Cover Rubbling and Stacking

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Abstract

A pebble distribution places a nonnegative number of pebbles on the vertices of a graph G. In graph rubbling, the pebbles can be redistributed using pebbling and rubbling moves, typically with the goal of reaching some target pebble distribution. In graph pebbling, only the pebbling move is allowed. The cover pebbling number is the smallest k such that from any initial distribution of k pebbles, it is possible that after a series of pebbling moves there is at least one pebble on every vertex of G. The Cover Pebbling Theorem asserts that to determine the cover pebbling number of a graph, it is sufficient to consider the pebbling distributions that initially place all pebbles on a single vertex. In this paper, we prove a rubbling analogue of the Cover Pebbling Theorem, providing an answer to an open question of Belford and Sieben (*Discrete Math.* 309 (2009) 3426-3446). In addition, we prove a stronger version of the Cover Rubbling Theorem for trees.

1. Introduction

Graph pebbling was originally proposed by Lagarias and Saks as a technique for solving a problem in number theory (see [7]). This problem was solved by Chung in [3], which is the first publication on graph pebbling. Since its introduction, graph pebbling and its variants have become a quite prolific area of graph theory. For a good overview of the history of graph pebbling and some of its variants, the reader is referred to [7]. Graph rubbling was introduced 20 years later in [2] and, to date, has remained relatively unstudied as compared to graph pebbling.

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Let G be a connected simple graph with vertex set V. A *pebble distribution* on G is a function f that maps V to the nonnegative integers, that is, $f : V \to \mathbb{Z}_{\geq 0}$. This function is viewed as a placement of $f(v) \geq 0$ pebbles on each vertex v of V. If $f(v) \geq 1$ for a vertex v, then we say that v is a *pebbled vertex*. The *size* of f is $|f| = \sum_{v \in V} f(v)$. Graph pebbling and graph rubbling are methods of moving the pebbles of a pebble distribution to specified locations using the following moves. The *pebbling move*, denoted $p(v \to u)$, removes two pebbles from a vertex v and places one pebble on an adjacent vertex u. That is, for a vertex v with $f(v) \geq 2$ and u a neighbor of v, the pebbling move produces a new distribution f' given by $f'(v) = f(v) - 2$, $f'(u) = f(u) + 1$, and $f'(x) = f(x)$ for all other vertices x. The *rubbling move*, denoted $r(u, v \to x)$, removes one pebble from a vertex u and one pebble from a vertex v and places one pebble on a common neighbor x of u and v. In other words, for two pebbled vertices u and v, and x adjacent to both u and v, the rubbling move produces a new distribution f' given by $f'(u) = f(u) - 1$, $f'(v) = f(v) - 1$, $f'(x) = f(x) + 1$, and $f'(y) = f(y)$ for all other vertices y. Since each move requires one pebble to be discarded and another to be moved to a new vertex, graph pebbling and graph rubbling provide models for transporting consumable products.

If a pebble can be placed on a vertex v from an initial pebble distribution f after a sequence of pebbling and rubbling moves, then we say that v can be *reached* from f. Two major reachability questions in graph pebbling involve the pebbling number and cover pebbling number. The *pebbling number* of G is the smallest integer k such that from any initial distribution of k pebbles, it is possible by a sequence of pebbling moves to place a pebble on any specified vertex of G. The *cover pebbling number* of G is the smallest k such that from any initial distribution of k pebbles, it is possible after a series of pebbling moves that every vertex of G has at least one pebble. The *cover rubbling number* is defined similarly while allowing series of pebbling and rubbling moves. The cover pebbling number was originally introduced in [4] and studied in [2], for example.

Note that in covering, the desired goal is to reach every vertex in G such that after a sequence of moves, every vertex of G is pebbled. In other words, the resulting configuration is a cover where every vertex has at least one pebble. There are also variants of cover pebbling and cover rubbling which seek to reach (leave a pebble on) the vertices of a subset satisfying certain properties. For example, in [8] the authors seek to reach a vertex cover of a graph G using pebbling moves, in [5] and [10] the authors seek to reach a dominating set of a graph G using pebbling moves, and in [1] the authors seek to reach a dominating set of a graph G using pebbling and rubbling moves.

1.1. Terminology and Notation

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and order $n = |V|$, and let v be a vertex in V. A *leaf* of G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex with at least two leaf-neighbors. A *star* is a tree with at most one vertex that is not a leaf.

The *distance* between two vertices u and v in a connected graph G, denoted by $d(u, v)$, is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. If $d(u, v) = \mathrm{diam}(G)$, then u and v are called *peripheral vertices* of G. A set S of vertices in a graph G is a *dominating set* of G if every vertex in $V \setminus S$ is adjacent to a vertex in S.

In addition to a pebble distribution, we will use a *weight function* $w : V \to \mathbb{Z}_{\geq 0}$. The purpose of this weight function is to serve as a target to be reached by a pebble distribution. In particular, we say that w is *reachable* from f if there is a sequence of pebbling and rubbling moves from f which results in a pebble distribution f' satisfying that $f'(v) \geq w(v)$ for every vertex v. We let the *w-cover rubbling number* of G, denoted $\rho_w(G)$, be the minimum number of pebbles, such that w is reachable from any pebble configuration of size $\rho_w(G)$. If only pebbling moves are allowed, the *w-cover pebbling number* of G, denoted $\pi_w(G)$, is defined analogously. We observe that $\rho_w(G) \leq \pi_w(G)$.

A pebble distribution f is called a *simple distribution* if f is non-zero at exactly one vertex. We set $\mathrm{st}_w(v)$ (respectively, $\mathrm{st}_w^r(v)$) to be the minimum size of a simple configuration which places pebbles on v and from which w is reachable via pebbling (respectively, pebbling and rubbling moves). Finally, we define the *w-rubbling stacking number* of G to be $\mathrm{st}_w^r(G) = \max_{v \in V}(\mathrm{st}_w^r(v))$, and we define the *$w$-pebbling stacking number* of G to be $\mathrm{st}_w(G) = \max_{v \in V}(\mathrm{st}_w(v))$.

1.2. Main Results

The Cover Pebbling Theorem, proven in [9], shows that to determine the w-cover pebbling number of a graph for w strictly positive on all of the vertices, it is sufficient to consider only simple initial pebbling distributions.

Theorem 1. [9] *If w is a strictly positive weight function on a connected graph G, then $\pi_w(G) = \mathrm{st}_w(G)$.*

Belford and Sieben [2] pose the question: is the cover rubbling number the same as the cover pebbling number for any graph?

Our main aim in this paper is two-fold. We first answer the question of Belford and Sieben in the affirmative by proving a rubbling analogue of the Cover Pebbling Theorem. Second, we strengthen this result for trees.

In particular, we prove a lemma showing that $\mathrm{st}_w^r(v) = \mathrm{st}_w(v)$, and use it to obtain the following result. A proof of Theorem 2 is given in Section 2.

Theorem 2. *If w is a strictly positive weight function on a connected graph G, then $\pi_w(G) = \rho_w(G)$.*

Next we prove a rubbling result analogous to Theorem 1 for trees and with weight functions which are not strictly positive on every vertex. Let S be a dominating set of a tree T, f be a pebble distribution on T, and $w_S : V(T) \to \mathbb{Z}_{\geq 0}$ be strictly positive on S. We show that it is sufficient to consider simple distributions on peripheral leaves to determine the number of pebbles necessary to reach a specified dominating set of a tree. Specifically, we prove the following two theorems.

Theorem 3. *Let T be a tree, S be a dominating set of T, and $w_S : V(T) \to \mathbb{Z}_{\geq 0}$ be strictly positive on S. If f is a pebble distribution on T from which w_S is not reachable, then there is a simple distribution f' (which depends on S) with $|f'| = |f|$, and from which w_S is not reachable.*

Theorem 4. *Let T be a non-trivial tree, S be a dominating set of T, and $w_S : V(T) \to \mathbb{Z}_{\geq 0}$ be strictly positive on S. Let f be a simple configuration with maximum size such that w_S is not reachable from f. If v is the vertex pebbled by f, then v is a peripheral vertex of T.*

Proofs of Theorems 3 and 4 are given in Section 3.

Before proceeding, we first explain a few implications of Theorem 3, as well as the necessity of the dominating condition and the rubbling move.

Since the only requirement for S is that it be a dominating set, we note that Theorems 3 and 4 hold for many variations of domination. For example, the theorems remain true if the weight function is strictly positive on a total dominating set, a paired dominating set, a connected dominating set, and even in the case that the weight function is a Roman dominating function. For more details concerning domination and its variants, the reader is referred to [6].

On the other hand, Theorem 3 does not hold if we relax the property that S is a dominating set. To see this let T be the subdivided star formed by subdividing each edge of a the star $K_{1,3}$ exactly once, and let c be the center vertex of T. Further, let $w(c) = 1$ and $w(v) = 0$ for all other vertices v of T. Note, w is strictly positive on $S = \{c\}$, but $\mathrm{st}_w^r(T) = 4$ and $\rho_w(T) = 6$. To see that $\rho_w(T) > 5$, observe that the pebble distribution placing three pebbles on a leaf of T along with one pebble on each of the other two leaves cannot reach w. We note that while S is not a dominating set, it is a distance 2-dominating set of T, that is, every vertex in T is within distance 2 from vertex in S.

We also mention that Theorem 3 does not hold if we do not allow the use of the rubbling move. To see this, one can simply consider the star with n leaves, denoted $K_{1,n}$. In this case, we could define a weight function w which is equal to 1 on the single nonleaf vertex and is equal to zero on all of the leaves. Note, w

is strictly positive on a dominating set of $K_{1,n}$. Moreover, one easily computes that $\mathrm{st}_w^r(G) = \mathrm{st}_w(G) = 2$. However, if we place one pebble on each leaf, then this is a pebble distribution of size n from which w is not reachable if we do not allow the rubbling move.

Finally, the case that the graph is not a tree is addressed in Section 4.

2. Proof of Theorem 2

In this section, we prove Theorem 2. It should be noted that our method is different than the method suggested in [2]. We first prove a preliminary lemma.

Lemma 5. *Let w be a weight function on a graph G. If w is reachable from a simple distribution on v using pebbling and rubbling moves, then w is reachable from the same simple distribution on v using only pebbling moves. In other words,* $\mathrm{st}_w^r(v) = \mathrm{st}_w(v)$.

Proof. Consider a simple distribution which initially places all pebbles on a vertex v. We show that in order to reach another vertex u, we need to consume at least $2^{d(v,u)}$ pebbles from v. Clearly, if $d(v, u) = 1$, two pebbles are necessary to reach u and the result holds. Let $d(v, u) = k \geq 2$, and for $1 \leq j < k$, assume that at least 2^j pebbles are consumed to reach a vertex at distance j from v.

Let s be a sequence of moves that reach u from v. Now the final move of s is either a pebbling move consuming two pebbles from a neighbor of u or a rubbling move consuming one pebble each from two neighbors of u. Since the distance from v to any neighbor of u is at least $k - 1$, our inductive hypothesis implies that it necessary to consume $2 \cdot 2^{k-1} = 2^k$ pebbles to reach u. But this number of pebbles on v is all we need to place a pebble on u even if rubbling moves are not allowed. □

The proof of Theorem 2 follows fairly easily from Lemma 5. We recall the statement of Theorem 2.

Theorem 2. *If w is a strictly positive weight function on a connected graph G, then $\pi_w(G) = \rho_w(G)$.*

Proof. Recall that $\rho_w(G) \leq \pi_w(G)$. By Theorem 1, $\pi_w(G) = \mathrm{st}_w(G)$. Since $\mathrm{st}_w(G) = \max_{v\in V}(\mathrm{st}_w(v))$ and $\mathrm{st}_w^r(G) = \max_{v\in V}(\mathrm{st}_w^r(v))$, Lemma 5 gives that $\mathrm{st}_w(G) = \mathrm{st}_w^r(G)$. As $\mathrm{st}_w^r(G) \leq \rho_w(G)$, we have that $\pi_w(G) = \mathrm{st}_w(G) = \mathrm{st}_w^r(G) \leq \rho_w(G)$. Hence, $\pi_w(G) = \rho_w(G)$. This completes the proof. □

3. Proofs of Theorems 3 and 4

We note that our proof of Theorem 3 is a somewhat involved modification of the argument used to prove Theorem 7 in [4]. Recall the statement of Theorem 3.

Theorem 3. *Let T be a tree, S be a dominating set of T, and $w_S : V(T) \to \mathbb{Z}_{\geq 0}$ be strictly positive on S. If f is a pebble distribution on T from which w_S is not reachable, then there is a simple distribution f' (which depends on S) with $|f'| = |f|$, and from which w_S is not reachable.*

Proof. Let f be a pebble distribution on T from which w_S is not reachable, and let p_f be the number of pebbled vertices under f. Further, among all initial pebbling distributions of size $|f|$ from which w_S is not reachable, we choose f so that p_f is minimized. If $p_f = 1$, then we are finished. Hence, we may assume that $p_f \geq 2$. We proceed by induction on the order n of T. The base cases can easily be verified for trees having order $n \leq 2$. Hence, we may assume that $n \geq 3$.

For induction, assume that the result holds for all trees of order less than n. Let T be a tree of order n and fix a dominating set S of T. We first prove three claims.

Claim 1. *If there is a leaf ℓ of T with $f(\ell) = 0$, then the result holds.*

Proof. Assume that $f(\ell) = 0$ and let v be the neighbor of ℓ in T. We remove ℓ from T and denote the resulting tree T'. Let f' be the restriction of f on T'. Then $|f'| = |f|$. We define a new weight function w on $V(T')$ by $w(v) = w_S(v) + 2w_S(\ell)$ and $w(x) = w_S(x)$ for all other $x \in V(T')$. Note that at least one of v and ℓ is in every dominating set of T, and so $w(v) \geq 1$ and w is strictly positive on a dominating set of T'. Also, w_S is reachable from f in T if and only if $w_{S'}$ is reachable from f' in T'. Thus, $w_{S'}$ is not reachable from f', and by the induction hypothesis, there is a simple configuration f'' on T', with $|f''| = |f'| = |f|$, from which $w_{S'}$ is not reachable in T', and hence from which w_S is not reachable in T, giving the result. (□)

Henceforth, by Claim 1, we may assume that every leaf of T is pebbled under f.

Claim 2. *If $p_f = 2$, then the result holds.*

Proof. Assume that $p_f = 2$. By Claim 1, every leaf is pebbled, implying that T is the path $(v_1, v_2, \ldots, v_n)$ and the two pebbled vertices are v_1 and v_n. Since w_S cannot be reached from f, there is a vertex v_k such that $w_S(v_k) > f(v_k)$ and moving pebbles from v_1 and v_n results in at most $w_S(v_k) - 1$ pebbles on v_k. Note that v_k could be one of v_1 and v_n. Relabeling the path if necessary, we may assume that $k - 1 \geq n - k$. Then, for every j in the range $k \leq j \leq n$, we have $d(v_1, v_j) \geq d(v_j, v_n)$. With this in mind, we define a simple configuration $f'(v_1) = f(v_1) + f(v_n)$ and $f'(v_i) = 0$ for $2 \leq i \leq n$. Hence, f' is a simple distribution on T such that w_S is not reachable from f' and $|f| = |f'|$, as desired. (□)

Claim 3. *If there is a leaf ℓ of T with $1 \le f(\ell) \le w_S(\ell)$, then the result holds.*

Proof. Assume that $f(\ell) \le w_S(\ell)$ and let v be the neighbor of ℓ in T. We define a pebble configuration, f', on T by $f'(v) = f(v) + f(\ell)$, $f'(\ell) = 0$, and $f'(x) = f(x)$ for all other x. Note, $|f'| = |f|$. Further, under f' at least $2w_S(\ell)$ pebbles must be moved from v to cover ℓ with $w_S(\ell)$ pebbles. Since $f'(v) = f(v) + f(\ell) \le f(v) + w_S(\ell)$ and w_S is not reachable from f, it follows that w_S is not reachable from f'. Since $f'(\ell) = 0$, we can apply Claim 1 to obtain the result. (□)

We now return our attention to the proof of Theorem 3. By Claim 2, we may assume that $p_f \ge 3$, for otherwise we are finished. By Claim 3, we may assume that for every leaf u, $f(u) > w_S(u)$.

Let v_1 be a leaf of T. If T is not a path, then let v_m be the vertex of degree at least 3 at the shortest distance from v_1. If T is a path, then let v_m be the other leaf of T. We call v_m the *split vertex* of v_1. Further, let $(v_1, v_2, \ldots, v_m)$ denote the v_1-v_m-path in T. If there are vertices in $\{v_2, \ldots, v_m\}$ which receive pebbles from f, then the one with the smallest subscript is called the *nearest pebbled* vertex of v_1. Since $p_f \ge 3$, if T is a path, at least one nonleaf vertex is pebbled.

Note that we can remove $f(v_1) - w_S(v_1)$ pebbles from v_1 and add $s_1 = \left\lfloor \frac{f(v_1) - w_S(v_1)}{2} \right\rfloor$ pebbles on v_2 using pebbling moves. Moreover, if $s_1 + f(v_2) > w_S(v_2)$, then $s_2 = \left\lfloor \frac{s_1 + f(v_2) - w_S(v_2)}{2} \right\rfloor$ pebbles can be moved to v_3, and so on. To aid in our discussion, we say that v_1 *supplies* v_r if f reaches v_r using this method, that is, if $s_{r-1} \ge 1$.

Let v_k be the vertex with the largest subscript among all vertices supplied by v_1 on $(v_1, v_2, \ldots, v_m)$. Note that this implies that every vertex in S on $(v_1, v_2, \ldots, v_k)$ is supplied by v_1.

Assume first that v_1 supplies its nearest pebbled vertex, say v_j. We define a new configuration f' of pebbles as $f'(v_1) = f(v_1) + f(v_j)$, $f'(v_j) = 0$, and $f'(v) = f(v)$ for all other vertices v of T. Note, $|f'| = |f|$ and w_S is not reachable from f' if w_S is not reachable from f. This contradicts the minimality of p_f.

Thus, we may assume that v_1 does not supply its nearest pebbled vertex for otherwise, the result holds. In particular, $f(v_k) = 0$. Similarly, if $f(v_{k+1}) \ge 1$, that is, if v_{k+1} is a pebbled vertex, then we define a new configuration f' of pebbles. Let $f'(v_1) = f(v_1) + f(v_{k+1})$, $f'(v_{k+1}) = 0$, and $f'(v) = f(v)$ for all other vertices v of T. Note, $|f'| = |f|$ and w_S is not reachable from f' if w_S is not reachable from f, contradicting that p_f is minimum.

Henceforth, we may assume that $f(v_i) = 0$ for $1 \le i \le k+1$. Since k is the largest subscript of a vertex on the v_1-v_m-path that can be supplied by v_1 and $f(v_k) = 0$, it follows that $s_{k-1} - w_S(v_k) \le 1$. We consider the possibilities in two cases based on k.

Case 1. $k < m - 1$.

Note that $\deg(v_{k+1}) = 2$. We consider two subcases.

Subcase A. $s_{k-1} - w_S(v_k) = 0, 1$.

First assume that $w_S(v_{k+1}) = 0$. We let T' be the tree formed by removing the path $(v_1, v_2, \ldots, v_{k+1})$ from T. Note, w_S restricted to T' is strictly positive on a dominating set. As $\deg(v_{k+1}) = 2$, we have that w_S restricted to T' is not reachable from f restricted to T' if and only if w_S is not reachable from f. Hence, w_S restricted to T' is not reachable from f restricted to T', and by the induction hypothesis there is a single vertex $x \in T'$ such that w_S restricted to T' is not reachable from the distribution f', defined by $f'(x) = |f| - f(v_1)$, and $f'(v) = 0$ for all other vertices v of T'. It follows that w_S is not reachable from the distribution f'', defined by $f''(x) = |f| - f(v_1)$, $f''(v_1) = f(v_1)$, and $f''(v) = 0$ for all other vertices v of T. Now $|f''| = |f|$ and f'' has exactly two pebbled vertices, contradicting that p_f is minimum.

Next assume that $w_S(v_{k+1}) \geq 1$.

If $s_{k-1} - w_S(v_k) = 1$, we let T' be the tree formed by removing the path $(v_1, v_2, \ldots, v_{k+1})$ from T, and we define a weight function on T' by $w(v_{k+2}) = w_S(v_{k+2}) + 1$ and $w(v) = w_S(v)$ for all other v. Note, w is strictly positive on a dominating set of T'. Moreover, the only way to move the extra pebble off of v_k is via the rubbling move $r(v_k, v_{k+2} \to v_{k+1})$. It follows that w is not reachable from f restricted to T' if and only w_S is not reachable on T from f. We obtain the result just as above.

If $s_{k-1} - w_S(v_k) = 0$, we let T' be the tree formed by removing the path $(v_1, v_2, \ldots, v_k)$ from T. Note, w_S is strictly positive on a dominating set of T' and w_S restricted to T' is not reachable from f restricted to T' if and only w_S is not reachable on T from f. The result follows just as above.

Subcase B. $s_{k-1} - w_S(v_k) < 0$, that is, $w_S(v_k) - s_{k-1} \geq 1$.

Let T' be the tree formed by removing the path $(v_1, v_2, \ldots, v_k)$ from T. If $s_{k-2} - w_S(v_{k-1}) - 2s_{k-1} = 1$, then we define $w(v_{k+1}) = w_S(v_{k+1}) + 2(w_S(v_k) - s_{k-1} - 1) + 1$ and $w(v) = w_S(v)$ for all other vertices v in T'. Otherwise, we define $w(v_{k+1}) = w_S(v_{k+1}) + 2(w_S(v_k) - s_{k-1})$ and $w(v) = w_S(v)$ for all other vertices v in T'. Note, by construction, w is strictly positive on a dominating set of T'. Then, w is not reachable from f restricted to T' if and only if w_S is not reachable from f in T. Thus, w is not reachable from f restricted to T'. Applying our inductive hypothesis, there is a simple configuration on T' of size $|f'| = |f| - f(v_1)$ from which w is not reachable. Hence, there is a configuration of size $|f|$ on T with two initially pebbled vertices from which w_S is not reachable, contradicting that $p_f \geq 3$.

Case 2. $k \geq m - 1$.

Since v_1 is an arbitrary leaf and the result holds for Case 1, we may assume that every leaf ℓ supplies its split vertex v or the neighbor of v on the ℓ-v-path. Further, since v_m is not a pebbled vertex and all leaves are pebbled, it follows that T is not a path.

We begin by rooting T at some leaf ℓ and let v be the vertex of degree at least three of farthest distance from ℓ. Now any two leaf descendants of v have v as their nearest split vertex. Label two of the leaf descendants of v as x and y, and let P be the unique x-y-path in T. Note that v is on P and that the only pebbled vertices on P are x and y.

Let v_x be the neighbor of v on the x-v-path, and let v_y be the neighbor of v on the v-y-path. By assumption we have that x supplies either v or v_x, and y supplies either v or v_y. Note, after x supplies v or v_x and y supplies v or v_y, it is possible that additional pebbles can be placed on v using a rubbling move $r(v_x, v_y \to v)$. Let s be the total number of pebbles which can be moved to v after possibly being supplied by x and y and after possibly using the previously mentioned rubbling move. Note, s could be equal to zero.

Relabeling the vertices x and y on path P if necessary, we may assume that it takes more pebbles to supply the x-v-path from x than to supply the y-v-path from y. Hence, the configuration f_P on P which places $f(x) + f(y)$ pebbles on x and 0 pebbles on y can, at best, reach w_S restricted to P and place s pebbles on v. If we extend this configuration to T, then if w_S is not reachable from f on T, we have that w_S is not reachable from the new configuration on T. But the new configuration has size $|f|$ and contradicts the minimiality of p_f.

This completes the proof of Theorem 3. □

It follows from Theorem 3 that if w_S is not reachable from a configuration f, then we may assume that f is a simple configuration. Our next result shows that the single vertex which receives a pebble from f can be chosen to be a peripheral vertex of T. Recall the statement of Theorem 4.

Theorem 4. *Let T be a non-trivial tree, S be a dominating set of T, and $w_S : V(T) \to \mathbb{Z}_{\geq 0}$ be strictly positive on S. Let f be a simple configuration with maximum size such that w_S is not reachable from f. If v is the vertex pebbled by f, then v is a peripheral vertex of T.*

Proof. Let f be a simple configuration with $f(v) = \mathrm{st}^r_{w_S}(v) - 1$. Note, by definition, f places the maximum number of pebbles on v while still satisfying that w_S is not reachable from f. Assume for contradiction that v is not a peripheral vertex of T. Recall that every peripheral vertex is a leaf of T. Define $s_w(v) = \sum_{u \in S} w_S(u) \cdot 2^{d(u,v)}$.

Suppose first that v is not a leaf of T. Root T at v and let $\{v_1, v_2, \ldots, v_k\}$ be the children of v. Since v is not a leaf, $k \geq 2$. Let T_i be the subtree of T obtained by removing the edge vv_i, and let T_i be rooted at v_i. Let S_i be the subset of S restricted to T_i, and let $s_{w_i} = \sum_{u \in S_i} w_S(u) \cdot 2^{d(u,v)}$ for $i \in [k]$. Relabeling the vertices if necessary, we may assume that the $s_{w_1} \leq s_{w_j}$ for all $j \geq 2$. But then placing $\frac{s_{w_1}}{2} + 2\sum_{j=2}^{k} s_{w_j}$ pebbles on v_1 gives a simple pebble configuration that cannot reach S having more pebbles than f, contradicting that f has maximal size.

Henceforth, we may assume that v is a leaf, for otherwise we are finished. Thus, the result holds for trees such that every leaf is a peripheral vertex. In particular, the result holds for paths giving us a base case. Assume that the result holds for all trees having $n' < n$ vertices.

Hence, T has at least three leaf vertices. Let x and y be peripheral leaves such that $d(x, y) = \text{diam}(T)$. Note that $v \notin \{x, y\}$ since by assumption v is not a peripheral vertex.

Consider the case where T has exactly three leaves, x, y, and v. Let m be the unique degree 3 vertex of T. It follows that $d(x, m) > d(v, m)$ and $d(y, m) > d(v, m)$, else $d(x, v)$ or $d(y, v)$ would be at least as large as the $\text{diam}(T)$, contradicting that v is not a peripheral vertex. Let T_1 be the x-m-path, T_2 denote the y-m-path, and T_3 denote the v-m-path in T. Without loss of generality, assume that $d(x, m) \leq d(y, m)$. Then $d(v, m) < d(x, m) \leq d(y, m)$. From this it is easily seen that for every $u \in S$, at least one of $d(x, u)$ and $d(y, u)$ is at least $d(v, u) + 1$. Consequently, $s_w(x) + s_w(y) > 2s_w(v)$, contradicting that $s_w(v)$ is largest.

Finally, assume that T has more than three leaves. Let $v' \notin \{v, x, y\}$ be a leaf of T (v' may or may not be a peripheral leaf). Consider $T' = T - v'$. Define the weight function w'_S on T' as follows: $w'_S(z) = w_S(z) + 2w_S(v')$, where z is the neighbor of v' in T. Note w'_S is reachable from f on T' if and only if w_S is reachable from f on T. Now T' has $n - 1$ vertices and v does not become a peripheral vertex in T'. By induction on n, the number of pebbles needed on some peripheral vertex of T' is larger than $s_w(v)$. This completes the proof. □

4. Open Problems

In this brief section we give some questions which follow naturally from the results in this paper.

1. Are there necessary and sufficient conditions on the weight function so that Theorem 2 holds?
2. Is it necessary that the graph in Theorem 3 be a tree? The authors strongly suspect that the answer is no and that a similar result can be proven for all connected graphs.
3. Can one formulate and prove a converse statement to Theorem 3? In particular, it would be interesting to know the precise condition on S so that Theorem 3 holds.

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