• **Functions in College Algebra:** Recall in college algebra, functions are denoted by

\[ f(x) = y \]

where \( f : \text{dom}(f) \rightarrow \text{range}(f) \).

• **Mappings:** In Linear Algebra, we have a similar notion, called a *map*:

\[ T : V \rightarrow W \]

where \( V \) is the domain of \( T \) and \( W \) is the codomain of \( T \) where both \( V \) and \( W \) are vector spaces.

• **Terminology:** If

\[ T(v) = w \]

then

- \( w \) is called the *image* of \( v \) under the mapping \( T \)
- \( v \) is called the *preimage* of \( w \)
- the set of all images of vectors in \( V \) is called the *range of \( T*)

• **Example:** Let

\[ T([v_1, v_2]) = [2v_2 - v_1, v_1, v_2] \]

then \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \).

- Find the image of \( v = [0, 6] \).

\[ T([0, 6]) = [2(6) - 0, 0, 6] = [12, 0, 6] \]

- Find the preimage of \( w = [3, 1, 2] \).

\[ [3, 1, 2] = [2v_2 - v_1, v_1, v_2] \]

which means

\[
\begin{align*}
2v_2 - v_1 &= 3 \\
v_1 &= 1 \\
v_2 &= 2
\end{align*}
\]

So, \( v = [1, 2] \).
**Example:** Let

\[ T([v_1, v_2, v_3]) = [2v_1 + v_2, v_1 - v_2] \]

Then \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \).

1. Find the image of \( v = [2, 1, 4] \):
   
   \[ T([2, 1, 4]) = [2(2) + 1, 2 - 1] = [5, 1] \]

2. Find the preimage of \( w = [-1, 2] \)

   \[ [-1, 2] = [2v_1 + v_2, v_1 - v_2] \]

   This leads to

   \[
   \begin{align*}
   2v_1 + v_2 &= -1 \\
   v_1 - v_2 &= 2
   \end{align*}
   \]

Recall that you are looking for \( v = [v_1, v_2, v_3] \). So, there are really 3 unknowns in the system:

\[
\begin{align*}
2v_1 + v_2 + 0v_3 &= -1 \\
v_1 - v_2 + 0v_3 &= 2
\end{align*}
\]

This leads to the solution

\[ v = \left[ \frac{1}{3}, -\frac{5}{3}, k \right] \]

where \( k \) is an real number.

**Definition:** Let \( V \) and \( W \) be vector spaces. The function \( T : V \rightarrow W \) is called a **linear transformation** of \( V \) into \( W \) if the following 2 properties are true for all \( u \) and \( v \) in \( V \) and for any scalar \( c \):

1. \( T(u + v) = T(u) + T(v) \)
2. \( T(cu) = cT(u) \)

**Example:** Determine whether \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by

\[ T([x, y, z]) = [x + y, x - y, z] \]

is a linear transformation.

1. Let \( u = [x_1, y_1, z_1] \) and \( v = [x_2, y_2, z_2] \). Then we want to prove \( T(u + v) = T(u) + T(v) \).

   \[
   \begin{align*}
   T(u + v) &= T([x_1, y_1, z_1] + [x_2, y_2, z_2]) \\
   &= T([x_1 + x_2, y_1 + y_2, z_1 + z_2]) \\
   &= [x_1 + x_2 + y_1 + y_2, x_1 + x_2 - (y_1 + y_2), z_1 + z_2]
   \end{align*}
   \]

   and

\[
\begin{align*}
T(u) + T(v) &= T([x_1, y_1, z_1]) + T([x_2, y_2, z_2]) \\
&= [x_1 + y_1, x_1 - y_1, z_1] + [x_2 + y_2, x_2 - y_2, z_2] \\
&= [x_1 + y_1 + x_2 + y_2, x_1 - y_1 + x_2 - y_2, z_1 + z_2] \\
&= [x_1 + x_2 + y_1 + y_2, x_1 + x_2 - (y_1 + y_2), z_1 + z_2]
\end{align*}
\]

Therefore, \( T(u + v) = T(u) + T(v) \).

2. We want to prove \( T(cu) = cT(u) \).

   \[
   \begin{align*}
   T(cu) &= T(c[x_1, y_1, z_1]) \\
   &= T([cx_1, cy_1, cz_1]) \\
   &= [cx_1 + cy_1, cx_1 - cy_1, cz_1]
   \end{align*}
   \]

   and

\[
\begin{align*}
cT(u) &= cT([x_1, y_1, z_1]) \\
&= c[x_1 + y_1, x_1 - y_1, z_1] \\
&= [c(x_1 + y_1), c(x_1 - y_1), cz_1] \\
&= [cx_1 + cy_1, cx_1 - cy_1, cz_1]
\end{align*}
\]

So, \( T(cu) = cT(u) \).
Therefore, $T$ is a linear transformation.

- Example: Determine whether $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by
  
  $$T([x, y]) = [x^2, y]$$

  is a linear transformation.

  1. Let $u = [x_1, y_1]$ and $v = [x_2, y_2]$. Then we want to prove $T(u + v) = T(u) + T(v)$.

     $$T(u + v) = T([x_1, y_1] + [x_2, y_2]) = T([x_1 + x_2, y_1 + y_2]) = ([x_1 + x_2]^2, y_1 + y_2) = [x_1^2 + 2x_1x_2 + x_2^2, y_1 + y_2]$$

     and

     $$T(u) + T(v) = T([x_1, y_1]) + T([x_2, y_2]) = [x_1^2, y_1] + [x_2^2, y_2] = [x_1^2 + x_2^2, y_1 + y_2]$$

  Since, $T(u + v) \neq T(u) + T(v)$, $T$ is not a linear transformation. There is no need to test the second criteria. However, you could have proved the same thing using the second criteria:

  2. We would want to prove $T(cu) = cT(u)$.

     $$T(cu) = T(c[x_1, y_1]) = T([cx_1, cy_1]) = ([cx_1]^2, cy_1) = [c^2x_1^2, cy_1]$$

     and

     $$cT(u) = cT([x_1, y_1]) = c[x_1^2, y_1] = [c^2x_1^2, cy_1]$$

  So, $T(cu) \neq cT(u)$ either. Thus, again, we would have showed, $T$ was not a linear transformation.

- Two Simple Linear Transformations:
  
  - Zero Transformation: $T: V \rightarrow W$ such that $T(v) = 0$ for all $v$ in $V$
  
  - Identity Transformation: $T: V \rightarrow V$ such that $T(v) = v$ for all $v$ in $V$

- Theorem: Let $T$ be a linear transformation from $V$ into $W$, where $u$ and $v$ are in $V$. Then

   1. $T(0) = 0$
   2. $T(-v) = -T(v)$
   3. $T(u - v) = T(u) - T(v)$
   4. If

      $$v = c_1v_1 + c_2v_2 + ... + c_nv_n$$

      then

      $$T(v) = c_1T(v_1) + c_2T(v_2) + ... + c_nT(v_n)$$

- Example: Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that

  $$T([1, 0, 0]) = [2, 4, -1] \quad T([0, 1, 0]) = [1, 3, -2] \quad T([0, 0, 1]) = [0, -2, 2]$$

  Find $T([-2, 4, -1])$. Since

  $$[-2, 4, -1] = -2[1, 0, 0] + 4[0, 1, 0] - 1[0, 0, 1]$$

  we can say

  $$T([-2, 4, -1]) = -2T([1, 0, 0]) + 4T([0, 1, 0]) - 1T([0, 0, 1]) = -2[2, 4, -1] + 4[1, 3, -2] - [0, -2, 2] = [0, 6, -8]$$
• **Theorem:** Let $A$ be a $mn$ matrix. The function $T$ defined by

$$T(v) = Av$$

is a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$.

• **Examples:**
  - If $T(v) = Av$ where
    $$A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}$$
    then $T : \mathbb{R}^2 \to \mathbb{R}^3$.
  - If $T(v) = Av$ where
    $$A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$$
    then $T : \mathbb{R}^5 \to \mathbb{R}^2$.

• **Standard Matrix:** Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has a $mn$ standard matrix $A$ associated with it where

$$T(v) = Av$$

To find the standard matrix, apply $T$ to the basis elements in $\mathbb{R}^n$. This produces vectors in $\mathbb{R}^m$ which become the *columns* of $A$:

For example, let

$$T([x_1, x_2, x_3]) = [2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3]$$

Then

$$T([1, 0, 0]) = [2, -1, 0] \quad T([0, 1, 0]) = [1, 3, 3] \quad T([0, 0, 1]) = [-1, -2, 4]$$

these vectors become the *columns* of $A$:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix}$$
• **Shortcut Method for Finding the Standard Matrix:** Two examples:

1. Let $T$ be the linear transformation from above, i.e.,
   
   $$T([x_1, x_2, x_3]) = [2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3]$$

   Then the first, second and third components of the resulting vector $w$, can be written respectively as
   
   $$w_1 = 2x_1 + x_2 - x_3$$
   $$w_2 = -x_1 + 3x_2 - 2x_3$$
   $$w_3 = 3x_2 + 4x_3$$

   Then the standard matrix $A$ is given by the coefficient matrix or the right hand side:
   
   $$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix}$$

   So,
   
   $$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2. **Example:** Let
   
   $$T([x, y, z]) = [x - 2y, 2x + y]$$

   Since $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $A$ is a 3x2 matrix:
   
   $$w_1 = x - 2y + 0z$$
   $$w_2 = 2x + y + 0z$$

   So,
   
   $$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

• **Geometric Operators:**
  
  – **Reflection Operators:**
    
    * **Reflection about the y-axis:** The schematic of reflection about the y-axis is given below. The transformation is given by
      
      $$w_1 = -x$$
      $$w_2 = y$$

      with standard matrix
      
      $$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

      ![Reflection about the y-axis](image)
* Reflection about the $x$-axis: The schematic of reflection about the $x$-axis is given below. The transformation is given by

\[ w_1 = x \]
\[ w_2 = -y \]

with standard matrix

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

* Reflection about the line $y = x$: The schematic of reflection about the line $y = x$ is given below. The transformation is given by

\[ w_1 = y \]
\[ w_2 = x \]

with standard matrix

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

– Projection Operators:

* Projected onto $x$-axis: The schematic of projection onto the $x$-axis is given below. The transformation is given by

\[ w_1 = x \]
\[ w_2 = 0 \]

with standard matrix

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]
* Projected onto y-axis: The schematic of projection onto the y-axis is given below. The transformation is given by

\[
\begin{align*}
  w_1 &= 0 \\
  w_2 &= y
\end{align*}
\]

with standard matrix

\[
A = \begin{bmatrix}
  0 & 0 \\
  0 & 1
\end{bmatrix}
\]

* In \( \mathbb{R}^3 \), you can project onto a plane. The standard matrices for the projection is given below.

  · *Projection onto xy-plane:*

\[
A = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 0
\end{bmatrix}
\]

  · *Projection onto xz-plane:*

\[
A = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

  · *Projection onto yz-plane:*

\[
A = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]
- **Rotation Operator:** We can consider rotating through an angle $\theta$.

If we look at a more detailed depiction of the rotation, as depicted below, we see how we can use trigonometric identities to recover the standard matrix.

Using trigonometric identities, we have

\[
x = r \cos(\phi) \\
y = r \sin(\phi)
\]

and

\[
w_1 = r \cos(\theta + \phi) \\
w_2 = r \sin(\theta + \phi)
\]

Using trigonometric identities on $w_1$ and $w_2$, we have

\[
w_1 = r \cos(\theta) \cos(\phi) - r \sin(\theta) \sin(\phi) \\
w_2 = r \sin(\theta) \cos(\phi) + r \cos(\theta) \sin(\phi)
\]

which equals

\[
w_1 = x \cos(\theta) - y \sin(\theta) \\
w_2 = x \sin(\theta) + y \cos(\theta)
\]

if we plug in $x$ and $y$ formulas from above. Therefore, the standard matrix is given by

\[
A = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}
\]
Dilation and Contraction Operators: We can consider the geometric process of dilating or contracting vectors. For example, in \( \mathbb{R}^2 \), the contraction of a vector is given below where \( 0 < k < 1 \).

\[
\begin{pmatrix}
k & 0 \\
0 & k
\end{pmatrix}
\]

In each case, the standard matrix is given by

If

* \( 0 < k < 1 \), we have contraction and
* \( k > 1 \), we have dilation

In \( \mathbb{R}^3 \), we have the standard matrix

\[
\begin{pmatrix}
k & 0 & 0 \\
0 & k & 0 \\
0 & 0 & k
\end{pmatrix}
\]

• **One-to-One linear transformations**: In college algebra, we could perform a horizontal line test to determine if a function was one-to-one, i.e., to determine if an inverse function exists. Similarly, we say a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) is one-to-one if \( T \) maps distinct vectors in \( \mathbb{R}^n \) into distinct vectors in \( \mathbb{R}^m \). In other words, a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) is one-to-one if for every \( w \) in the range of \( T \), there is exactly one \( v \) in \( \mathbb{R}^n \) such that \( T(v) = w \).

• **Examples**:

1. The rotation operator is one-to-one, because there is only one vector \( v \) which can be rotated through an angle \( \theta \) to get any vector \( w \).
2. The projection operator is not one-to-one. For example, both \( [2, 4] \) and \( [2, -1] \) can be projected onto the \( x \)-axis and result in the vector \( [2, 0] \).

• **Linear system equivalent statements**: Recall that for a linear system, the following are equivalent statements:

1. \( A \) is invertible
2. \( Ax = b \) is consistent for every \( n \times 1 \) matrix \( b \)
3. \( Ax = b \) has exactly one solution for every \( n \times 1 \) matrix \( b \)

• Recall, that for every linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \), we can represent the linear transformation as

\[
T(v) = Av
\]

where \( A \) is the \( m \times n \) standard matrix associated with \( T \). Using the above equivalent statements with this form of the linear transformation, we have the following theorem.
**Theorem:** If $A$ is an $n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$T(v) = Av$$

then the following is equivalent.

1. $A$ is invertible
2. For every $w$ in $\mathbb{R}^n$, there is some vector $v$ in $\mathbb{R}^n$ such that $T(v) = w$, i.e., the range of $T$ is $\mathbb{R}^n$.
3. For every $w$ in $\mathbb{R}^n$, there is a unique vector $v$ in $\mathbb{R}^n$ such that $T(v) = w$, i.e., $T$ is one-to-one.

**Examples:**

1. **Rotation Operator:** The standard matrix for the rotation operator is given by

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

To determine if $A$ is invertible, we can find the determinant of $A$:

$$|A| = \cos^2(\theta) + \sin^2(\theta) = 1 \neq 0$$

so $A$ is invertible. Therefore, the range of the rotation operator in $\mathbb{R}^2$ is all of $\mathbb{R}^2$ and it is one-to-one.

2. **Projection Operators:** For each projection operator, we can easily show that $|A| = 0$. Therefore, the projection operator is not one-to-one.

**Inverse Operator:** If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one transformation given by

$$T(v) = Av$$

where $A$ is the standard matrix, then there exists an inverse operator $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and is given by

$$T^{-1}(w) = A^{-1}v$$

**Examples:**

1. The standard matrix for the rotation operator through an angle $\theta$ is

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

The inverse operator can be found by rotating back through an angle $-\theta$, i.e.,

$$A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

Using trigonometric identities, we can see this is the same as

$$A^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

2. Let

$$T([x, y]) = [2x + y, 3x + 4y]$$

Then $T$ has the standard matrix

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

Thus, $|A| = 5 \neq 0$, so $T$ is one-to-one and has an inverse operator with standard matrix

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 4/5 & -1/5 \\ -3/5 & 2/5 \end{bmatrix}$$

So, the inverse operator is given by

$$T^{-1}(w) = A^{-1} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4/5 w_1 - 1/5 w_2, -3/5 w_1 + 2/5 w_2 \end{bmatrix}$$
• **Kernel of** $T$: One of the properties of linear transformations is that

$$T(0) = 0$$

There may be other vectors $v$ in $V$ such that $T(v) = 0$. The *kernel of* $T$ is the set of all vectors $v$ in $V$ such that

$$T(v) = 0$$

It is denoted $\ker(T)$.

• **Example:** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T([x_1, x_2]) = [x_1 - 2x_2, 0, -x_1]$$

To find $\ker(T)$, we need to find all vectors $v = [x_1, x_2]$ in $\mathbb{R}^2$, such that $T(v) = 0 = [0, 0, 0]$ in $\mathbb{R}^3$. In other words,

$$x_1 - 2x_2 = 0$$

$$0 = 0$$

$$-x_1 = 0$$

The only solution to this system if $[0, 0]$. Thus

$$\ker(T) = \{[0, 0]\} = \{0\}$$

• **Example:** Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x) = Ax$ where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$$

To find $\ker(T)$, we need to find all $v = [x_1, x_2, x_3]$ such that $T(v) = [0, 0]$. In other words, we need to solve the system

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting this in augmented form, we have

$$\begin{bmatrix} 1 & -1 & -2 & | & 0 \\ -1 & 2 & 3 & | & 0 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}$$

Therefore, $x_3 = t$ is a free parameter, so the solutions is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} t$$

Therefore, $\ker(T) = \text{span} \{[1, -1, 1]\}$.

• **Corollary:** If $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$T(v) = Av$$

then $\ker(T)$ is equal to the nullspace of $A$.

• **Example:** Given $T(v) = Av$ where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

find a basis for $\ker(T)$.

Solving the system, we have

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1/2 \end{bmatrix}$$

Therefore, a basis for $\ker(T)$ is given by a basis for the nullspace of $A$: $\{[-2, -1/2, 1]\}$. 


• **Example:** Given \( T(v) = Av \) where

\[
A = \begin{bmatrix}
1 & 2 & 0 & 1 & -1 \\
2 & 1 & 3 & 1 & 0 \\
-1 & 0 & -2 & 0 & 1 \\
0 & 0 & 0 & 2 & 8
\end{bmatrix}
\]

find a basis for \( ker(T) \).

Ans: \([[-2, 1, 1, 0, 0], [1, 2, 0, -4, 1]]\)

• **Terminology:** The dimension of \( ker(T) \) is called the nullity of \( T \). In the previous example, the nullity of \( T \) is 2.

• **Range of \( T \):** The range of \( T \) is the set of all vectors \( w \) such that \( T(v) = w \). If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is given by

\[
T(v) = Av
\]

then the range of \( T \) is the column space of \( A \).

• **Onto:** If \( T : V \rightarrow W \) is a linear transformation from a vector space \( V \) to a vector space \( W \), then \( T \) is said to be onto (or onto \( W \)) if every vector in \( W \) is the image of at least one vector in \( V \), i.e., the range of \( T = W \).

• **Equivalence Statements for One-to-One, Kernel:** If \( T : V \rightarrow W \) is a linear transformation, then the following are equivalent:

1. \( T \) is one-to-one
2. \( ker(T) = \{0\} \)

• **Equivalence Statements for One-to-One, Kernel, and Onto:** If \( T : V \rightarrow V \) is a linear transformation and \( V \) is finite-dimensional, then the following are equivalent:

1. \( T \) is one-to-one
2. \( ker(T) = \{0\} \)
3. \( T \) is onto

• **Isomorphism:** If a linear transformation \( T : V \rightarrow W \) is both one-to-one and onto, then \( T \) is said to be an *isomorphism* and the vector spaces \( V \) and \( W \) are said to be *isomorphic*. 