Matrices and Matrix Operations

Linear Algebra

MATH 2010

- **Basic Definition and Notation for Matrices**

  – If m and n are positive integers, then an mxn **matrix** is a rectangular array of numbers (entries)

$$m \text{ rows}\left\{ \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \right.$$

  where a_{ij} is the number corresponding to the i^{th} row and j^{th} column. i is the row subscript and j is the column subscript.

  – The **size** of the matrix is mxn.

  – Matrices are denoted by capital letters: A, B, C, etc.

  – If $m = n$, then the matrix is said to be **square**.

  – For a square matrix, a_{11}, a_{22}, a_{33}, ..., a_{nn} is called the **main diagonal**.

  – $tr(A)$ denotes the **trace** of A which is the sum of the diagonal elements. For example, if

$$A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & -1 & 4 \\ 3 & 1 & 2 \end{bmatrix}$$

  then the diagonal elements are 3, -1, and 2, so

$$tr(A) = 3 + (-1) + 2 = 4$$

  – A **column vector** is a matrix with only 1 column, i.e., it has size mx1. Example:

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

  – A **row vector** is a matrix with only 1 row, i.e., it has size 1xn. Example:

$$\begin{bmatrix} 0 & -1 & 2 & 8 \end{bmatrix}$$

  – Two matrices are **equal** if they are the same size and all the entries are the exact same. For example, if

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} a & 4 \\ -1 & b \end{bmatrix},$$

  what do a and b have to equal for $A = B$?

- **Adding and Subtracting Matrices**: IMPORTANT!!! In order to add/subtract matrices, matrices must be the **SAME** size. If two matrices are the same size, then to add (subtract) them, we simply add (subtract) corresponding elements.

  Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}$$

  Then

$$\begin{aligned} A + B &= \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2+2 & 1+(-3) & 1+4 \\ -1+(-3) & -1+1 & 4+(-2) \end{bmatrix} \\ &= \begin{bmatrix} 4 & -2 & 5 \\ -4 & 0 & 2 \end{bmatrix} \end{aligned}$$

  and

$$\begin{aligned} A - B &= \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2-2 & 1-(-3) & 1-4 \\ -1-(-3) & -1-1 & 4-(-2) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4 & -3 \\ 2 & -2 & 6 \end{bmatrix} \end{aligned}$$

- **Scalar Multiplication:** Example, $2A$. In order to do scalar multiplication, multiply all entries by the scalar. For example, using the matrix A from above, i.e.,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix},$$

  we can calculate $2A$ as

$$2A = \begin{bmatrix} 2(2) & 2(1) & 2(1) \\ 2(-1) & 2(-1) & 2(4) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ -2 & -2 & 8 \end{bmatrix}$$

- **Linear Combination:** If A_1, A_2, ..., A_n are matrices of the same size and c_1, c_2, ..., c_n are scalars, then

$$c_1A_1 + c_2A_2 + ... + c_nA_n$$

  is called a **linear combination** of A_1, A_2, ..., A_n with coefficients c_1, c_2, ..., c_n. For example, if

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 5 & -1 \\ 1 & 0 & -4 \end{bmatrix}$$

then

$$\begin{aligned}
2A - 3B + C &= 2\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix} - 3\begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 5 & -1 \\ 1 & 0 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 2(2) & 2(1) & 2(1) \\ 2(-1) & 2(-1) & 2(4) \end{bmatrix} - \begin{bmatrix} 3(2) & 3(-3) & 3(4) \\ 3(-3) & 3(1) & 3(-2) \end{bmatrix} + \begin{bmatrix} 0 & 5 & -1 \\ 1 & 0 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 2 & 2 \\ -2 & -2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & -9 & 12 \\ -9 & 3 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 5 & -1 \\ 1 & 0 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 4-6+0 & 2-(-9)+5 & 2-12+(-1) \\ -2-(-9)+1 & -2-3+0 & 8-(-6)+(-4) \end{bmatrix} \\
&= \begin{bmatrix} -2 & 16 & -11 \\ 8 & -5 & 10 \end{bmatrix}
\end{aligned}$$

- **Properties of Matrix Addition and Substraction:** Let A, B and C be mxn matrices and c and d be scalars, then
  1. $A + B = B + A$ Commutative Property of Addition
  2. $A + (B + C) = (A + B) + C$ Associative Property of Addition
  3. $(cd)A = c(dA)$ Associative Property of Scalar Multiplication
  4. $1A = A$ Multiplicative Identity
  5. $c(A + B) = cA + cB$ Distributive Property
  6. $(c + d)A = cA + dA$ Distributive Property
- **Transposes:**
  - The transpose of a matrix is denoted A^T. To find the transpose of a matrix, you interchange the rows and columns. In other words, you can think about it as write all the rows as columns or all the columns as rows. For example, if

    $$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix},$$

    then

    $$A^T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

    Find the transpose of

    $$A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 3 & -2 \end{bmatrix}$$

  - Notice that if A is mxn, then A^T is nxm.
  - Some Properties of Transposes
    1. $(A^T)^T = A$
    2. $(A + B)^T = A^T + B^T$
    3. $(cA)^T = cA^T$
  - A matrix is said to be **symmetric** if $A = A^T$.
- Sample Problems:
  1. Let

     $$A = \begin{bmatrix} 2 & -1 & 5 \\ 0 & 3 & 5 \end{bmatrix}$$

     Find A^T.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 5 & 4 \end{bmatrix}$$

   Find $(A + (2B)^T)^T$.

3. Find c and d so that

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & c \\ -d & 4 & 0 \end{bmatrix}$$

   is symmetric.

- **Special Matrices:** There are two special matrices,
  - **Identity Matrix** is denoted by I or I_n where n denotes a square matrix of size nxn. The identitity matrix is a square matrix with 1's on the diagonal and 0's as all other elements:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - **Zero Matrix**
    * The *zero matrix* is denoted by O or O_{mxn} where O is a matrix of size mxn. This is simply a matrix with all zeros. Example:

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{or} \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

    * Properties of the Zero Matrix
      1. $A + O = A$ where it is understood that O has the same size as A.
      2. $A + (-A) = O$
      3. If $cA = O$, then $c = 0$ or $A = O$.

- **Matrix Multiplication:** Matrix multiplication is more involved. You can **NOT** multiply corresponding entries!!
  - To help understand the process of matrix multiplication, we will first examine an applied problem which uses the same strategy as is used in matrix multiplication. Assume you are at a football stadium where there are three different refreshment centers, the south stand, north stand and west stand. At each stand, they are selling peanuts, hot dogs and soda. See the figure below.

Football Stadium

	Number items sold: peanuts	hot dogs	soda	Selling price	
South stand	120	250	305	2.00	peanuts
North stand	207	140	419	3.00	hot dogs
West stand	29	120	190	2.75	soda

Assume you want to know how much total the south stand made. You need to multiply the number of each of the items sold by the south stand (in the first row of the matrix) by the selling price of each item (given in the column vector containing selling price). In other words, you need to isolate the first row and multiply by the corresponding items in the column and then add:

$$\begin{bmatrix} 120 & 250 & 305 \end{bmatrix} \begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix} = 120(2.00) + 250(3.00) + 305(2.75) = 1828.75.$$

So, the south stand sold a total of \$ 1828.75.
Similarly, the north stand sold

$$\begin{bmatrix} 207 & 140 & 419 \end{bmatrix} \begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix} = 207(2.00) + 140(3.00) + 419(2.75) = 1986.25.$$

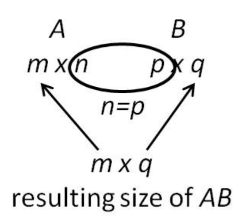
And the west stand sold

$$\begin{bmatrix} 29 & 120 & 190 \end{bmatrix} \begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix} = 29(2.00) + 120(3.00) + 190(2.75) = 940.50.$$

What we just did was matrix multiplication. We multiplied a 3x3 matrix by a 3x1 matrix to get a 3x1 matrix:

$$\begin{bmatrix} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{bmatrix} \begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix} = \begin{bmatrix} 1828.75 \\ 1986.25 \\ 940.5 \end{bmatrix}$$

– **Size of matrices is important!** Notice above, that we multiplied two matrices together, one was size 3x3 and the other was size 3x1. They are NOT the same size. Let A be a mxn matrix and B be a pxq matrix. In order to multiply AB, *the number of columns of* A *must equal the number of rows of* B. The schematic below will help.



So, if A be a mxn matrix and B be a pxq, then in order to multiply AB, n must equal p and the resulting size of AB is mxq.

– **Examples:** First, determine if it is possible to find AB and BA by looking at the sizes of the matrix. If so, what is the size of the resulting matrix? Find the resulting matrix.

1. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 & 2 & 3 \\ 3 & 4 & 0 & 1 \end{bmatrix}$
3. $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$

- **Properties of Matrix Multiplication:** Let A, B, and C be matrices of appropriate size for matrix multiplication and c be a scalar, then the following properties hold.
  1. $A(BC) = (AB)C$ Associative property of multiplication
  2. $A(B + C) = AB + AC$ Distributive property
  3. $(A + B)C = AC + BC$ Distributive property
  4. $c(AB) = (cA)B = A(cB)$
- **Commutativity:** In general, $AB \neq BA$!. Note that if AB is defined, BA may not be defined.
- **Cancelation:** If $AC = BC$, you can NOT say $A = B$. You can not simply cancel like in scalar multiplication. There are conditions on C which must be met in order to apply the cancelation principle. We will discuss these conditions in a later section. As an example, given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix}$$

  find AC and BC. You will notice they are equal even though $A \neq B$.
- **Properties for the Identity Matrix**
  1. $AI = A$
  2. $IA = A$
- **Multiplication Property of Transposes:** $(AB)^T = B^T A^T$. Note that the order reverses!!!!
- Prove AA^T is symmetric for any matrix A. To prove AA^T is symmetric, I need to prove $(AA^T)^T = AA^T$, so starting with $(AA^T)^T$ I have:

$$\begin{aligned} (AA^T)^T &= (A^T)^T A^T && \text{by properties of transposes: } (AB)^T = B^T A^T \\ &= AA^T && \text{by properties of transposes: } (A^T)^T = A \end{aligned}$$

- **Powers:**
  * $A^0 = \mathrm{I}$
  * $A^n = \underbrace{A \cdot A \cdot ... \cdot A}_{n \text{ times}} \ (n > 0)$
- **Finding a single row or column of AB:**
  * If you only need to find the i^{th} row of AB, then multiply the i^{th} row of A by all of B.

$$\left[i^{th} \text{ row of } A \right] [B]$$

  * If you only need to find the j^{th} column of AB, then multiply A by the j^{th} column of B.

$$[A] \left[j^{th} \text{ column of } B \right]$$

  * **Example:** Let

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$$

    Then the second row of AB is given by

$$\begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 12 \end{bmatrix}$$

    and the first column of AB is

$$\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$