Subspaces
Linear Algebra
MATH 2010

- **Definition of Subspace:** Let V be a vector space and W be a nonempty subset of V. If W is a vector space with respect to the operations in V, then W is a **subspace** of V.
- **Example:** Every vector space has at least two subspaces:
  1. itself
  2. the *zero subspace* consisting of just $\{0\}$, the zero element.
- **Theorem:** Let V be a vector space with operations $\oplus$ and $\odot$ and let W be a nonempty subst of V. Then W is a subspace of V if and only if the following conditions hold:
  A) If u and v are any elements of V then $u \oplus v$ is in V. (V is said to be closed under the operation $\oplus$.)
  S) If u is any element of V and c is any real number, then $c \odot u$ is in V. (V is said to be closed under the operation $\odot$.)

  Note that all the other properties are satisfied for W, since every element in W is an element of V and thus satisfies [A1]-[A4] and [S1]-[S4].
- **Example:** Let W be a subset of $\Re^3$ consisting of vectors of the form $(a, 0, b)$ with usual vector addition and scalar multiplication. (I.e., the second component must equal 0.) Is W a subspace of V? To determine whether or not W is a subspace, you must determine whether or not it is closed under addition ([A]) and closed under scalar multiplication ([S]).

  A) Need to show if u and v are any elements of W then $u \oplus v$ is in W. (W is closed under addition.)

  Let $u = (x, 0, z)$ and $v = (x', 0, z')$ be two vectors in W. (Note: this is the required form to be in W). Then

  $$u \oplus v = (x, 0, z) \oplus (x', 0, z') = (x, 0, z) + (x', 0, z') = (x + x', 0 + 0, z + z') = (x + x', 0, z + z')$$

  by using the standard definition of addition for $\Re^3$. Since $(x+x', 0, z+z')$ has a second component of 0, it has the right form and is in W. So, **property A is satisfied**!

  S) If u is any element of W and c is any real number, then $c \odot u$ is in W. (W is closed under scalar multiplication.)

  Let $u = (x, 0, z)$ be in W. Then

  $$c \odot u = c(x, 0, z) = (cx, c(0), cz) = (cx, 0, cz)$$

  by using the standard definition of scalar multiplication . Since $(cx, 0, cz)$ has a second component of 0, it has the right form and is in W. So, **property S is satisfied**!

  Since W is closed under addition and scalar multiplication, W is said to be a *subspace* of $\Re^3$.

- **Example:** Let W be a subset of $\Re^3$ consisting of vectors of the form $(a, 1, b)$ with usual vector addition and scalar multiplication. (I.e., the second component must equal 1.) Is W a subspace of V? To determine whether or not W is a subspace, you must determine whether or not it is closed under addition ([A]) and closed under scalar multiplication ([S]).

  A) Need to show if u and v are any elements of W then $u \oplus v$ is in W. (W is closed under addition.)

  Let $u = (x, 1, z)$ and $v = (x', 1, z')$ be two vectors in W. (Note: this is the required form to be in W). Then

$$u \oplus v = (x, 1, z) \oplus (x', 1, z') = (x, 1, z) + (x', 1, z') = (x + x', 1 + 1, z + z') = (x + x', 2, z + z')$$

  by using the standard definition of addition for $\Re^3$. Since $(x+x', 2, z+z')$ does **not** have a second component of 1, it is **not** in W. So, **property A is not satisfied**!

  Since W is not closed under addition, we do not need to go any further. W **is not a subspace.**

- **Example:** Let W be the set of all 2x3 matrices of the form

$$\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}$$

  where a, b, c, and d are arbitrary real numbers. Show W is a subspace of M_{23}if $\oplus$ and $\odot$ are the standard matrix addition and scalar multiplication.

  A) Let

$$u = \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix} \text{ and } v = \begin{bmatrix} a_2 & b_2 & 0 \\ 0 & c_2 & d_2 \end{bmatrix}$$

  be two elements in W. (Note: this is the required form to be in W). Then

$$\begin{aligned} u \oplus v &= \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix} \oplus \begin{bmatrix} a_2 & b_2 & 0 \\ 0 & c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & 0 \\ 0 & c_2 & d_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & 0 + 0 \\ 0 + 0 & c_1 + c_2 & d_1 + d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & 0 \\ 0 & c_1 + c_2 & d_1 + d_2 \end{bmatrix} \end{aligned}$$

  by using the standard definition of addition for M_{23}. Since $u \oplus v$ has the right form, it is in W. So, **property A is satisfied**!

  S) Let

$$u = \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix}$$

  be in W. Then

$$k \odot u = k \odot \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix} = k \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & c_1 & d_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & k(0) \\ k(0) & kc_1 & kd_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & 0 \\ 0 & kc_1 & kd_1 \end{bmatrix}$$

  by using the standard definition of scalar multiplication . Since $k \odot u$ has the right form, it is in W. So, **property S is satisfied**!

  Since W is closed under addition and scalar multiplication, W is said to be a *subspace* of M_{23}.

- **Example:** Consider the homogeneous system $Ax = 0$ where A is a mxn matrix and x in $\Re^n$ is a solution vector. Let W be the subset of $\Re^n$ that consists of all solutions to this homogeneous system. Determine whether or not W is a subspace of $\Re^n$ where $\oplus$ and $\odot$ are the standard vector addition and scalar multiplication in $\Re^n$. (In other words, if x is an element of W, then it satisfies $Ax = 0$.)

  A) Let x and y be elements of W, then $Ax = 0$ and $Ay = 0$ since this is the requirement to be in W. To prove [A], we must determine whether or not $x \oplus y$ is in W. Since we are using standard addition, $x \oplus y = x + y$. To determine whether or not $x + y$ is in W, we must determine if $A(x + y) = 0$ (the requirement to be in W).

  $$\begin{aligned} A(x+y) &= Ax + Ay && \text{by distributing property of matrix multiplication} \\ &= 0 + 0 && \text{since } x \text{ and } y \text{ are in } W,\ Ax = 0 \text{ and } Ay = 0 \\ &= 0 && \text{addition of zero vectors} \end{aligned}$$

  Therefore, $A(x + y) = 0$, so $x + y$ is in W. Thus, W **is closed under addition.**

  S) Let x be in W and c be a scalar. Since x is in W, $Ax = 0$ (requirement of elements in W). We need to show cx is also in W, i.e., we need to show $A(cx) = 0$

  $$\begin{aligned} A(cx) &= c(Ax) && \text{property of matrix multiplication} \\ &= c(0) && \text{since } x \text{ is in } W,\ Ax = 0 \\ &= 0 && \text{property of zero vector} \end{aligned}$$

  So, $A(cx) = 0$; therefore, cx is in W. Hence, W **is closed under scalar multiplication**.

  Since W is closed under both addition and scalar multiplication, W **is a subspace of** $\Re^n$.

- **Example:** Which of the following are subspaces of $\Re^2$ with normal addition and scalar multiplication.
  1. The set of points (x, y) in $\Re^2$ which lie on the line $x + 2y = 0$.
  2. The set of points (x, y) in $\Re^2$ which lie on the line $x + 2y = 1$.

- If W is a subset of $\Re^2$, then it is a subspace of $\Re^2$ if and only if one of the three possibilities is true:
  1. W consists of just $(0, 0)$.
  2. W consists of all points on a line that pass through the origin.
  3. W consists of all of $\Re^2$.

- **Example:** Which of the following subsets of $\Re^3$ is a subspace of $\Re^3$ with normal addition and scalar multiplication?
  1. $W = \{(x_1, x_2, 1); x_1 \text{ and } x_2 \text{ are real numbers}\}$
  2. $W = \{(x_1, x_1 + x_3, x_3); x_1 \text{ and } x_3 \text{ are real numbers}\}$