Span, Linear Independence and Basis
Linear Algebra
MATH 2010

- Span:
  - **Linear Combination**: A vector \( v \) in a vector space \( V \) is called a *linear combination* of vectors \( u_1, u_2, \ldots, u_k \) in \( V \) if there exists scalars \( c_1, c_2, \ldots, c_k \) such that \( v \) can be written in the form
    \[
    v = c_1 u_1 + c_2 u_2 + \ldots + c_k u_k
    \]
  - **Example**: Is \( v = [2, 1, 5] \) is a linear combination of \( u_1 = [1, 2, 1], \ u_2 = [1, 0, 2], \ u_3 = [1, 1, 0] \).

    To determine whether or not \( v \) is a linear combination of \( u_1, u_2, \) and \( u_3 \), it is necessary to determine if there exists scalars \( c_1, c_2, \) and \( c_3 \), such that
    \[
    c_1 u_1 + c_2 u_2 + c_3 u_3 = v
    \]

    In other words, is there a solution to
    \[
    c_1[1, 2, 1] + c_2[1, 0, 2] + c_3[1, 1, 0] = [2, 1, 5]?
    \]

    Equating corresponding elements, this leads to the system
    \[
    \begin{align*}
    c_1 + c_2 + c_3 &= 2 \\
    2c_1 + c_3 &= 1 \\
    c_1 + 2c_2 &= 5
    \end{align*}
    \]

    Solving the system, we have
    \[
    \begin{bmatrix}
    1 & 1 & 1 & | & 2 \\
    2 & 0 & 1 & | & 1 \\
    1 & 2 & 0 & | & 5
    \end{bmatrix} \rightarrow \begin{bmatrix}
    1 & 1 & 1 & | & 2 \\
    0 & -2 & -1 & | & -3 \\
    0 & 1 & -1 & | & 3
    \end{bmatrix} \\
    \rightarrow \begin{bmatrix}
    1 & 0 & 2 & | & -1 \\
    0 & 1 & -1 & | & 3 \\
    0 & 0 & -3 & | & 3
    \end{bmatrix} \\
    \rightarrow \begin{bmatrix}
    1 & 0 & 0 & | & 1 \\
    0 & 1 & 0 & | & 2 \\
    0 & 0 & 1 & | & -1
    \end{bmatrix}
    \]

- **Span**: The vectors \( v_1, v_2, \ldots, v_k \) in a vector space \( V \) are said to span \( V \) if every vector in \( V \) is a linear combination of \( v_1, v_2, \ldots, v_k \). If \( S = \{v_1, v_2, \ldots, v_k\} \), then we say that \( S \) spans \( V \) or \( V \) is spanned by \( S \).

- **Procedure**: To determine if \( S \) spans \( V \):
  1. Choose an arbitrary vector \( v \) in \( V \).
  2. Determine if \( v \) is a linear combination of the given vectors in \( S \).
     * If it is, then \( S \) spans \( V \).
     * If it is not, then \( S \) does not span \( V \).
Example: Let $V$ be the vector space $\mathbb{R}^3$ and let
\[ v_1 = [1, 2, 1] \quad v_2 = [1, 0, 2] \quad v_3 = [1, 1, 0] \]
Does $S = \{v_1, v_2, v_3\}$ span $V$?
1. Let $v = [x, y, z]$ be an arbitrary vector in $V = \mathbb{R}^3$.
2. Are there constants $c_1$, $c_2$ and $c_3$ such that
   \[ c_1 v_1 + c_2 v_2 + c_3 v_3 = v \]
   for all $v = (x, y, z)$? Then
   \[ c_1 [1, 2, 1] + c_2 [1, 0, 2] + c_3 [1, 1, 0] = [x, y, z] \]
results in the system
\[
\begin{align*}
  c_1 + c_2 + c_3 &= x \\
  2c_1 + c_3 &= y \\
  c_1 + 2c_2 &= z
\end{align*}
\]
Solving the system, we have
\[
\begin{bmatrix}
  1 & 1 & 1 & | & x \\
  2 & 0 & 1 & | & y \\
  1 & 2 & 0 & | & z
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 1 & 1 & | & x \\
  0 & -2 & -2 & | & y - 2x \\
  0 & 1 & -1 & | & z - x
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 1 & 1 & | & x \\
  0 & 1 & -1 & | & y - 2x + 2(z - x) \\
  0 & 0 & -2 & | & z - x \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 1 & 1 & | & x \\
  0 & 1 & -1 & | & z - x \\
  0 & 0 & 1 & | & 1/3(y - 2x + 2(z - x))
\end{bmatrix}
\]
Notice, that for any $x$, $y$, and $z$, there is a solution to the above system! Therefore, for any arbitrary $v$, we can write
\[ v = c_1 v_1 + c_2 v_2 + c_3 v_3 \]
so $S$ spans $V$. We can also say $\text{span}\{v_1, v_2, v_3\} = \mathbb{R}^3$.

Example: $v_1 = [1, 0, 0]$, $v_2 = [0, 1, 0]$ and $v_3 = [0, 0, 1]$ trivially span $\mathbb{R}^3$, because for any vector $v = [x, y, z]$ in $\mathbb{R}^3$, we can write
\[ [x, y, z] = x[1, 0, 0] + y[0, 1, 0] + z[0, 0, 1] \]

Example: Let $V$ be the vector space $P_2$. Let $S = \{p_1(t), p_2(t)\}$ where
\[ p_1(t) = t^2 + 2t + 1 \quad p_2(t) = t^2 + 2 \]
Does $S$ span $V$?
1. Let $p(t) = at^2 + bt + c$ be any arbitrary polynomial in $P_2$.
2. Does there exist $c_1$ and $c_2$ such that
   \[ p(t) = c_1 p_1(t) + c_2 p_2(t) \]
or
   \[ c_1 (t^2 + 2t + 1) + c_2 (t^2 + 2) = at^2 + bt + c \]
Equating coefficients, we have the system
\[
\begin{align*}
  c_1 + c_2 &= a \\
  2c_1 &= b \\
  c_1 + 2c_2 &= c
\end{align*}
\]
So we have
\[
\begin{bmatrix}
1 & 1 & a \\
2 & 0 & b \\
1 & 2 & c
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & a \\
0 & -2 & b - 2a \\
0 & 1 & c - a
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & a \\
0 & 1 & c - a \\
0 & 0 & b - 2a + 2(c - a)
\end{bmatrix}
\]

There is no solution for EVERY \(a, b,\) and \(c\). Therefore, \(S\) does not span \(V\).

- **Theorem** If \(S = \{v_1, v_2, \ldots, v_n\}\) is a basis for a vector space \(V\), then every vector in \(V\) can be written in *one and only one* way as a linear combination of vectors in \(S\).

- **Example:** \(S = \{[1, 2, 3], [0, 1, 2], [-2, 0, 1]\}\) is a basis for \(\mathbb{R}^3\). Then for any \(u\) in \(\mathbb{R}^3\),

\[
u = c_1v_1 + c_2v_2 + c_3v_3\]

has a unique solution for \(c_1, c_2, c_3\).

\([a, b, c] = c_1[1, 2, 3] + c_2[0, 1, 2] + c_3[-2, 0, 1]\)

results in the system
\[
\begin{align*}
c_1 & - 2c_3 = a \\
2c_1 + c_2 & = b \\
3c_1 + 2c_2 + c_3 & = c
\end{align*}
\]
or
\[
Ac = u
\]
where
\[
A = \begin{bmatrix}
1 & 0 & -2 \\
2 & 1 & 0 \\
3 & 2 & 1
\end{bmatrix}
\]
The unique solution is
\[
c = A^{-1}u
\]
So, if
\[
A^{-1} = \begin{bmatrix}
-1 & 4 & -2 \\
2 & -7 & 4 \\
-1 & 2 & -1
\end{bmatrix}
\]
then
\[
c = A^{-1} \begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
\]
which means that
\[
\begin{align*}
c_1 &= -a + 4b - 2c \\
c_2 &= 2a - 7b + 4c \\
c_3 &= -a + 2b - c
\end{align*}
\]
So, if \(u = [1, 1, 0]\), then
\[
\begin{align*}
c_1 &= -1 + 4 = 3; \\
c_2 &= 2 - 7 = -5; \\
c_3 &= -1 + 2 = 1,
\end{align*}
\]
so
\[
u = 3v_1 - 5v_2 + v_3
\]

- **Example:** The set \(S = \{1, t, t^2\}\) spans \(P_2\):

\[
at^2 + bt + c = cv_1 + bv_2 + av_3
\]

- **Theorem** If \(S = \{v_1, v_2, \ldots, v_k\}\) is a set of vectors in vector space \(V\), then \(\text{span}(S)\) is a subspace of \(V\). Moreover, \(\text{span}(S)\) is the smallest subspace of \(V\) that contains \(S\).
- **Linear Independence**

  - **Definition:** The vectors \( v_1, v_2, \ldots, v_k \) in a vector space \( V \) are said to be **linearly independent** if the only \( c_1, c_2, \ldots, c_k \) that make

  \[
  c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0
  \]

  are \( c_1 = c_2 = \ldots = c_k = 0 \). Otherwise, \( v_1, v_2, \ldots, v_k \) are **linearly dependent**.

  - **Example:** Are the vectors \( v_1 = [1, 0, 1, 2], v_2 = [0, 1, 1, 2] \) and \( v_3 = [1, 1, 1, 3] \) in \( \mathbb{R}^4 \) linearly independent or linearly dependent?

    Solve

    \[
    c_1 v_1 + c_2 v_2 + c_3 v_3 = c_1[1, 0, 1, 2] + c_2[0, 1, 1, 2] + c_3[1, 1, 1, 3] = [0, 0, 0, 0].
    \]

    This leads to the system

    \[
    \begin{align*}
    c_1 + c_3 &= 0 \\
    c_2 + c_3 &= 0 \\
    c_1 + c_2 + c_3 &= 0 \\
    2c_1 + 2c_2 + 3c_3 &= 0
    \end{align*}
    \]

    We have the system

    \[
    \begin{bmatrix}
    1 & 0 & 1 & 0 \\
    0 & 1 & 1 & 0 \\
    1 & 1 & 1 & 0 \\
    2 & 2 & 3 & 0
    \end{bmatrix} \rightarrow \begin{bmatrix}
    1 & 0 & 1 & 0 \\
    0 & 1 & 1 & 0 \\
    0 & 1 & 1 & 0 \\
    0 & 2 & 1 & 0
    \end{bmatrix} \rightarrow \begin{bmatrix}
    1 & 0 & 1 & 0 \\
    0 & 1 & 1 & 0 \\
    0 & 0 & -1 & 0 \\
    0 & 0 & 0 & 0
    \end{bmatrix} \rightarrow \begin{bmatrix}
    1 & 0 & 1 & 0 \\
    0 & 1 & 1 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0
    \end{bmatrix}
    \]

    The solution is \( c_1 = 0, c_2 = 0, \) and \( c_3 = 0 \), thus, \( v_1, v_2, \) and \( v_3 \) are linearly independent.

  - **Example:** Determine if the elements of \( S \) in \( M_{2,2} \) is linearly independent or linearly dependent where

    \[
    S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 370 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}
    \]

    Solve the system

    \[
    c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}
    \]

    This leads to the system

    \[
    \begin{align*}
    2c_1 + 3c_2 + c_3 &= 0 \\
    c_1 &= 0 \\
    2c_2 + 2c_3 &= 0 \\
    c_1 &= 0 \\
    c_2 &= 0
    \end{align*}
    \]

    This leads to \( c_1 = 0, c_2 = 0, \) and \( c_3 = 0 \). Therefore, the elements are linearly independent.

  - **Theorem** A set \( S = \{v_1, v_2, \ldots, v_k\} \), \( k \geq 2 \), is linearly dependent if and only if at least one of the vectors \( v_j \) can be written as a linear combination of the other vectors in \( S \).

  - **Example:** Let \( v_1 = [1, 2, -1], v_2 = [1, -2, 1], v_3 = [-3, 2, -1], \) and \( v_4 = [2, 0, 0] \) in \( \mathbb{R}^3 \). Is \( S = \{v_1, v_2, v_3, v_4\} \) linearly dependent or linearly independent?

    This leads to the system

    \[
    \begin{align*}
    c_1 + c_2 - 3c_3 + 2c_4 &= 0 \\
    2c_1 - 2c_2 + 2c_3 &= 0 \\
    -c_1 + c_2 - c_3 &= 0
    \end{align*}
    \]

    The solution is \( c_1 = s, c_2 = 2s, c_3 = s, \) and \( c_4 = 0 \) where \( s \) is a free parameter, so there are an infinite number of solutions. Hence, \( S \) is linearly dependent. If we let \( s = 1 \), we can write

    \[
    v_1 + 2v_2 + v_3 = 0
    \]

    or

    \[
    v_3 = -v_1 - 2v_2
    \]

    is a linear combination of the other vectors in \( S \).
- **Example:** Let \( p_1(t) = t^2 + t + 2, p_2(t) = 2t^2 + t \) and \( p_3(t) = 3t^2 + 2t + 2. \) Is \( S = \{p_1(t), p_2(t), p_3(t)\} \) linearly independent or linearly dependent? Answer: linearly dependent.

- **Corollary** Two vectors \( u \) and \( v \) in a vector space \( V \) are linearly dependent if and only if one is a scalar multiple of the other.

- **Example:** \( S = \{[1, 2, 0], [-2, 2, 1]\}. \) Since \( v_1 \neq cv_2, v_1 \) and \( v_2 \) are linearly independent.

- **Example:** \( S = \{[4, -4, -2], [-2, 2, 1]\}. \) Since \( v_1 = -2v_2, v_1 \) and \( v_2 \) are linearly dependent.

- **Basis**

  - **Definition:** The set of vectors \( S = \{v_1, v_2, v_3, ..., v_n\} \) in a vector space \( V \) is called a **basis** for \( V \) if
    1. \( S \) spans \( V \)
    2. \( S \) is linearly independent

- **Standard Basis for \( \mathbb{R}^2 \)** \( S = \{[1, 0], [0, 1]\} \) is a standard basis.

  - \( [x, y] = x[1, 0] + y[0, 1] \)

  - so \( S \) spans \( \mathbb{R}^2 \) and

  \[
  c_1[1, 0] + c_2[0, 1] = [0, 0]
  \]

  - leads to \( c_1 = c_2 = 0 \), so \( S \) is linearly independent. Therefore, \( S \) is a basis.

- **Nonstandard Basis for \( \mathbb{R}^2 \):**
  1. Determine whether \( S = \{[1, 2], [1, -1]\} \) is a basis for \( \mathbb{R}^2 \).
     (a) Does \( S \) span \( \mathbb{R}^2 \)? Let \( v = [a, b] \) be a vector in \( \mathbb{R}^2 \). Then we want \( c_1 \) and \( c_2 \) such that

     \[
     c_1[1, 2] + c_2[1, -1] = [a, b]
     \]

     - In other words, we need to solve

     \[
     \begin{align*}
     c_1 + c_2 &= a \\
     2c_1 - c_2 &= b
     \end{align*}
     \]

     - We end up with the row echelon form:

     \[
     \begin{bmatrix}
     1 & 1 & | & a \\
     0 & 1 & | & -1/3(b - 2a)
     \end{bmatrix}
     \]

     - It has a solution for every \( a \) and \( b \), so \( S \) spans \( \mathbb{R}^2 \).

     (b) Is \( S \) linearly independent? We need to solve

     \[
     c_1[1, 2] + c_2[1, -1] = [0, 0]
     \]

     - Notice, this is the exact same system as above with the right hand side zero, so it reduces to

     \[
     \begin{bmatrix}
     1 & 1 & | & 0 \\
     0 & 1 & | & 0
     \end{bmatrix}
     \]

     - which has the trivial solution, so it’s linearly independent.

     Since \( S \) spans \( \mathbb{R}^2 \) and is linearly independent, \( S \) is a basis for \( \mathbb{R}^2 \).

  2. Determine whether \( S = \{[-1, 2], [1, -2], [2, 4]\} \) is a basis for \( \mathbb{R}^2 \).

     (a) Does \( S \) span \( \mathbb{R}^2 \)? Let \( v = [a, b] \) be a vector in \( \mathbb{R}^2 \). Then we want \( c_1 \) and \( c_2 \) such that

     \[
     c_1[-1, 2] + c_2[1, -2] + c_3[1, -2] = [a, b]
     \]

     - In other words, we need to solve

     \[
     \begin{bmatrix}
     -1 & 1 & 2 & | & a \\
     2 & -2 & 4 & | & b
     \end{bmatrix} \rightarrow \begin{bmatrix}
     1 & -1 & -2 & | & -a \\
     0 & 0 & 1 & | & 1/8(b + 2a)
     \end{bmatrix}
     \]

     - So, \( S \) spans \( \mathbb{R}^2 \) since there is a solution for every vector \([a, b]\).
(b) Is $S$ linearly independent. Again, this is the process of solving the same system as above with zeros on the right hand side to get

$$
\begin{bmatrix}
1 & -1 & -2 & | & 0 \\
0 & 0 & 1 & | & 0
\end{bmatrix}
$$

Since there is a free parameter ($c_2 = t$), there is not simply the trivial solution. Therefore, $S$ is NOT linearly independent.

Since $S$ isn’t linearly independent, $S$ is NOT a basis for $\mathbb{R}^2$.

- **Theorem** If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space $V$, then every set containing more than $n$ vectors is linearly dependent.

Since $S = \{[1,0],[0,1]\}$ is a basis for $\mathbb{R}^2$ and it contains 2 vectors, then we can use the theorem to say that since the previous example had 3 vectors, it was linearly dependent and thus not a basis.

- **Theorem** If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space $V$, then every set containing less than $n$ vectors does not span $V$.

- **Theorem** If a vector space $V$ has one basis with $n$ vectors, then every basis for $V$ has $n$ vectors. $n$ is called the *dimension* of $V$ and denoted $\dim(V)$.

- **Standard Basis for Several Vector Spaces:**
  - Standard basis for $\mathbb{R}^3$: $S = \{[1,0,0],[0,1,0],[0,0,1]\}$
  - Standard basis for $\mathbb{R}^n$: $S = \{[1,0,\ldots,0],[0,1,\ldots,0],[0,\ldots,0,1]\}$
  - Standard basis for $P_3$: $S = \{1,x,x^2,x^3\}$
  - Standard basis for $P_n$: $S = \{1,x,x^2,\ldots,x^n\}$
  - Standard basis for $M_{2,2}$: $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

- **Dimensions:**
  - $\dim(\mathbb{R}^3) = 3$
  - $\dim(\mathbb{R}^n) = n$
  - $\dim(P_3) = 4$
  - $\dim(P_n) = n + 1$
  - $\dim(M_{2,2}) = 4$
  - $\dim(M_{m,n}) = mn$

- Note that if we have the correct dimension, then to determine if the vectors in $S$ are a basis, we can look at the determinant of the coefficient matrix to determine if $S$ is a basis. If the determinant doesn’t equal 0, then $A$ is invertible, so we get the trivial solution for the homogeneous problem (and hence it is linearly independent) and a unique solution (hence at least one solution) for every element in the space (and hence it spans).