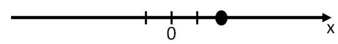
Vectors in Euclidean Space

Linear Algebra

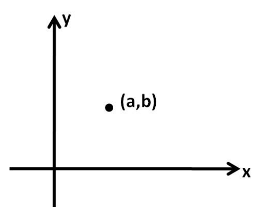
MATH 2010

- **Euclidean Spaces:** First, we will look at what is meant by the different Euclidean Spaces.

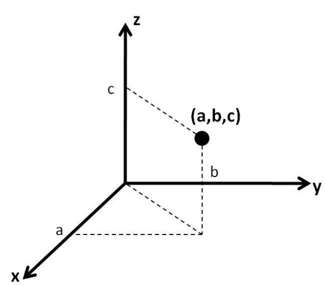
  - Euclidean 1-space $\Re^1$: The set of all real numbers, i.e., the real line. For example, 1, $\frac{1}{2}$, -2.45 are all elements of $\Re^1$.



  - Euclidean 2-space $\Re^2$: The collection of ordered pairs of real numbers, (x_1, x_2), is denoted $\Re^2$. Euclidean 2-space is also called *the plane*. For example, $(0, -1)$ and $(5, \frac{1}{2})$ are elements of $\Re^2$.



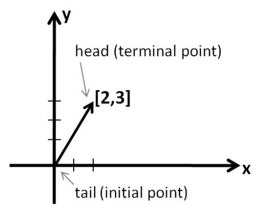
  - Euclidean 3-space $\Re^3$: The collection of all ordered triplets, (x_1, x_2, x_3), of real numbers is denoted $\Re^3$. Euclidean 3-space is also called *space*. For example, $(-1, 2, 4)$ is in $\Re^3$.



  - Although it is harder to visualize, we can extend the notation above, to the set of all ordered n-tuples, $(x_1, x_2, ..., x_n)$. This space is called Euclidean n-space and is denoted $\Re^n$.

- **Introduction to Vectors:** Vectors are used in many disciplines such as physics and engineering. Let's first consider vectors in $\Re^2$.

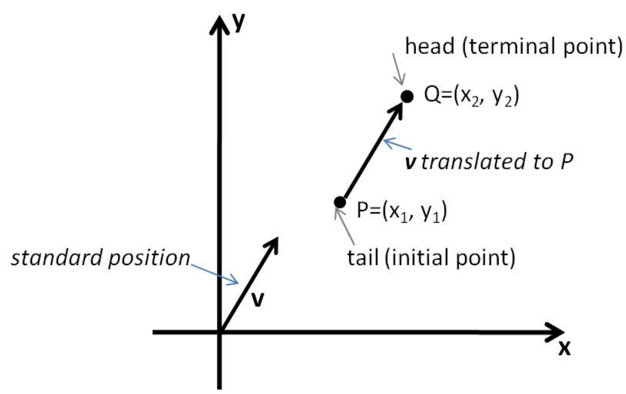
  - **Definition:** Vectors are *directed* line segments that have both a magnitude and a direction.
    - The length of the vector denotes the *magnitude*. For example in Physics, the length of the vector will denote the amount of force on an object.
    - The *direction* of the vector is denoted by the arrow at the terminal point. In Physics, the arrow will denote the direction of the force.
    - Below is the vector pointing to the point (2,3).



- **Position:** Typically the tail of the vector is at the origin, as in the figure above. This is called *standard position*. However, sometimes, the vector has a tail not at the origin. For example, consider the vector $\mathbf{v} = \vec{PQ}$ where P is the point (x_1, y_1), and Q is the point (x_2, y_2) in $\Re^2$. The figure shows the vector $\mathbf{v}$ in its *standard position* as well as $\mathbf{v}$ *translated to* P. The standard position of $\mathbf{v}$ is represented by

$$\mathbf{v} = [x_2 - x_1, y_2 - y_1]$$

the coordinates are given by the head point (Q) minus the tail point (P).



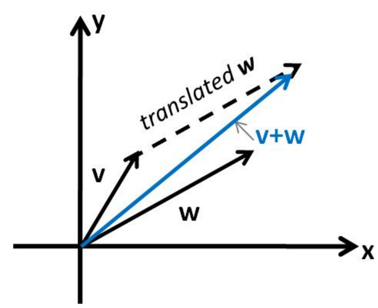
- **Terminology:**
  * If $\mathbf{x} = [x_1, x_2, ..., x_n]$, then x_i is called the i^{th} *component* of $\mathbf{x}$ relative to the coordinate system.
  * $\mathbf{0} = [0, 0, ..., 0]$ is called the *zero vector*.
  * Two vectors $\mathbf{v} = [v_1, v_2, ..., v_n]$ and $\mathbf{w} = [w_1, w_2, ..., w_m]$ are *equal* is $n = m$ (same length) and $v_i = w_i$ for all i (all components are equal).
- **Notation:** A point in $\Re^n$ is denoted by the ordered pair $(x_1, x_2, ..., x_n)$; however, depending on the context, this notation can also be used to represent a vector. For example $(2, 3)$ is a point in $\Re^2$ or a vector in $\Re^2$ depending on the context. The different notations for vectors are as follows s
  * $(x_1, x_2, ..., x_n)$ is the comma-delimited form of a vector
  * $[x_1, x_2, ..., x_n]$ is the bracketed comma-delimited form of vector. For example, $[2, 3]$.
  * A bold letter: $\mathbf{v} = [x_1, x_2]$ represents a vector.
  * A letter with an arrow over top: $\vec{v}$ also represents a vector.
  * A vector can also be considered a row-matrix:

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$$

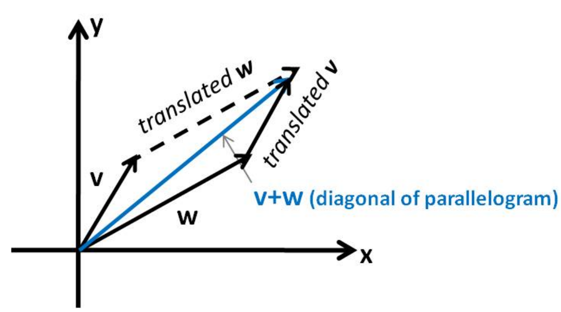
  * A vector can also be written as a column matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- **Manipulation of Vectors**
  - **Addition of vectors:** Consider two vectors $\mathbf{v}$ and $\mathbf{w}$. We want to find $\mathbf{v}+\mathbf{w}$.
    - **Geometrically** (see the figure below), we can translate $\mathbf{w}$ to the head of $\mathbf{v}$, denoted as *translated* $\mathbf{w}$. Then the resulting vector found with tail at the origin and head at the terminal point of *translated* $\mathbf{w}$ is $\mathbf{v}+\mathbf{w}$.



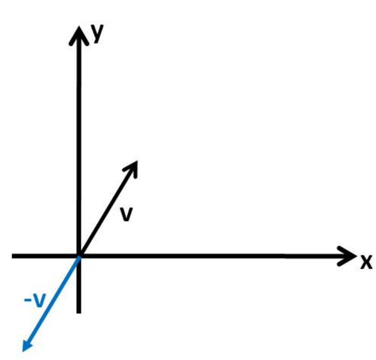
Alternatively, you can also view the sum of $\mathbf{v}$ and $\mathbf{w}$ as the diagonal of the parallelogram found by translating both $\mathbf{v}$ and $\mathbf{w}$ as shown in the figure below.



    - **Numerically** you just add the components of the vectors. If $\mathbf{u}=[1,2]$ and $\mathbf{v}=[3,-4]$, then

$$\mathbf{u}+\mathbf{v}=[1+3,2+(-4)]=[4,-2].$$

    - **Zero vector addition:** $\mathbf{0}+\mathbf{v}=\mathbf{v}$.
  - **Negative of v:** The negative of $\mathbf{v}$ is denoted $-\mathbf{v}$ and is a vector of the same length as $\mathbf{v}$ in the oppposite direction of $\mathbf{v}$.



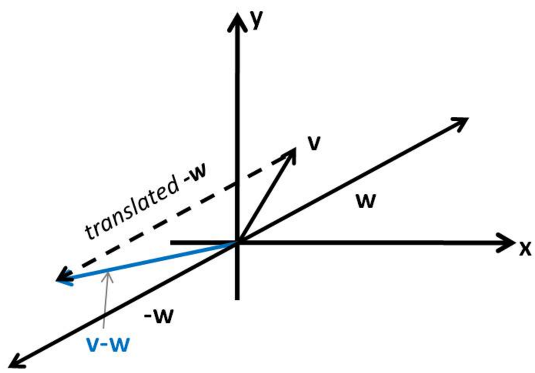
If $\mathbf{v} = [v_1, v_2, ..., v_n]$, then $-\mathbf{v} = [-v_1, -v_2, ..., -v_n]$, and

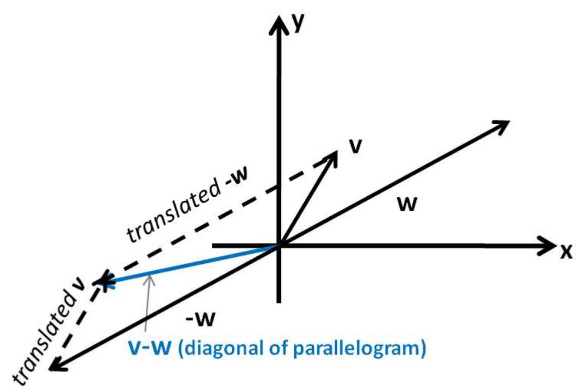
$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

Example: If $\mathbf{v} = [3, 2, -1]$, then $-\mathbf{v}$ is given as $-\mathbf{v} = [-3, -2, 1]$.

– **Subtraction of Vectors:**

  * **Geometrically**, there are a couple of different ways to think about $\mathbf{v} - \mathbf{w}$. We can first find $-\mathbf{w}$, and then look at the addition of $\mathbf{v}$ and $-\mathbf{w}$ as we did above, either using the just translated vector of $-\mathbf{w}$ or the parallelogram formed by $\mathbf{v}$ and $-\mathbf{w}$ (see below)





Alternatively, you can also view the $\mathbf{v} - \mathbf{w}$ as the *off-diagonal* of the original parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$. This gives the vector $\mathbf{v} - \mathbf{w}$ in a translated position where we can simply find the standard position by placing the initial point at the origin.

  * **Numerically** you just subtract the components of the vectors. If $\mathbf{u} = [1, 3, -4]$ and $\mathbf{v} = [2, 0, -1]$, then

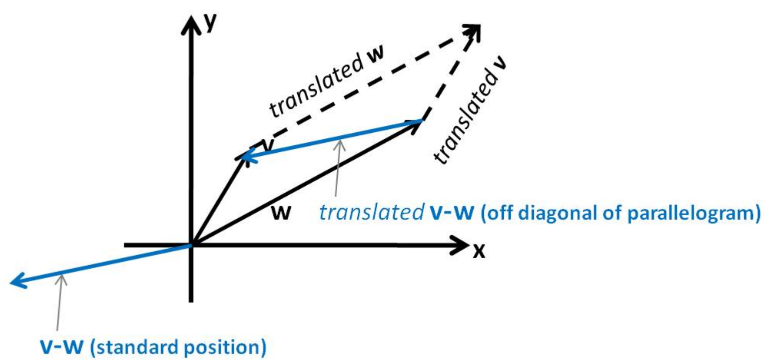
$$\mathbf{u} - \mathbf{v} = [1 - 2, 3 - 0, -4 - (-1)] = [-1, 3, -3].$$

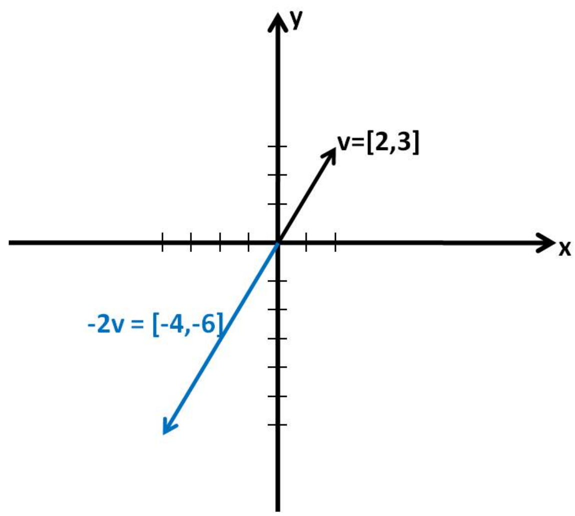
– **Parallel Vectors:**

  * Two vectors $\mathbf{v}$ and $\mathbf{w}$ are *parallel* if one vector is a scalar multiple of the other, i.e.,

$$\mathbf{v} = k\mathbf{w}$$

    · If $k > 0$, then the vectors are in the same direction.
    · If $k < 0$, then the vectors are in opposite directions.
    · If $0 < |k| < 1$, the length (force) is decreased.





- If $|k| > 1$, the length (force) is increased

* The notation is: $\mathbf{v}||\mathbf{w}$.
* An example is shown for $\mathbf{v} = [2, 3]$

– **Sample Problems**

1. Given $\mathbf{u} = [-2, 3, 1]$ and $\mathbf{w} = [-3, -2, -1]$, find $\frac{1}{2}(3\mathbf{u} + \mathbf{w})$. Ans: $[-\frac{9}{2}, \frac{7}{2}, 1]$
2. Find all scalars c, if any exist, such that $[c^2, -4]||[1, -2]$. Ans: $c = \pm\sqrt{2}$.

- **Properties of Vector Algebra in $\Re^n$**: Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be any vectors in $\Re^n$ and let r and s be any scalars in $\Re$.

– **Properties of Vector Addition**

A1) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ Associative Law
A2) $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ Commutative Law
A3) $\mathbf{0} + \mathbf{v} = \mathbf{v}$ Additive Identity of $\mathbf{0}$
A4) $\mathbf{v} + -\mathbf{v} = \mathbf{0}$ Additive Inverse of $\mathbf{v}$

– **Properties Involving Scalar Multiplication**

S1) $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$ Distributive Law
S2) $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$ Distributive Law
S3) $r(s\mathbf{v}) = (rs)\mathbf{v}$ Associative Law
S4) $1\mathbf{v} = \mathbf{v}$ Preservation of Scale

– **Additional Properties**

1. $0\mathbf{v} = \mathbf{0}$
2. $r\mathbf{0} = \mathbf{0}$
3. $(-1)\mathbf{u} = -\mathbf{u}$

- **Norm of a vector** The *length* of a vector, also called the *norm* of a vector is denoted $||\mathbf{x}||$ and given by

$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

  – *Example:* Let $\mathbf{x} = [2, 3, 1, 0]$, then

$$||\mathbf{x}|| = \sqrt{2^2 + 3^2 + 1^1 + 0^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

  – *Properties of norm* If $\mathbf{x}$ is a vector in $\Re^n$, and if r is any scalar, then

  1. $||\mathbf{x}|| \geq 0$
  2. $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
  3. $||r\mathbf{x}|| = |r|||\mathbf{x}||$

- **Unit vector:** A vector with length 1 is called a *unit vector.* If $\mathbf{x}$ is any vector in $\Re^n$, then

$$\mathbf{u} = \frac{1}{||\mathbf{x}||}\mathbf{x}$$

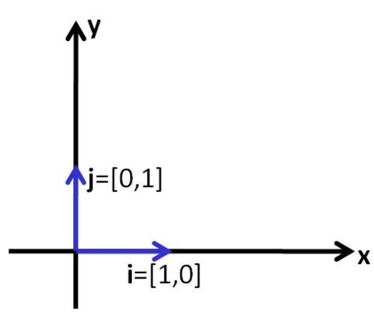
  is a unit vector in the direction of $\mathbf{x}$

  For example, for the vector above, $\mathbf{x} = [2, 3, 1, 0]$, we found that $||\mathbf{x}|| = \sqrt{14}$. Therefore, the vector

$$\mathbf{u} = \frac{1}{\sqrt{14}}[2, 3, 1, 0] = \left[\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, 0\right] = \left[\frac{2\sqrt{14}}{14}, \frac{3\sqrt{14}}{14}, \frac{\sqrt{14}}{14}, 0\right]$$

  is a unit vector in the direction of $\mathbf{x}$

- **Standard unit vectors:** The standard unit vectors are the vectors of length 1 along the coordinate axis. The picture below shows the standard unit vectors in $\Re^2$.



  – *Standard unit vectors in* $\Re^2$*:* The standard unit vectors in $\Re^2$ are given by

$$\hat{i} = [1, 0] \quad \hat{j} = [0, 1]$$

  – *Standard unit vectors in* $\Re^3$*:* The standard unit vectors in $\Re^2$ are given by

$$\hat{i} = [1, 0, 0] \quad \hat{j} = [0, 1, 0] \quad \hat{j} = [0, 0, 1]$$

  – *Standard unit vectors in* $\Re^n$*:* In general, the standard unit vectors in $\Re^n$ are given by

$$\mathbf{e}_1 = [1, 0, 0, \ldots, 0, 0] \;\; \mathbf{e}_2 = [0, 1, 0, \ldots, 0, 0] \;\; \cdots \;\; \mathbf{e}_n = [0, 0, 0, \ldots, 0, 1]$$

  where $\mathbf{e}_i$ has a 1 in the i^{th} components and all the other components are 0.

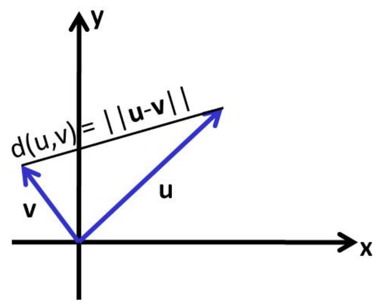
- Every vector in $\Re^n$ can be written a a linear combination of the standard unit vectors

  $$\mathbf{x} = [x_1, x_2, ..., x_n] = x_1[1, 0, 0, \ldots, 0, 0] + x_2[0, 1, 0, \ldots, 0, 0] + \cdots + x_n[0, 0, 0, \ldots, 0, 1] = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

- *Examples:*
  1. $\mathbf{x} = -1[1, 0, 0, 0] + 3[0, 1, 0, 0] + 4[0, 0, 1, 0] - 2[0, 0, 0, 1] = -1\mathbf{e}_1 + 3\mathbf{e}_2 + 4\mathbf{e}_3 - 2\mathbf{e}_4$
  2. $\mathbf{x} = [2, -1, 5] = 2[1, 0, 0] - 1[0, 1, 0] + 5[0, 0, 1] = 2\hat{i} - \hat{j} + 5\hat{k}$

- **Distance between two vectors:** Let $\mathbf{u} = [u_1, u_2, ..., u_n]$ and $\mathbf{v} = [v_1, v_2, ..., v_n]$ be two vectors in $\Re^n$, then the distance between the two vectors is given by the formula:

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + ... + (u_n - v_n)^2}$$

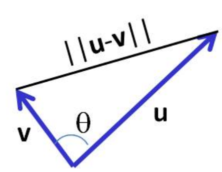


- *Example:* Let $\mathbf{u} = [2, 5]$ and $\mathbf{v} = [-1, 0]$, then

  $$d(\mathbf{u}, \mathbf{v}) = \sqrt{(2 - (-1))^2 + (5 - 0)^2} = \sqrt{9 + 25} = \sqrt{34}$$

- *Properties:* If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\Re^n$, then
  1. $d(\mathbf{u}, \mathbf{v}) \geq 0$
  2. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
  3. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u}$

- **Angle between two vectors:** We are interested in finding the angle between two given vectors are pictured in the schematic below:



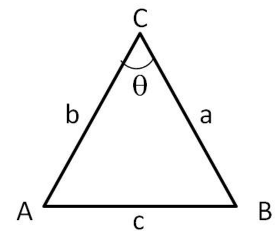
In order to do this, we need the *Law of Cosines.*

If we have the schematic above, then the Law of Cosines is given by

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Using the law of cosines with our vector schematic, we have

$$||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}||\cos(\theta)$$



Let's consider $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$ in $\Re^2$. Then

$$\begin{aligned} ||\mathbf{u} - \mathbf{v}||^2 &= (u_1 - v_1)^2 + (u_2 - v_2)^2 \\ &= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 \\ &= u_1^2 + u_2^2 + v_1^2 + v_2^2) - 2(u_1v_1 + u_2v_2) \\ &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2(u_1v_1 + u_2v_2) \end{aligned}$$

Combining this with

$$||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}|| \cos(\theta)$$

we have

$$||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2(u_1v_1 + u_2v_2) = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}|| \cos(\theta)$$

The $||\mathbf{u}||^2 + ||\mathbf{v}||^2$ cancels out leaving

$$-2(u_1v_1 + u_2v_2) = -2||\mathbf{u}||||\mathbf{v}|| \cos(\theta)$$

Solving for $\cos(\theta)$, we have

$$\cos(\theta) = \frac{u_1v_1 + u_2v_2}{||\mathbf{u}||||\mathbf{v}||}$$

The numerator is defined as the **dot product** between $\mathbf{u}$ and $\mathbf{v}$. Let's examine the dot product. Afterwards, we will continue looking at the angle between two vectors.

- **Dot Product** The dot product between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\Re^n$ is denoted $\mathbf{u} \cdot \mathbf{v}$ and is defined by

  $$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots u_nv_n$$

  Therefore, in $\Re^2$, the dot product is simply

  $$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$$

  - *Examples*
    1. Let $\mathbf{u} = [1, 2]$ and $\mathbf{v} = [-1, 3]$, then

       $$\mathbf{u} \cdot \mathbf{v} = 1(-1) + 2(3) = -1 + 6 = 5$$

    2. Let $\mathbf{u} = [1, -2, 3, 4]$ and $\mathbf{v} = [2, 3, -2, 1]$, then

       $$\mathbf{u} \cdot \mathbf{v} = 1(2) + (-2)(3) + 3(-2) + 4(1) = 2 - 6 - 6 + 4 = -6$$

  - *Properties of the Dot Product:* Let $\mathbf{u}$,$\mathbf{v}$, and $\mathbf{w}$ be vectors in $\Re^n$ and c be a scalar in $\Re$. Then
    1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
    2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
    3. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
    4. $\mathbf{v} \cdot \mathbf{v} = v_1v_1 + v_2 + v_2 = ||\mathbf{v}||^2$
    5. $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$

- *Examples:* For the given $\mathbf{u}$ and $\mathbf{v}$, find
  a) $\mathbf{u} \cdot \mathbf{v}$
  b) $\mathbf{u} \cdot \mathbf{u}$
  c) $||\mathbf{u}||^2$
  d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$
  e) $\mathbf{u} \cdot (5\mathbf{v})$

  1. $\mathbf{u} = [-1, 2]$, $\mathbf{v} = [2, -2]$
  2. $\mathbf{u} = [2, -1, 1]$, $\mathbf{v} = [0, 2, -1]$

  Answers

  1. a) $\mathbf{u} \cdot \mathbf{v} = -6$
     b) $\mathbf{u} \cdot \mathbf{u} = 5$
     c) $||\mathbf{u}||^2 = 5$
     d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = [-12, 12]$
     e) $\mathbf{u} \cdot (5\mathbf{v}) = -30$
  2. a) $\mathbf{u} \cdot \mathbf{v} = -3$
     b) $\mathbf{u} \cdot \mathbf{u} = 6$
     c) $||\mathbf{u}||^2 = 6$
     d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = [0, -6, 3]$
     e) $\mathbf{u} \cdot (5\mathbf{v}) = -15$

- *Example:* Find

$$(3\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - 3\mathbf{v})$$

  given that

$$\mathbf{u} \cdot \mathbf{u} = 8 \quad \mathbf{u} \cdot \mathbf{v} = 7 \quad \mathbf{v} \cdot \mathbf{v} = 6$$

  Solution:

$$\begin{aligned} (3\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - 3\mathbf{v}) &= 3\mathbf{u} \cdot \mathbf{u} - 3\mathbf{u} \cdot (3\mathbf{v}) - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot (3\mathbf{v}) \\ &= 3\mathbf{u} \cdot \mathbf{u} - 9\mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + 3\mathbf{v} \cdot \mathbf{v} \\ &= 3(8) - 9(7) - 7 + 3(6) \\ &= -26 \end{aligned}$$

- **Back to the angle between vectors:** The angle θ between vectors $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}, \quad 0 \le \theta \le \pi$$

  - **Example:** Let $\mathbf{u} = [1, 0, 0, 1]$ and $\mathbf{v} = [0, 1, 0, 1]$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 1}{\sqrt{1^2 + 1^2}\sqrt{1^2 + 1^2}} = \frac{1}{2}$$

    So,

$$\theta = \frac{\pi}{3} \text{ radians or } 60^\circ$$

  - **Theorem** If $\mathbf{u}$ and $\mathbf{v}$ are nonzero and θ is the angle between them, then
    * θ is acute if and only if $\mathbf{u} \cdot \mathbf{v} > 0$
    * θ is obtuse if and only if $\mathbf{u} \cdot \mathbf{v} < 0$
  - **Example:** Let $\mathbf{u} = [1, -1, 0, 1]$ and $\mathbf{v} = [-1, 2, -1, 0]$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \cdot \mathbf{v} = -1 - 2 = -3, \quad ||\mathbf{u}|| = \sqrt{3}, \quad \mathbf{v} = \sqrt{6}$$

    so

$$\cos(\theta) = \frac{-3}{\sqrt{3}\sqrt{6}} = -\frac{\sqrt{3}}{\sqrt{6}} = -\frac{\sqrt{2}}{2}$$

    Then,

$$\theta = \frac{3\pi}{4}$$

– **Example:** Let $\mathbf{u} = [2, 3, 1]$ and $\mathbf{v} = [-3, 2, 0]$. Then $\mathbf{u} \cdot \mathbf{v} = -6 + 6 + 0 = 0$. Then

$$\cos(\theta) = \frac{0}{||\mathbf{u}||||\mathbf{v}||} = 0$$

So, $\theta = \frac{\pi}{2}$ or $90°$.

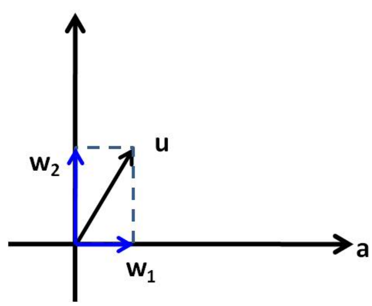
– **Orthogonal:** If $\cos\theta = 0$, i.e. $\theta = \frac{\pi}{2}$, then $\mathbf{u}$ and $\mathbf{v}$ are said to be *orthogonal* (or perpendicular). Therefore, two vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

– **Examples:**

1. Determine all vectors orthogonal to $u = [2, 7]$. Ans: All vectors $\mathbf{v} = t[-7/2, 1]$ where t is any real number.
2. Determine all vectors orthogonal to $u = [2, -1, 1]$. Ans: All vectors $\mathbf{v} = [1/2(s-t), s, t]$ where s and t are any real number.

- **Projections:** Sometimes it is necessary to decompose a vector into a combination of two vectors which are orthogonal to one another. A trivial case is decomposing a vector $\mathbf{u} = [u_1, u_2]$ in $\Re^2$ into its $\hat{i}$ and $\hat{j}$ directions, i.e., $\mathbf{u} = u_1\hat{i} + u_2\hat{j}$. However, sometimes it is necessary to decompose it along a direction different than the standard coordinate directions. Say, we need to decompose a vector into components along a vector $\mathbf{a}$, say $\mathbf{w}_1$ and along a vector, $\mathbf{w}_2$, on an axis orthogonal to $\mathbf{a}$. See the image below.



– In the above figure $\mathbf{w}_1$ is called the *orthogonal projection of* $\mathbf{u}$ *on* $\mathbf{a}$ or the *vector component of* $\mathbf{u}$ *along* $\mathbf{a}$ and is given by

$$\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{||\mathbf{a}||^2}\mathbf{a} \text{ (vector component of } \mathbf{u} \text{ along } \mathbf{a})$$

– $\mathbf{w}_2$ is called the *vector component of* $\mathbf{u}$ *orthogonal to* $\mathbf{a}$ and is given by

$$\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{||\mathbf{a}||^2}\mathbf{a}$$

– **Example:** Let $\mathbf{u} = [2, -1, 3]$ and $\mathbf{a} = [4, -1, 2]$. Find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

∗ vector component of $\mathbf{u}$ along $\mathbf{a}$:

$$\mathbf{u} \cdot \mathbf{a} = 2(4) + (-1)(-1) + 3(2) = 15$$

and

$$||\mathbf{a}||^2 = 4^2 + (-1)^2 + 2^2 = 21$$

Then

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{||\mathbf{a}||^2}\mathbf{a} = \frac{15}{21}[4, -1, 2] = \left[\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right]$$

* vector component of **u** orthogonal to **a**:

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = [2, -1, 3] - \left[\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right] = \left[-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right]$$

– Formula for the length of the projection of **u** along **a**.

$$\begin{aligned} ||\text{proj}_{\mathbf{a}}\mathbf{u}|| &= \left|\left|\tfrac{\mathbf{u}\cdot\mathbf{a}}{||\mathbf{a}||^2}\mathbf{a}\right|\right| \\ &= \left|\tfrac{\mathbf{u}\cdot\mathbf{a}}{||\mathbf{a}||^2}\right| ||\mathbf{a}|| \\ &= \tfrac{|\mathbf{u}\cdot\mathbf{a}|}{||\mathbf{a}||^2} ||\mathbf{a}|| \\ &= frac|\mathbf{u}\cdot\mathbf{a}|||\mathbf{a}|| \\ &= ||\mathbf{u}||| \cos(\theta)| \end{aligned}$$