Span, Linear Independence and Basis

Linear Algebra

MATH 2010

- **Span:**

  - **Linear Combination:** A vector v in a vector space V is called a *linear combination* of vectors u_1, u_2, ..., u_k in V if there exists scalars c_1, c_2, ..., c_k such that v can be written in the form

$$v = c_1 u_1 + c_2 u_2 + ... + c_k u_k$$

  - **Example:** Is $v = [2, 1, 5]$ is a linear combination of $u_1 = [1, 2, 1]$, $u_2 = [1, 0, 2]$, $u_3 = [1, 1, 0]$.

    To determine whether or not v is a linear combination of u_1, u_2, and u_3, it is necessary to determine if there exists scalars c_1, c_2, and c_3, such that

$$c_1 u_1 + c_2 u_2 + c_3 u_3 = v$$

    In other words, is there a solution to

$$c_1[1, 2, 1] + c_2[1, 0, 2] + c_3[1, 1, 0] = [2, 1, 5]?$$

    Equating corresponding elements, this leads to the system

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & + & c_3 & = & 2 \\ 2c_1 & & & + & c_3 & = & 1 \\ c_1 & + & 2c_2 & & & = & 5 \end{array}$$

    Solving the system, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 5 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & -3 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

  - **Span:** The vectors v_1, v_2, ..., v_k in a vector space V are said to **span** V if *every* vector in V is a linear combination of v_1, v_2, ..., v_k. If $S = \{v_1, v_2, ..., v_k\}$, then we say that S *spans* V or V is *spanned by* S.
  - **Procedure:** To determine if S spans V:
    1. Choose an *arbitray* vector v in V.
    2. Determine if v is a linear combination of the given vectors in S.
       * If it is, then S spans V.
       * If it is not, then S does *not* span V.

- **Example:** Let V be the vector space $\Re^3$ and let

$$v_1 = [1,2,1] \;\; v_2 = [1,0,2] \;\; v_3 = [1,1,0]$$

Does $S = \{v_1, v_2, v_2\}$ span V?

1. Let $v = [x, y, z]$ be an arbitrary vector in $V = \Re^3$.
2. Are there constants c_1, c_2 and c_3 such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = v$$

for all $v = (x, y, z)$? Then

$$c_1[1,2,1] + c_2[1,0,2] + c_3[1,1,0] = [x, y, z]$$

results in the system

$$\begin{array}{rcrcrcl} c_1 & + & c_2 & + & c_3 & = & x \\ 2c_1 & & & + & c_3 & = & y \\ c_1 & + & 2c_2 & & & = z & \end{array}$$

Solving the system, we have

$$\left[\begin{array}{ccccc} 1 & 1 & 1 & | & x \\ 2 & 0 & 1 & | & y \\ 1 & 2 & 0 & | & z \end{array}\right] \rightarrow \left[\begin{array}{ccccc} 1 & 1 & 1 & | & x \\ 0 & -2 & -1 & | & y-2x \\ 0 & 1 & -1 & | & z-x \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc} 1 & 1 & 1 & | & x \\ 0 & 1 & -1 & | & z-x \\ 0 & 0 & -3 & | & y-2x+2(z-x) \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc} 1 & 1 & 1 & | & x \\ 0 & 1 & -1 & | & z-x \\ 0 & 0 & 1 & | & 1/3(y-2x+2(z-x)) \end{array}\right]$$

Notice, that for any x, y, and z, there is a solution to the above system! Therefore, for any arbitrary v, we can write

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3$$

so S spans V. We can also say $span\{v_1, v_2, v_3\} = \Re^3$.

- **Example:** $v_1 = [1,0,0]$, $v_2 = [0,1,0]$ and $v_3 = [0,0,1]$ trivially span $\Re^3$, because for any vector $v = [x, y, z]$ in $\Re^3$, we can write

$$[x, y, z] = x[1,0,0] + y[0,1,0] + z[0,0,1]$$

- **Example:** Let V be the vector space P_2. Let $S = \{p_1(t), p_2(t)\}$ where

$$p_1(t) = t^2 + 2t + 1 \;\; p_2(t) = t^2 + 2$$

Does S span V?

1. Let $p(t) = at^2 + bt + c$ be any arbitrary polynomial in P_2.
2. Does there exist c_1 and c_2 such that

$$p(t) = c_1 p_1(t) + c_2 p_2(t)?$$

or

$$c_1(t^2 + 2t + 1) + c_2(t^2 + 2) = at^2 + bt + c$$

Equating coefficients, we have the system

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & a \\ 2c_1 & & & = & b \\ c_1 & + & 2c_2 & = & c \end{array}$$

So we have

$$\begin{bmatrix} 1 & 1 & | & a \\ 2 & 0 & | & b \\ 1 & 2 & | & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & | & a \\ 0 & -2 & | & b-2a \\ 0 & 1 & | & c-a \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & | & a \\ 0 & 1 & | & c-a \\ 0 & 0 & | & b-2a+2(c-a) \end{bmatrix}$$

There is no solution for EVERY a, b, and c. Therefore, S does not span V.

- **Theorem** If $S = \{v_1, v_2, ..., v_n\}$ is a basis for a vector space V, then every vector in V can be written in *one and only one* way as a linear combination of vectors in S.
- **Example:** $S = \{[1, 2, 3], [0, 1, 2], [-2, 0, 1]\}$ is a basis for $\Re^3$. Then for any u in $\Re^3$,

$$u = c_1 v_1 + c_2 v_2 + c_3 v_3$$

has a unique solution for c_1, c_2, c_3.

$$[a, b, c] = c_1[1, 2, 3] + c_2[0, 1, 2] + c_3[-2, 0, 1]$$

results in the system

$$\begin{array}{rcrcrcl} c_1 & & & - & 2c_3 & = & a \\ 2c_1 & + & c_2 & & & = & b \\ 3c_1 & + & 2c_2 & + & c_3 & = & c \end{array}$$

or

$$Ac = u$$

where

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

The unique solution is

$$c = A^{-1}u$$

So, if

$$A^{-1} = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 4 \\ -1 & 2 & -1 \end{bmatrix}$$

then

$$c = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

which means that

$$\begin{array}{rcrcrcr} c_1 & = & -a & + & 4b & - & 2c \\ c_2 & = & 2a & - & 7b & + & 4c \\ c_3 & = & -a & + & 2b & - & c \end{array}$$

So, if $u = [1, 1, 0]$, then

$$c_1 = -1 + 4 = 3; \;\; c_2 = 2 - 7 = -5; \;\; c_3 = -1 + 2 = 1,$$

so

$$u = 3v_1 - 5v_2 + v_3$$

- **Example:** The set $S = \{1, t, t^2\}$ spans P_2:

$$at^2 + bt + c = cv_1 + bv_2 + av_3$$

- **Theorem** If $S = \{v_1, v_2, ..., v_k\}$ is a set of vectors in vector spave V, then span(S) is a subspace of V. Moreover, span(S) is the smallest subspace of V that contains S.

- **Linear Independence**
  - **Definition:** The vectors v_1, v_2, ..., v_k in a vector space V are said to be **linearly independent** if the only c_1, c_2, ..., c_k that make

$$c_1v_1 + c_2v_2 + ... + c_kv_k = 0$$

    are $c_1 = c_2 = ... = c_k = 0$. Otherwise, v_1, v_2, ..., v_k are **linearly dependent**.
  - **Example:** Are the vectors $v_1 = [1, 0, 1, 2]$, $v_2 = [0, 1, 1, 2]$ and $v_3 = [1, 1, 1, 3]$ in $\Re^4$ linearly independent or linearly dependent?

    Solve

$$c_1v_1 + c_2v_2 + c_3v_3 = c_1[1, 0, 1, 2] + c_2[0, 1, 1, 2] + c_3[1, 1, 1, 3] = [0, 0, 0, 0].$$

    This leads to the system

$$\begin{array}{rcrcrcl} c_1 & & & + & c_3 & = & 0 \\ & & c_2 & + & c_3 & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \\ 2c_1 & + & 2c_2 & + & 3c_3 & = & 0 \end{array}$$

    We have the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

    The solution is $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$, thus, v_1, v_2, and v_3 are linearly independent.
  - **Example:** Determine if the elements of S in $M_{2,2}$ is linearly independent or linearly dependent where

$$S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 370 & \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$$

    Solve the system

$$c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

    This leads to the system

$$\begin{array}{rcrcrcl} 2c_1 & + & 3c_2 & + & c_3 & = & 0 \\ c_1 & & & & & = & 0 \\ & & 2c_2 & + & 2c_3 & = & 0 \\ c_1 & + & c_2 & & & = & 0 \end{array}$$

    This leads to $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Therefore, the elements are linearly independent.
  - **Theorem** A set $S = \{v_1, v_2, ..., v_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors v_j can be written as a linear combination of the other vectors in S.
  - **Example:** Let $v_1 = [1, 2, -1]$, $v_2 = [1, -2, 1]$, $v_3 = [-3, 2, -1]$, and $v_4 = [2, 0, 0]$ in $\Re^3$. Is $S = \{v_1, v_2, v_3, v_4\}$ linearly dependent or linearly independent?

    This leads to the system

$$\begin{array}{rcrcrcrcl} c_1 & + & c_2 & - & 3c_3 & + & 2c_4 & = & 0 \\ 2c_1 & - & 2c_2 & + & 2c_3 & & & = & 0 \\ -c_1 & + & c_2 & - & c_3 & & & = & 0 \end{array}$$

    The solution is $c_1 = s$, $c_2 = 2s$, $c_3 = s$, and $c_4 = 0$ where is a free parameter, so there are an infinite number of solutions. Hence, S is linearly dependent. If we let $s = 1$, we can write

$$v_1 + 2v_2 + v_3 = 0$$

    or

$$v_3 = -v_1 - 2v_2$$

    is a linearly combination of the other vectors in S.

- **Example:** Let $p_1(t) = t^2+t+2$, $p_2(t) = 2t^2+t$ and $p_3(t) = 3t^2+2t+2$. Is $S = \{p_1(t), p_2(t), p_3(t)\}$ linearly independent or linearly dependent? Answer: linearly dependent.
- **Corollary** Two vectors u and v in a vector space V are linearly dependent if and only if one is a scalar mutliple of the other.
- **Example:** $S = \{[1, 2, 0], [-2, 2, 1]\}$. Since $v_1 \neq cv_2$, v_1 and v_2 are linearly independent.
- **Example:** $S = \{[4, -4, -2], [-2, 2, 1]\}$. Since $v_1 = -2v_2$, v_1 and v_2 are linearly dependent.

- **Basis**
  - **Definition:** The set of vectors $S = \{v_1, v_2, v_3, ..., v_n\}$ in a vector space V is called a **basis** for V if
    1. S spans V
    2. S is linearly independent
  - **Standard Basis for** $\Re^2$ $S = \{[1, 0], [0, 1]\}$ is a standard basis.

$$[x, y] = x[1, 0] + y[0, 1]$$

so S spans $\Re^2$ and

$$c_1[1, 0] + c_2[0, 1] = [0, 0]$$

leads to $c_1 = c_2 = 0$, so S is linearly independent. Therefore, S is a basis.

  - **Nonstandard Basis for** $\Re^2$**:**
    1. Determine whether $S = \{[1, 2], [1, -1]\}$ is a basis for $\Re^2$.
       (a) Does S span $\Re^2$? Let $v = [a, b]$ be a vector in $\Re^2$. Then we want c_1 and c_2 such that

$$c_1[1, 2] + c_2[1, -1] = [a, b]$$

In other words, we need to solve

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & a \\ 2c_1 & - & c_2 & = & b \end{array}$$

We end up with the row echelon form:

$$\begin{bmatrix} 1 & 1 & | & a \\ 0 & 1 & | & -1/3(b-2a) \end{bmatrix}$$

It has a solution for every a and b, so S spans $\Re^2$.

       (b) Is S linearly independent? We need to solve

$$c_1[1, 2] + c_2[1, -1] = [0, 0]$$

Notice, this is the exact same system as above with the right hand side zero, so it reduces to

$$\begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

which has the trivial solution, so it's linearly independent.

Since S spans $\Re^2$ and is linearly independent, S is a basis for $\Re^2$.

    2. Determine whether $S = \{[-1, 2], [1, -2], [2, 4]\}$ is a basis for $\Re^2$.
       (a) Does S span $\Re^2$? Let $v = [a, b]$ be a vector in $\Re^2$. Then we want c_1 and c_2 such that

$$c_1[-1, 2] + c_2[1, -2] + c_3[1, -2] = [a, b]$$

In other words, we need to solve

$$\begin{bmatrix} -1 & 1 & 2 & |a & \\ 2 & -2 & 4 & | & b \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 & | & -a \\ 0 & 0 & 1 & | & 1/8(b+2a) \end{bmatrix}$$

So, S spans $\Re^2$ since there is a solution for every vector $[a, b]$.

(b) Is S linearly independent. Again, this is the process of solving the same system as above with zeros on the right hand side to get

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since there is a free parameter ($c_2 = t$), there is not simply the trivial solution. Therefore, S is NOT linearly independent.

Since S isn't linearly independent, S is NOT a basis for $\Re^2$.

- **Theorem** If $S = \{v_1, v_2, ..., v_n\}$ is a basis for a vector space V, then every set containing more than n vectors is linearly dependent.

  Since $S = \{[1,0],[0,1]\}$ is a basis for $\Re^2$ and it contains 2 vectors, then we can use the theorem to say that since the previous example had 3 vectors, it was linearly dependent and thus not a basis.

- **Theorem** If $S = \{v_1, v_2, ..., v_n\}$ is a basis for a vector space V, then every set containing less than n vectors does not span V.

- **Theorem** If a vector space V has one basis with n vectors, then every basis for V has n vectors. n is called the *dimension* of V and denoted $\dim(V)$.

- **Standard Basis for Several Vector Spaces:**
  * Standard basis for $\Re^3$: $S = \{[1,0,0],[0,1,0],[0,0,1]\}$
  * Standard basis for $\Re^n$: $S = \{[1,0,...,0],[0,1,...,0],[0,...,0,1]\}$
  * Standard basis for P_3: $S = \{1, x, x^2, x^3\}$
  * Standard basis for P_n: $S = \{1, x, x^2, ..., x^n\}$
  * Standard basis for $M_{2,2}$: $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

- **Dimensions:**
  * $\dim(\Re^3) = 3$
  * $\dim(\Re^n) = n$
  * $\dim(P_3) = 4$
  * $\dim(P_n) = n + 1$
  * $\dim(M_{2,2}) = 4$
  * $\dim(M_{m,n}) = mn$

- Note that if we have the correct dimension, then to determine if the vectors in S are a basis, we can look at the determinant of the coefficient matrix to determine if S is a basis. If the determinant doesn't equal 0, then A is invertible, so we get the trivial solution for the homogeneous problem (and hence it is linearly independent) and a unique solution (hence at least one solution) for every element in the space (and hence it spans).