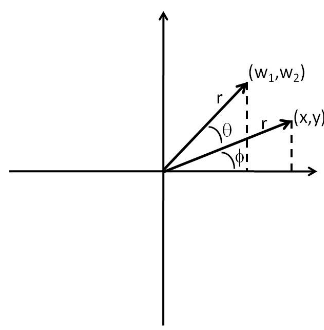
  · *Projection onto yz-plane:*

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– **Rotation Operator:** We can consider rotating through an angle θ.



If we look at a more detailed depiction of the rotation, as depicted below, we see how we can use trignometric identities to recover the standard matrix.



Using trigonometric identities, we have

$$\begin{aligned} x &= r\cos(\phi) \\ y &= r\sin(\phi) \end{aligned}$$

and

$$\begin{aligned} w_1 &= r\cos(\theta+\phi) \\ w_2 &= r\sin(\theta+\phi) \end{aligned}$$

Using trigonometric identities on w_1 and w_2, we have

$$\begin{aligned} w_1 &= r\cos(\theta)\cos(\phi) - r\sin(\theta)\sin(\phi) \\ w_2 &= r\sin(\theta)\cos(\phi) + r\cos(\theta)\sin(\phi) \end{aligned}$$

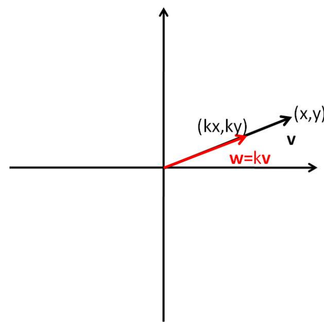
which equals

$$\begin{aligned} w_1 &= x\cos(\theta) - y\sin(\theta) \\ w_2 &= x\sin(\theta) + y\cos(\theta) \end{aligned}$$

if we plug in x and y formulas from above. Therefore, the standard matrix is given by

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

– **Dilation and Contraction Operators:** We can consider the geometric process of dilating or contracting vectors. For example, in $\Re^2$, the contraction of a vector is given below where $0 < k < 1$.



If

  * $0 < k < 1$, we have *contraction* and
  * $k > 1$, we have *dilation*

In each case, the standard matrix is given by

$$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

In $\Re^3$, we have the standard matrix

$$A = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

- **One-to-One linear transformations:** In college algebra, we could perform a horizontal line test to determine if a function was one-to-one, i.e., to determine if an inverse function exists. Similarly, we say a linear transformation $T : \Re^n \rightarrow \Re^m$ is *one-to-one* if T maps distincts vectors in $\Re^n$ into distinct vectors in $\Re^m$. In other words, a linear transformation $T : \Re^n \rightarrow \Re^m$ is *one-to-one* if for every w in the range of T, there is *exactly* one v in $\Re^n$ such that $T(v) = w$.

- **Examples:**

  1. The rotation operator is one-to-one, because there is only one vector v which can be rotated through an angle θ to get any vector w.
  2. The projection operator is not one-to-one. For example, both $[2, 4]$ and $[2, -1]$ can be projected onto the x-axis and result in the vector $[2, 0]$.

- **Linear system equivalent statements:** Recall that for a linear system, the following are equivalent statements:

  1. A is invertible
  2. $Ax = b$ is consistent for every nx1 matrix b
  3. $Ax = b$ has exactly one solution for every nx1 matrix b

- Recall, that for every linear transformation $T : \Re^n \rightarrow \Re^m$, we can represent the linear transformation as

$$T(v) = Av$$

where A is the mxn standard matrix associated with T. Using the above equivalent statements with this form of the linear transformation, we have the following theorem.

- **Theorem:** If A is an nxn matrix and $T : \Re^n \to \Re^n$ is given by

$$T(v) = Av$$

then the following is equivalent.

1. A is invertible
2. For every w in $\Re^n$, there is some vector v in $\Re^n$ such that $T(v) = w$, i.e., the range of T is $\Re^n$.
3. For every w in $\Re^n$, there is a unique vector v in $\Re^n$ such that $T(v) = w$, i.e., T is one-to-one.

- **Examples:**

1. *Rotation Operator:* The standard matrix for the rotation operator is given by

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

To determine if A is invertible, we can find the determinant of A:

$$|A| = \cos^2(\theta) + \sin^2(\theta) = 1 \neq 0$$

so A is invertible. Therefore, the range of the rotation operator in $\Re^2$ is all of $\Re^2$ and it is one-to-one.

2. *Projection Operators:* For each projection operator, we can easily show that $|A| = 0$. Therefore, the projection operator is not one-to-one.

- **Inverse Operator:** If $T : \Re^n \to \Re^n$ is a one-to-one transformation given by

$$T(v) = Av$$

where A is the standard matrix, then there exists an inverse operator $T^{-1} : \Re^n \to \Re^n$ and is given by

$$T^{-1}(w) = A^{-1}v$$

- **Examples:**

1. The standard matrix for the rotation operator through an angle θ is

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

The inverse operator can be found by rotating back through an angle $-\theta$, i.e.,

$$A = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

Using trigonometric idenitities, we can see this is the same as

$$A^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

2. Let

$$T([x, y]) = [2x + y, 3x + 4y]$$

Then T has the standard matrix

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

Thus, $|A| = 5 \neq 0$, so T is one-to-one and has an inverse operator with standard matrix

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 4/5 & -1/5 \\ -3/5 & 2/5 \end{bmatrix}$$

So, the inverse operator is given by

$$T^{-1}(w) = A^{-1}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = [\frac{4}{5}w_1 - \frac{1}{5}w_2, -\frac{3}{5}w_1 + \frac{2}{5}w_2]$$

- **Kernel of T:** One of the properties of linear transformations is that

$$T(0) = 0$$

There may be other vectors v in V such that $T(v) = 0$. The *kernel of* T is the set of all vectors v in V such that

$$T(v) = 0$$

It is denoted $ker(T)$.

- **Example:** Let $T : \Re^2 \to \Re^3$ be given by

$$T([x_1, x_2]) = [x_1 - 2x_2, 0, -x_1]$$

To find $ker(T)$, we need to find all vectors $v = [x_1, x_2]$ in $\Re^2$, such that $T(v) = 0 = [0, 0, 0]$ in $\Re^3$. In other words,

$$\begin{array}{rcrcl} x_1 & - & 2x_2 & = & 0 \\ & & 0 & = & 0 \\ -x_1 & & & = & 0 \end{array}$$

The only solution to this system if $[0, 0]$. Thus

$$ker(T) = \{[0, 0]\} = \{\mathbf{0}\}$$

- **Example:** Let $T : \Re^3 \to \Re^2$ be given by $T(x) = Ax$ where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$$

To find $ker(T)$, we need to find all $v = [x_1, x_2, x_3]$ such that $T(v) = [0, 0]$. In other words, we need to solve the system

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting this in augmented form, we have

$$\left[\begin{array}{rrr|r} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{array}\right]$$

which reduces to

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Therefore, $x_3 = t$ is a free parameter, so the solutions is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} t$$

Therefore, $ker(T) = span(\{[1, -1, 1]\})$.

- **Corollary:** If $T : \Re^n \to \Re^m$ is given by

$$T(v) = Av$$

then $ker(T)$ is equal to the nullspace of A.

- **Example:** Given $T(v) = Av$ where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

find a basis for $ker(T)$.

Solving the system, we have

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1/2 \end{bmatrix}$$

Therefore, a basis for $ker(T)$ is given by a basis for the nullspace of A: $\{[-2, -1/2, 1]\}$.

- **Example:**Given $T(v) = Av$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

  find a basis for $ker(T)$.

  Ans: $\{[-2, 1, 1, 0, 0], [1, 2, 0, -4, 1]\}$

- **Terminology:** The dimension of $ker(T)$ is called the nullity of T. In the previous example, the nullity of T is 2.

- **Range of T:** The range of T is the set of all vectors w such that $T(v) = w$. If $T : \Re^n \to \Re^m$ is given by

$$T(v) = Av$$

  then the range of T is the column space of A.

- **Onto:** If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be *onto* (or *onto* W) if every vector in W is the image of at least one vector in V, i.e., the range of $T = W$.

- **Equivalence Statements for One-to-One, Kernel:** If $T : V \to W$ is a linear transformation, then the following are equivalent:

  1. T is one-to-one
  2. $ker(T) = \{0\}$

- **Equivalence Statements for One-to-One, Kernel, and Onto:** If $T : V \to V$ is a linear transformation *and V is finite-dimensional*, then the following are equivalent:

  1. T is one-to-one
  2. $ker(T) = \{0\}$
  3. T is onto

- **Isomorphism:** If a linear transformation $T : V \to W$ is both one-to-one and onto, then T is said to be an *isomorphism* and the vector spaces V and W are said to be *isomorphic*.