Vector Spaces
Linear Algebra
MATH 2010

- Recall that when we discussed vector addition and scalar multiplication, that there were a set of properties, such as distributive property, associative property, etc. *Any* set that satisfies these properties is called a *vector space* and the objects in the set are called *vectors*.

- **Definition of Vector Space:** A **real vector space** is a set of elements V together with two operations $\oplus$ and $\odot$ satisfying the following properties:

  A) If u and v are any elements of V then $u \oplus v$ is in V. (V is said to be closed under the operation $\oplus$.)

  A1) $u \oplus v = v \oplus u$ for u and v in V. (commutative property)

  A2) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ for u, v, and w in V. (associative property)

  A3) There is an element 0, called the zero vector, in V such that

  $$u \oplus 0 = 0 \oplus u = u$$

  for all u in V. (additive identity)

  A4) For each u in V, there is an element $-u$, called the negative of u, in V such that

  $$u \oplus -u = 0$$

  (additive inverse)

  S) If u is any element of V and c is any real number, then $c \odot u$ is in V. (V is said to be closed under the operation $\odot$.)

  S1) $c \odot (u \oplus v) = (c \odot u) \oplus (c \odot v)$ for all real numbers c and all u and v in V. (distributive property)

  S2) $(c + d) \odot u = (c \odot u) \oplus (d \odot u)$ for all real numbers c and d and all u in V. (distributive property)

  S3) $c \odot (d \odot u) = (cd) \odot u$ for all real numbers c and d and all u in V. (associative property)

  S4) $1 \odot u = u$ for all u in V. (scalar identity)

- **Example:** Consider the set V of all ordered triples of real numbers of the form $(x, y, 0)$ and define the operations $\oplus$ and $\odot$ by

  $$(x, y, 0) \oplus (x', y', 0) = (x + x', y + y', 0)$$

  and

  $$c \odot (x, y, 0) = (cx, cy, 0)$$

  Determine whether or not V is a vector space.

  To prove that V is a vector space, it is necessary to show all 10 properties are satisfied. *Intuitive perspective*: this is the regular vector addition and scalar multiplication on vectors in which the third component is 0, so one should *expect* all the properties to be satisfied. Now, you must **prove** all of them are satisfied.

  A) If u and v are any elements of V then $u \oplus v$ is in V. (V is said to be closed under the operation $\oplus$.)

  Let $u = (x, y, 0)$ and $v = (x', y', 0)$ be two vectors in V. (Note: this is the required form to be in V). Then

  $$u \oplus v = (x, y, 0) \oplus (x', y', 0) = (x + x', y + y', 0)$$

  by using the *definition* of $\oplus$. Since $(x + x', y + y', 0)$ has a third component of 0, it has the right form and is in V. So, **property A is satisfied**!

A1) $u \oplus v = v \oplus u$ for u and v in V. (commutative property)

Consider the u and v from above, i.e, $u = (x, y, 0)$ and $v = (x', y', 0)$, as two vectors in V. Then

$$\begin{aligned} u \oplus v &= (x, y, 0) \oplus (x', y', 0) && \text{plugging in } u \text{ and } v \\ &= (x + x', y + y', 0) && \text{using the definition of } \oplus \\ &= (x' + x, y' + y, 0) && \text{by the commutative property of scalars since } x,\ x',\ y, \text{ and } y' \\ & && \text{are all scalars} \end{aligned}$$

Similarly,

$$\begin{aligned} v \oplus u &= (x', y', 0) \oplus (x, y, 0) && \text{plugging in } v \text{ and } u \\ &= (x' + x, y' + y, 0) && \text{using the definition of } \oplus \end{aligned}$$

Comparing the two above, we have

$$u \oplus v = v \oplus u$$

Thus **A1 is satisfied**.

A2) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ for u, v, and w in V. (associative property)

Let $u = (x, y, 0)$, $v = (x', y', 0)$, and $w = (x'', y'', 0)$ be three vectors in V. Then the left hand side looks like

$$\begin{aligned} u \oplus (v \oplus w) &= (x, y, 0) \oplus ((x', y', 0) \oplus (x'', y'', 0)) && \text{plugging in } u, v, \text{ and } w \\ &= (x, y, 0) \oplus (x' + x'', y' + y'', 0) && \text{by definition of } \oplus \text{ on } v \text{ and } w \\ &= (x + (x' + x''), y + (y' + y''), 0) && \text{by definition of } \oplus \text{ on } u \text{ and } (x' + x'', y' + y'', 0) \\ &= ((x + x') + x'', (y + y') + y'', 0) && \text{by associative property of scalars} \end{aligned}$$

The right hand side is given by

$$\begin{aligned} (u \oplus v) \oplus w &= ((x, y, 0) \oplus (x', y', 0)) \oplus (x'', y'', 0) && \text{plugging in } u, v, \text{ and } w \\ &= (x + x', y + y', 0) \oplus (x'', y'', 0) && \text{by definition of } \oplus \text{ on } u \text{ and } v \\ &= ((x + x') + x'', (y + y') + y'', 0) && \text{by definition of } \oplus \text{ on } (x + x', y + y', 0) \text{ and } w \end{aligned}$$

Thus, comparing the left side and right side, they are equal. So, **A2 is satisfied**.

A3) There is an element 0, called the zero vector, in V such that

$$u \oplus 0 = 0 \oplus u = u$$

for all u in V. (additive identity)

Let $u = (x, y, 0)$ and $0 = (0, 0, 0)$. Notice that this is a vector in V since the last component is a 0 (that is the only requirement to be in V. Then

$$\begin{aligned} u \oplus 0 &= (x, y, 0) \oplus (0, 0, 0) && \text{plugging in } u, \text{ and } 0 \\ &= (x + 0, y + 0, 0) && \text{by definition of } \oplus \\ &= (x, y, 0) && \text{by the property of 0 in the real numbers} \\ &= u && \text{since } u = (x, y, 0) \end{aligned}$$

Thus, **A3 is satisfied**.

A4) For each u in V, there is an element $-u$, called the negative of u, in V such that

$$u \oplus -u = 0$$

(additive inverse)

Let $-u = (-x, -y, 0)$ be in V. Then

$$\begin{aligned} u \oplus -u &= (x, y, 0) \oplus (-x, -y, 0) && \text{by substituting in } u \text{ and } -u \\ &= (x + (-x), y + (-y), 0) && \text{by definition of } \oplus \\ &= (0, 0, 0) && \text{by additive inverse in the real numbers} \end{aligned}$$

Therefore, **A4 is satisfied**.

S) If u is any element of V and c is any real number, then $c \odot u$ is in V. (V is said to be closed under the operation $\odot$.)

Let $u = (x, y, 0)$ be in V. Then

$$c \odot u = (cx, cy, 0)$$

by using the *definition* of $\odot$. Since $(cx, cy, 0)$ has a third component of 0, it has the right form and is in V. So, **property S is satisfied**!

S1) $c \odot (u \oplus v) = (c \odot u) \oplus (c \odot v)$ for all real numbers c and all u and v in V. (distributive property)

Let $u = (x, y, 0)$ and $v = (x', y', 0)$ be in V, then looking at the left hand side above, we have

$$\begin{aligned} c \odot (u \oplus v) &= c \odot ((x, y, 0) \oplus (x', y', 0)) && \text{substituting in } u \text{ and } v \\ &= c \odot (x + x', y + y', 0) && \text{by definition of } \oplus \\ &= (c(x + x'), c(y + y'), 0) && \text{by definition of } \odot \end{aligned}$$

Now, looking at the right hand side above

$$\begin{aligned} (c \odot u) \oplus (c \odot v) &= (c \odot (x, y, 0)) \oplus (c \odot (x', y', 0)) && \text{substituting in } u \text{ and } v \\ &= (cx, cy, 0) \oplus (cx', cy', 0)) && \text{by definition of } \odot \\ &= (cx + cx', cy + cy', 0) && \text{by definition of } \oplus \\ &= (c(x + x'), c(y + y'), 0) && \text{by distributive property of real numbers} \end{aligned}$$

Since the left and right hand sides are equal, **property S1 has been satisfied**.

S2) $(c + d) \odot u = (c \odot u) \oplus (d \odot u)$ for all real numbers c and d and all u in V. (distributive property)

Let $u = (x, y, 0)$ be in V. Then the left hand side is

$$\begin{aligned} (c + d) \odot u &= (c + d) \odot (x, y, 0) && \text{by substituting in } u \\ &= ((c + d)x, (c + d)y, 0) && \text{by definition of } \odot \\ &= (cx + dx, cy + dy, 0) && \text{by distributive property of real numbers} \end{aligned}$$

The right hand side is simply

$$\begin{aligned} (c \odot u) \oplus (d \odot u) &= (c \odot (x, y, 0)) \oplus (d \odot (x, y, 0)) && \text{by substituting in } u \\ &= (cx, cy, 0) \oplus (dx, dy, 0) && \text{by definition of } \odot \\ &= (cx + dx, cy + dy, 0) && \text{by definition of } \oplus \end{aligned}$$

This is the same as the left hand side, so textbfproperty S2 has been satisfied.

S3) $c \odot (d \odot u) = (cd) \odot u$ for all real numbers c and d and all u in V. (associative property)

Again, let $u = (x, y, 0)$ be in V and c and d be scalars. Then

$$\begin{aligned}
c \odot (d \odot u) &= c \odot (d \odot (x, y, 0) && \text{by substituting in } u \\
&= c \odot (dx, dy, 0) && \text{by definition of } \odot \\
&= (c(dx), c(dy), 0) && \text{by definition of } \odot \\
&= ((cd)x, (cd)y, 0) && \text{by associative property of real numbers}
\end{aligned}$$

The right hand side

$$(cd) \odot u = (cd) \odot (x, y, 0) = ((cd)x, (cd)y, 0)$$

by substitution and definition of $\odot$. Therefore, textbfproperty S3 has been satisfied.

S4) $1 \odot u = u$ for all u in V. (scalar identity)

Finally,

$$1 \odot u = 1 \odot (x, y, 0) = (1x, 1y, 0) = (x, y, 0) = u$$

by substitution and property of multiplication by 1 for any real number. Therefore, **property S4 has been satisfied** as well.

Since all 10 properties have been proven to be satisfied, V is said to be a *vector space*.

- **Example:** Consider the set V of all ordered triples of real numbers of the form (x, y, z) with operations $\oplus$ and $\odot$ defined by

$$(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$$

and

$$c \odot (x, y, z) = (cx, y, z)$$

Determine whether or not V is a vector space.

To prove that V is a vector space, it is necessary to show all 10 properties are satisfied. *Intuitive perspective*: this is the regular vector addition but the scalar multiplication is different that standard scalar multiplication. Therefore, one should *expect* that all the addition properties will be satisfied, but the scalar multiplication properties may or may not be satisfied. If V is a vector space, then we have to prove that all 10 properties are satisfied. However, if even one property fails, then V is NOT a vector space. Therefore, it is beneficial to start with the properties that you think may fail. In this case, that would be the scalar multiplication properties.

S) If u is any element of V and c is any real number, then $c \odot u$ is in V. (V is said to be closed under the operation $\odot$.)

Let $u = (x, y, z)$ be an element of V. (Note: there is no special form to the elements of V - they are regular vectors of $\Re^3$.) Then

$$c \odot (x, y, z) = (cx, y, z)$$

Since (cx, y, z) is in $\Re^3$ and there is no special form to vectors in V other than being in $\Re^3$, then $c \odot u$ is in V. Thus **property S is satisfied**.

S1) $c \odot (u \oplus v) = (c \odot u) \oplus (c \odot v)$ for all real numbers c and all u and v in V. (distributive property)

Let $u = (x, y, z)$ and $v = (x', y', z')$ be in V, then

$$\begin{aligned} c \odot (u \oplus v) &= c \odot ((x, y, z) \oplus (x', y', z')) && \text{substituting in } u \text{ and } v \\ &= c \odot (x + x', y + y', z + z') && \text{definition of } \oplus \\ &= (c(x + x'), y + y', z + z') && \text{definition of } \odot \end{aligned}$$

The right hand side is given by

$$\begin{aligned} (c \odot u) \oplus (c \odot v) &= (c \odot (x, y, z)) \oplus (c \odot (x', y', z')) && \text{substituting in } u \text{ and } v \\ &= (cx, y, z) \oplus (cx', y', z') && \text{definition of } \odot \\ &= (cx + cx', y + y', z + z') && \text{definition of } \oplus \\ &= (c(x + x'), y + y', z + z') && \text{distributive property of real numbers} \end{aligned}$$

The left and right hand side are the same; therefore, **property S1 is satisfied**.

S2) $(c + d) \odot u = (c \odot u) \oplus (d \odot u)$ for all real numbers c and d and all u in V. (distributive property)

Let u be as above and c and d be scalars, then the left hand side is given by

$$\begin{aligned} (c + d) \odot u &= (c + d) \odot (x, y, z) && \text{substituting in } u \\ &= ((c + d)x, y, z) && \text{definition of } \odot \end{aligned}$$

The right hand side is given by

$$\begin{aligned} (c \odot u) \oplus (d \odot u) &= (c \odot (x, y, z)) \oplus (d \odot (x, y, z)) && \text{substituting in } u \\ &= (cx, y, z) \oplus (dx, y, z) && \text{definition of } \odot \\ &= (cx + dx, y + y, z + z) && \text{definition of } \oplus \\ &= ((c + d)x, 2y, 2z) && \text{distributive property of real numbers} \end{aligned}$$

Notice, that the left hand side and right hand side are NOT equal. Thus, **property S2 is NOT satisfied**.

Since we have found one property which is not satisfied, we do not need to go any further. V **is not a vector space.**

- **Important Vector Spaces** There are several important vector spaces worth noting that satisfy all 10 properties.

  1. The set of all n tuples, $\Re^n$, is a vector space with $\oplus$ and $\odot$ defined as the standard operations for addition and scalar multiplication.
  2. The set of all polynomials of degree less than or equal to n, denoted P_n where $\oplus$ and $\odot$ are the standard operations. An element in P_n has the form

     $$p(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_1 t + a_0$$

     The zero polynomial is

     $$0 = 0t^n + 0t^{n-1} + ... + 0t + 0$$

     Then if

     $$q(t) = b_n t^n + b_{n-1} t^{n-1} + ... + b_1 t + b_0$$

the $\oplus$ operation is defined by

$$p(t) \oplus q(t) = (a_n + b_n)t^n + (a_{n-1} + b_{n-1})t^{n-1} + ... + (a_1 + b_1)t + (a_0 + b_0)$$

and the $\odot$ operation is given by

$$c \odot p(t) = (ca_n)t^n + (ca_{n-1})t^{n-1} + ... + (ca_1)t + ca_0$$

3. The set of all continuous functions on the real number line, denoted $C(-\infty, \infty)$ (from Calculus) is a vector space with $\oplus$ given by

$$(f \oplus g)(x) = f(x) + g(x)$$

and $\odot$ is given by

$$(c \odot f)(x) = c(f(x))$$

where f and g are functions in $C(-\infty, \infty)$ and c is a scalar. Note that from Calculus, you can recall that the sum of two continuous functions is continuous and the product of a scalar and continuous function is continuous. Also, the zero function is given by

$$f_0(x) = 0 \text{ for all } x$$

4. The set of all mxn matrices, denoted $M_{m,n}$, is a vector space with typical matrix addition and scalar multiplication.

- **Example:** Let V be the set of all real numbers with the operations

$$u \oplus v = u - v \text{ ordinary subtraction}$$

and

$$c \odot u = cu \text{ ordinary multiplication}$$

Is V a vector space? If not, name a property which fails. *Intuition:* if a property will fail, it will probably be $\oplus$ since this is not the normal definition of addition.

A) If u and v are any elements of V then $u \oplus v$ is in V.

Note that the only requirement to be in V is to a real number. Thus, $u \oplus v = u - v$ is still a real number, so V is closed under addition and **property A is satisfied**.

A1) $u \oplus v = v \oplus u$ for u and v in V. (commutative property)
So, by definition $u \oplus v = u - v = -v + u = -(v - u)$ by definition of $\oplus$ and distributive property of real numbers. However, $v \oplus u = v - u$ by definition of $\oplus$, so $u \oplus v \neq v \oplus u$. Property **A1 is NOT satisfied**.

Thus V **is NOT a vector space.**

- **Example:** Let V be the set of all 2nd degree polynomials with normal addition and scalar multiplication (not lesser than or equal to 2, but second degree polynomial). Notice, that V is NOT closed under addition. Let

$$p(t) = a_2t^2 + a_1t + a_0$$

and

$$q(t) = b_2t^2 + b_1t + b_0$$

be in V. Since they are elements of V, $a_2 \neq 0$ and $b_2 \neq 0$ by the property of V. However,

$$p(t) \oplus q(t) = (a_2 + b_2)t^2 + (a_1 + b_1)t + (a_0 + b_0)$$

by the standard definition of $\oplus$ for polynomials. In order for V to be closed under addition, $p(t) \oplus q(t)$ must ALWAYS be a second degree polynomial. That means, $a_2 + b_2 \neq 0$. However, if $a_2 = -b_2$ or $b_2 = -a_2$, then $p(t) \oplus q(t)$ is a first degree polynomial since the coefficient of t^2 is zero, hence V is not closed under addition. Therefore, **A is NOT satisfied** and V is NOT a vector space.

- **Theorem:** Let v be any element of a vector space V and let c be any scalar, then the following properties are true:

  1. $0\mathbf{v} = 0$
  2. $c\mathbf{0} = \mathbf{0}$
  3. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$
  4. $(-1)\mathbf{v} = -\mathbf{v}$