Eigenvalues of Ikeda Lifts

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Abstract

In this paper we compute explicit formulas for the Hecke eigenvalues of Ikeda lifts. These formulas, though complicated, are obtained by purely elementary techniques.

1 Introduction

It is well-known in the setting of elliptic newforms that the Hecke eigenvalues encode important arithmetic information. For example, if we attach a Galois representation to our newform (for a survey of this material see [5]), then the Hecke eigenvalues at p determine the characteristic polynomial of the image of the Frobenius element at p.

Similarly, by the work of Laumon in [4] and Weissauer in [9], we have Galois representations attached to degree 2 Siegel eigenforms, and just as in the elliptic setting, the Hecke eigenvalues at p determine the characteristic polynomial of the image of the Frobenius element at p.

In the setting of higher genus Siegel eigenforms, it is reasonable to assume that similar arithmetic information is contained in the Hecke eigenvalues. Note, in this setting we are hindered by the lack of Galois representations attached to higher genus Siegel eigenforms.

We will consider the Ikeda lift, i.e., a map which sends an elliptic eigenform to an even degree Siegel eigenform. In this setting, we derive explicit formulas for the Hecke eigenvalues at p of the lifted form in terms of the Hecke eigenvalues at p of the original elliptic eigenform.

Our paper is organized as follows. In Section 2 we give the necessary background on Siegel modular forms. In Section 3 we present the properties of the Satake parameters and Hecke operators which we will need. Section 4 gives a brief introduction the Ikeda lift and some required results. Finally, in Section 5 we derive the following expressions for the Hecke eigenvalues of Ikeda lifts. The expression for $\lambda(p; I_n(f))$ is

$$p^{\frac{nk}{2}-\frac{n(n+1)}{4}}\left(\sum_{j=1}^{\frac{n}{2}}\sum_{r=0}^{\lfloor\frac{j}{2}\rfloor}\binom{j-r}{r}\frac{j(-1)^r p^c\prod^{n}(p)}{(j-r)\prod^{\frac{n}{2}+j}(p)\prod^{\frac{n}{2}-j}(p)}\lambda(p;f)^{j-2r}+\frac{p^{-\frac{n^2}{8}}\prod^{n}(p)}{\prod^{\frac{n}{2}}(p)^2}\right) \tag{1}$$

and the expression for $\lambda_r(p^2; I_n(f))$ is

$$\begin{aligned} &p^{nk-\frac{n(n+1)}{2}} \sum_{a=1}^{n-r} \sum_{\substack{b=r \\ b\equiv a \pmod 2}}^{n-a} \sum_{c=0}^{\lfloor \frac{a}{2} \rfloor} \frac{a(-1)^c \binom{a-c}{c} p^d m_b(r) \prod^{n}(p)^2}{(a-c) \prod^{\frac{n+a+b}{2}}(p) \prod^{\frac{n-a-b}{2}}(p) \prod^{\frac{n+a-b}{2}}(p) \prod^{\frac{n-a+b}{2}}(p)} \lambda(p;f)^{a-2c} \\ &+ p^{nk-\frac{n(n+1)}{2}} \sum_{\substack{b=r \\ b\equiv 0 \pmod 2}}^{n} \frac{m_b(r) p^{\frac{5b^2-n^2}{4}} \prod^{n}(p)^2}{\prod^{\frac{n+b}{2}}(p) \prod^{\frac{n-b}{2}}(p) \prod^{\frac{n-b}{2}}(p) \prod^{\frac{n+b}{2}}(p)}, \end{aligned} \tag{2}$$

where $d = \frac{a^2+5b^2-n^2+2(n-2k+1)(a-2r)}{4}$.

2 Siegel Modular Forms

Given a ring R we set $M_n(R)$ to be the set of n by n matrices with entries in R.

Define the degree n symplectic group to be

$$\mathrm{GSp}_{2n} := \{g \in \mathrm{GL}_{2n} : {}^t g J_n g = \mu_n(g) J_m, \mu_n(g) \in \mathrm{GL}_1\},$$

where $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. We define Sp_{2n} to be the kernel of μ_n. Throughout we will denote $\mathrm{Sp}_{2n}(\mathbb{Z})$ by Γ_n.

The genus n Siegel upper half-plane is given by

$$\mathfrak{h}^n = \{Z \in M_n(\mathbb{C} : {}^t Z = Z, \mathrm{Im}(Z) > 0\}.$$

We have an action of $\mathrm{GSp}_{2n}^+(\mathbb{R}) := \{g \in \mathrm{GSp}_2\, n(\mathbb{R}) : \mu_n(g) > 0\}$ on $\mathfrak{h}^n$ given by

$$gZ = (aZ+b)(cZ+d)^{-1},$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

For $g \in \mathrm{GSp}_{2n}^+(\mathbb{R})$ and $Z \in \mathfrak{h}^n$, we set $j(g,Z) = \det(cZ+d)$. Let k be a positive integer. For a function $F : \mathfrak{h}^n \to \mathbb{C}$ we define the weight k slash operator on F by

$$(F|_k g_(Z)) = \mu_n(g)^{\frac{nk}{2}} j(g,Z)^{-k} f(gZ).$$

We say that F is a genus n Siegel modular form of weight k if F is holomorphic and $(F|_k\gamma)(Z) = F(Z)$ for all $\gamma \in \Gamma_n$. Note, if $n = 1$ then we must also require F to be holomorphic at the cusps. Denote the space of all such forms by $M_k(\Gamma_n)$. It is a basic result from the theore of Siegel modular forms that every $F \in M_k(\Gamma_n)$ has a Fourier expansion of the form

$$F(Z) = \sum_{\Lambda_n} a_F(T) e(\mathrm{Tr}(TZ)),$$

where Λ_n is the set of n by n half-integral positive semi-definite symmetric matrices and $e(W) := e^{2\pi i W}$. If $a_F(T) = 0$ unless T is strictly positive definite we say that F is a cusp form, and we denote the space of all such cusp forms by $S_k(\Gamma_n)$.

3 Hecke Operators and Satake Parameters

Let $g \in \mathrm{GSp}_{2n}(\mathbb{Q})$, then we write $T(g)$ to denote the double coset $\Gamma_n g \Gamma_n$. Then, we call $T(g)$ a Hecke operator and we have an action of $T(g)$ on $F \in M_k(\Gamma_n)$ given by

$$T(g)F = \sum F|_k g_i,$$

where we are summing over a set of coset representatives for $\Gamma_n \backslash \Gamma_n g \Gamma_n$. Let p be a prime and define

$$T^n(p) = T(\mathrm{diag}(1_n, p1_n)),$$

and for $i = 1, \dots, n$ define

$$T_i^n(p^2) = T(\mathrm{diag}(1_{n-i}, p1_i, p^2 1_{n-i}, p1_i)).$$

Note, we will drop the n from the superscript when the genus is clear from context. It is well known that these operators generate the local Hecke algebra at p. We call $F \in M_k(\Gamma_n)$ an eigenform if F is a simultaneous eigenvector for each of the Hecke operators.

Let $F \in S_k(\Gamma_n)$ be an eigenform. Then, there is a set of complex numbers $\alpha_i(p; F)$ for $0 \le i \le n$, called the Satake p-parameters of F, which completely determine the eigenvalues of F with respect to $T(p)$ and $T_i(p^2)$ for $1 \le i \le n$, which we denote by $\lambda(p; F)$ and $\lambda_i(p^2; F)$, respectively.

In fact, from [7], we have the following relationships between the eigenvalues of F, and the Satake parameters of F,

$$\lambda(p; F) = \alpha_0(p; F)(1 + \sigma_1 + \cdots + \sigma_n), \tag{3}$$

$$\lambda_r(p^2; F) = \alpha_0(p; F)^2 \sum_{\substack{i-j \ge r \\ i,j \ge 0}}^{n} m_{i-j}(r) p^{\binom{i-j+1}{2}} \sigma_i \sigma_j, \tag{4}$$

where $m_h(r) := \#\{M \in M_h(\mathbb{F}_p) : {}^tA = A, \mathrm{corank}(A) = r\}$, σ_i is the degree i elementary symmetric function in $\alpha_1(p; F), \dots, \alpha_n(p; F)$, and $1 \le i \le n$. Note, we have normalized our Satake parameters to satisfy

$$\alpha_0(p; F)^2 \alpha_1(p; F) \dots \alpha_n(p; F) = p^{nk - \frac{n(n+1)}{2}}.$$

Suppose $f \in S_k(\Gamma_1)$ is a normalized eigenform, i.e., $a_f(1) = 1$. Then we will make use of the following relationship as well,

$$\alpha(p; f) + \alpha(p; f)^{-1} = p^{\frac{1-k}{2}} \lambda(p; f), \tag{5}$$

where we simply use $\alpha(p;f)$ to denote $\alpha_0(p;f)$.

Using these Satake parameters we can associate two L-functions to an eigenform $F \in S_k(\Gamma_n)$. First, in order to define the standard L-function of F we need the following local L-factor

$$L_p(s,F;\mathrm{st}) = (1-p^{-s})\prod_{i=1}^{n}(1-\alpha_i(p;F)p^{-s})(1-\alpha_i(p;F)^{-1}p^{-s}).$$

Then, the standard L-function associated to F is given by the following Euler product

$$L(s,F;\mathrm{st}) = \prod_p L_p(s,F;\mathrm{st})^{-1}.$$

Second, in order to define the spinor L-function associated to F we need the following local L-factor

$$L_p(s,F;\mathrm{spin}) = (1-\alpha_0(p;F)p^{-s})\prod_{i=1}^{n}\prod_{1\le i_1\le\cdots\le i_j\le n}(1-\alpha_0(p;F)\alpha_{i_1}(p;F)\ldots\alpha_{i_j}(p;F)p^{-s}).$$

Then, the spinor L-function associated to F is given by

$$L(s,F;\mathrm{spin}) = \prod_p L_p(s,F;\mathrm{spin})^{-1}.$$

Note, when F is of genus 1, this is simply the usual Dirichlet series associated to F.

4 The Ikeda Lift

In this section we introduce the Ikeda lift as presented in [3].

Let $T > 0$ be in Λ_n. Set D_T to be the determinant of $2T$, Δ_T to be the absolute value of the discriminant of $\mathbb{Q}(\sqrt{D_T})$, χ_T to be the primitive Dirichlet character associated to $\mathbb{Q}(\sqrt{D_T})/\mathbb{Q}$, and $\mathfrak{f}_T$ to be the rational number satisfying $D_T = \Delta_T\mathfrak{f}_T^2$.

Let $S_n(R)$ denote the set of symmetric $n\times n$ matrices over a ring R. For a rational prime p, let $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ be the unique additive character given by

$$\psi_p(x) = \exp(-\{x\}_p),$$

where $\{x\}_p \in \mathbb{Z}[\frac{1}{p}]$ is the p-adic fractional part of x. The Siegel series for T is defined to be

$$b_p(T,s) := \sum_{S\in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} \psi_p(\mathrm{Tr}(TS))|\nu(S)|_p p^{-s}, \text{ for } \mathrm{Re}(s) >> 0,$$

where $\nu(S) := \det(S_1)\cdot\mathbb{Z}_p$, and S_1 is from the factorization $S = S_1^{-1}S_2$ for a symmetric coprime pair of matrices S_1, S_2. We have a factorization of the Siegel series given by

$$b_p(T,s) = \gamma_p(T,p^{-s})F_p(T,p^{-s}),$$

where

$$\gamma_p(T,X) = \frac{1-X}{1-p^{\frac{n}{2}}\chi_T(p)X}\prod_{i=1}^{n/2}(1-p^{2i}X^2),$$

and $F_p(T,X) \in \mathbb{Z}[X]$ has constant term 1 and $\deg(F_p(T,X)) = 2\,\mathrm{ord}_p(\mathfrak{f}_T)$. Using this polynomial $F_p(T,X)$ we define

$$\tilde{F}_p(T,X) := X^{-\mathrm{ord}_p(\mathfrak{f}_T)}F_p(T, p^{-\frac{n}{2}-\frac{1}{2}}X).$$

We will make use of this $\tilde{F}_p(T,X)$ later.

Let $f(\tau) \in S_{2k-n}(\Gamma_1))$ be a normalized eigenform and let

$$h(\tau) = \sum_{\substack{m>0 \\ (-1)^k m \equiv 0,1(4)}} c(N)q^N \in S^+_{k-\frac{n}{2}+\frac{1}{2}}(\Gamma_0(4))$$

be an eigenform corresponding to f via. the Shimura correspondence.

For each $T > 0$ in Λ_n, define

$$a(T) = c(|\Delta_T|)\mathfrak{f}_T^{k-\frac{n+1}{2}}\prod_p \tilde{F}_p(T, \alpha(p;f)), \tag{6}$$

and form the following series

$$I_n(f)(z) = \sum_T a(T)e(\mathrm{Tr}(Tz)).$$

We have the following theorem.

Theorem 1. *([3, Thm. 3.2, 3.3]) The series $I_n(f)(Z)$, referred to as the Ikeda lift of f, is an eigenform in $S_k(\Gamma_n)$ whose standard L-function factors as*

$$L(s,F;\mathrm{st}) = \zeta(s-n)\prod_{i=1}^{n} L(s+k-n-i,f).$$

Note, if $n = 2$ then this L-function factorization agrees with that of the Saito-Kurokawa lift. From the factorization of the standard L-function given in the previous theorem, we are able to obtain an expression for $\alpha_i(p;I_n(f))$ in terms of $\alpha(p;f)$ for $1 \leq i \leq n$. However, in order to determine the eigenvalue of $I_n(f)$ we must also have information about $\alpha_0(p;I_n(f))$. This can be achieved by determining the factorization of the spinor L-function of $I_n(f)$. This factorization can be found in [6]. In summary, we obtain the following relations between Satake parameters

$$\alpha_0(p;I_n(f)) = p^{\frac{nk}{2}-\frac{n(n+1)}{4}}\alpha(p;f)^{-\frac{n}{2}}, \tag{7}$$

$$\alpha_j(p;I_n(f)) = p^{j-\frac{(n+1)}{2}}\alpha(p;f), \text{ for } j = 1,\dots,n. \tag{8}$$

Note, we our Satake parameters are normalized differently than the Satake parameters in [6].

5 Eigenvalues of the Ikeda lift

In this section we will derive Equations 1 and 2.

We begin by determining a nice expression for the symmetric polynomials in Equations 3 and 4. Using Equation 8 we obtain the following expression,

$$\sigma_j = p^{-\frac{j(n+1)}{2}} \alpha(p;f)^j \sum_{i=0}^{\frac{n(n+1)}{2}} q_{\leq n}(i,j)p^i, \tag{9}$$

where we are using $q_{\leq n}(i,j)$ to denote the number of partitions of i into j distinct parts where each part is no greater than n and our symmetric polynomial is in $\alpha_1(p;I_n(f)),\ldots,\alpha_n(p;I_n(f))$.
Combining Equation (7.3) and Theorem 8 of [1] we have the following generating series for this partition function

$$\begin{aligned}
\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q_{\leq n}(i,j)z_1^j z_2^i &= \prod_{\ell=1}^{n}(1+z_1 z_2^{\ell}) \\
&= \sum_{j=0}^{\infty}\left(\frac{\prod^{n}(z_2)}{\prod^{n-j}(z_2)\prod^{j}(z_2)}\right) z_2^{\frac{j(j+1)}{2}} z_1^j,
\end{aligned}$$

where $\prod^n(x) := \prod_{m=1}^{n}(1-x^m)$. If we fix a $j \leq n$ on the left hand side and evaluate at $z_2 = p$ we obtain

$$\left(\sum_{i=0}^{\frac{n(n+1)}{2}} q_{\leq n}(i,j)p^i\right) z_1^j,$$

where we need only take the summation to $n(n+1)/2$ since we have restricted to $j \leq n$. Matching coefficients we have

$$\sum_{i=0}^{\frac{n(n+1)}{2}} q_{\leq n}(i,j)p^i = \frac{p^{\frac{j(j+1)}{2}}\prod^{n}(p)}{\prod^{n-j}(p)\prod^{j}(p)}. \tag{10}$$

The following lemma follows immediately from this equality.

Lemma 2. *For any $j \leq n$,*

$$p^{\frac{-j(n+1)}{2}} \sum_{i=0}^{\frac{n(n+1)}{2}} q_{\leq n}(i,j)p^i = p^{\frac{-(n-j)(n+1)}{2}} \sum_{i=0}^{\frac{n(n+1)}{2}} q_{\leq n}(i,n-j)p^i.$$

In summary we have

$$\sigma_j = \frac{p^{\frac{j(j-n)}{2}} \prod^{n}(p)}{\prod^{n-j}(p) \prod^{j}(p)} \alpha(p; f)^j.$$

We are now prepared to compute $\lambda(p; I_n(f))$. Combining Equations 3, 9, and 10 and pairing our terms using Lemma 2 we obtain the following expression for $\lambda(p; I_n(f))$,

$$p^{\frac{nk}{2} - \frac{n(n+1)}{4}} \left(\sum_{j=1}^{\frac{n}{2}} \frac{p^{\frac{-(\frac{n}{2}-j)(\frac{n}{2}+j)}{2}} \prod^{n}(p)}{\prod^{\frac{n}{2}+j}(p) \prod^{\frac{n}{2}-j}(p)} \left(\alpha(p)^j + \alpha(p)^{-j} \right) + \frac{p^{-\frac{n^2}{8}} \prod^{n}(p)}{\prod^{\frac{n}{2}}(p)^2} \right).$$

Applying a theorem of Waterson from [8] along with Equation 5 we have

$$\alpha(p)^j + \alpha(p)^{-j} = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^r \frac{j}{j-r} \binom{j-r}{r} \left(p^{\frac{n-2k+1}{2}} \lambda(p; f) \right)^{j-2r}.$$

Inserting this into the previous equation and rearranging we obtain the following expression for $\lambda_{I_n(f)}(p)$

$$p^{\frac{nk}{2} - \frac{n(n+1)}{4}} \left(\sum_{j=1}^{\frac{n}{2}} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j-r}{r} \frac{j(-1)^r p^c \prod^{n}(p)}{(j-r) \prod^{\frac{n}{2}+j}(p) \prod^{\frac{n}{2}-j}(p)} \lambda(p; f)^{j-2r} + \frac{p^{-\frac{n^2}{8}} \prod^{n}(p)}{\prod^{\frac{n}{2}}(p)^2} \right),$$

where $c = \frac{-(\frac{n}{2}-j)(\frac{n}{2}+j)+j(n-2k+1)+2r(2k-n-1)}{2}$.

In order to compute a similar formula for $\lambda_r(p^2; I_n(f))$ for $r < n$, we first need the following expression for $m_h(r)$ from [2]

$$m_h(r) = \frac{p^{\binom{h}{2}} \prod^{h}(p) \prod_{\ell=0}^{\lfloor \frac{h}{2} \rfloor} (1 - p^{1-2\ell})}{\prod^{h-r}(p) \prod^{r}(p)}.$$

Applying this formula along with similar techniques used above we have the following expression for $\lambda_r(p^2; I_n(f))$,

$$p^{nk - \frac{n(n+1)}{2}} \sum_{\substack{i-j \geq r \\ i,j \geq 0}}^{n} \frac{p^{\frac{j(j-n)+i(i-n)+2(i-j)^2}{2}} \prod^{n}(p)^2 \prod^{i-j}(p) \prod_{\ell=0}^{\lfloor \frac{i-j}{2} \rfloor} (1 - p^{1-2\ell})}{\prod^{i-j-r}(p) \prod^{r}(p) \prod^{n-j}(p) \prod^{j}(p) \prod^{n-i}(p) \prod^{i}(p)} \alpha(p; f)^{i+j-n}. \blacksquare$$

By making the substitutions $a = i + j$ and $b = i - j$ and simplifying we obtain

$$= p^{nk-\frac{n(n+1)}{2}} \sum_{a=r}^{n} \sum_{\substack{b=r \\ b\equiv a \pmod 2}}^{a} \frac{p^{\frac{a^2+5b^2-n^2}{4}} \prod^{n}(p)^2 \prod^{b}(p) \prod_{\ell=0}^{\lfloor \frac{b}{2} \rfloor}(1-p^{1-2\ell})}{\prod^{b-r}(p) \prod^{r}(p) \prod^{n-\frac{a-b}{2}}(p) \prod^{\frac{a-b}{2}}(p) \prod^{n-\frac{a+b}{2}}(p) \prod^{\frac{a+b}{2}}(p)} \alpha(p;f)^{a-n}$$

$$+ p^{nk-\frac{n(n+1)}{2}} \sum_{a=n+1}^{2n-r} \sum_{\substack{b=r \\ b\equiv a \pmod 2}}^{2n-a} \frac{p^{\frac{5b^2-n^2}{4}} \prod^{n}(p)^2 \prod^{b}(p) \prod_{\ell=0}^{\lfloor \frac{b}{2} \rfloor}(1-p^{1-2\ell})}{\prod^{b-r}(p) \prod^{r}(p) \prod^{n-\frac{a-b}{2}}(p) \prod^{\frac{a-b}{2}}(p) \prod^{n-\frac{a+b}{2}}(p) \prod^{\frac{a+b}{2}}(p)} \alpha(p;f)^{a-n} \qquad \blacksquare$$

Just as above, we can shift each summation, pair appropriately, and then apply the theorem of Waterson to obtain our final expression for $\lambda_r(p^2; I_n(f))$

$$= p^{nk-\frac{n(n+1)}{2}} \sum_{a=1}^{n-r} \sum_{\substack{b=r \\ b\equiv a \pmod 2}}^{n-a} \sum_{c=0}^{\lfloor \frac{a}{2} \rfloor} \frac{a(-1)^c \binom{a-c}{c} p^d m_b(r) \prod^{n}(p)^2}{(a-c) \prod^{\frac{n+a+b}{2}}(p) \prod^{\frac{n-a-b}{2}}(p) \prod^{\frac{n+a-b}{2}}(p) \prod^{\frac{n-a+b}{2}}(p)} \lambda(p;f)^{a-2c}$$

$$+ p^{nk-\frac{n(n+1)}{2}} \sum_{\substack{b=r \\ b\equiv 0 \pmod 2}}^{n} \frac{m_b(r) p^{\frac{5b^2-n^2}{4}} \prod^{n}(p)^2}{\prod^{\frac{n+b}{2}}(p) \prod^{\frac{n-b}{2}}(p) \prod^{\frac{n-b}{2}}(p) \prod^{\frac{n+b}{2}}(p)}.$$

where $d = \frac{a^2+5b^2-n^2+2(n-2k+1)(a-2r)}{4}$.

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