Congruence Primes for Ikeda Lifts and the Ikeda ideal

Jim Brown and Rodney Keaton

Abstract

Let f be a newform of level 1 and weight $2\kappa - n$ for κ and n positive even integers. In this paper we study congruence primes for the Ikeda lift of f. In particular, we consider a conjecture of Katsurada stating that primes dividing certain L-values of f are congruence primes for the Ikeda lift. Instead of focusing on a congruence to a single eigenform, we deduce a lower bound on the number of all congruences between the Ikeda lift and forms not lying in the space spanned by Ikeda lifts.

1 Introduction

Let κ be an integer and let χ be a Dirichlet character of conductor N satisfying $\chi(-1) = (-1)^\kappa$. One has an associated Eisenstein series $E_{\kappa,\chi}$. It is a well-known fact that for a prime $\ell \nmid N$ there exists a cuspidal eigenform f of level M with $N|M$ such that $f \equiv E_{\kappa,\chi} \pmod{\lambda}$ for λ a prime dividing ℓ in a suitably large extension of $\mathbb{Z}$. Such congruences between cusp forms and Eisenstein series have been studied by many authors. For instance, one can use such congruences to make deductions on the structure of the residual Galois representation of the cusp form, which can then be used to study Selmer groups associated to the cusp form. For instance, one can see [26, 35, 41] for some prominent examples of this type of argument.

If we view the Eisenstein series above as a "lift" of the Dirichlet character χ from GL(1) to GL(2), then we can fit the congruences mentioned above into a more general framework. Namely, one can consider more general automorphic forms and lifting them to automorphic forms on other algebraic groups. This approach has also received considerable attention as this method can also be used to study Selmer groups of higher degree Galois representations, see for example [3, 22, 34] for specific examples and [25] for a survey of this method. This makes classifying primes for which one will have a congruence between a lifted form and a non-lifted form a natural question to study. In this paper we investigate this problem for Ikeda lifts.

Let κ and n be positive even integers, $f \in S_{2\kappa-n}(\mathrm{SL}_2(\mathbb{Z}))$ be a newform, and $I_n(f) \in S_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$ the Ikeda lift of f. In [20] Katsurada states a conjecture on when a prime λ will satisfy that there is an eigenform $F \in S_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$ that is not an Ikeda lift and is congruent to $I_n(f)$ modulo λ. The conjecture is in terms of divisibilities of special values of L-functions of f by λ. One can see Conjecture 9 for the precise statement. To provide evidence for his conjecture he proves that if a prime divides the required L-values and does not divide other L-values then one indeed does have such a congruence (see Theorem 10.) In this paper we also consider Ikeda lifts, but instead of focusing on producing one congruence we introduce the Ikeda ideal. This ideal is an analogue of the Eisenstein ideal in the $\mathrm{GL}(2)$ case and measures congruences between $I_n(f)$ and all other eigenforms. We then show that under similar hypotheses as given in [20], we can do better and bound from below the congruences between $I_n(f)$ and all other eigenforms that are not lifts. One can see Theorem 14 for the precise result.

One thing to note here is that while the Saito-Kurokawa lift is useful for studying the p-adic Bloch-Kato conjecture for the L-value $L_{\mathrm{alg}}(\kappa, f)$ due to the fact that the value $L_{\mathrm{alg}}(\kappa, f)$ "controls" the congruence between the Saito-Kurokawa lift and a non-lifted form (see [3, 9] for example), the L-values that control the congruence for an Ikeda lift are given by $L_{\mathrm{alg}}(\kappa, f)\prod_{j=1}^{n/2-1} L_{\mathrm{alg}}(2j+1, \mathrm{ad}^0 f)$. This indicates that if one knew the existence of Galois representations for automorphic forms on GSp_{2n} as well as expected properties of these representations, one could use the congruence results produced in this paper to study the p-adic Bloch-Kato conjecture not only for $L_{\mathrm{alg}}(\kappa, f)$, but also for the values $L_{\mathrm{alg}}(2j+1, \mathrm{ad}^0 f)$ for $j = 1, \dots, n/2-1$. This makes such congruences particularly interesting.

The structure of the paper is as follows. Section 2 recalls the basic definitions we will need throughout the paper. We recall the Ikeda lift and some necessary properties in Section 3. In the following section we state Katsurada's conjecture and result, introduce the Ikeda ideal, and show how Katsurada's congruence can be recovered by studying the Ikeda ideal. We then state our main result and discuss the major hypotheses in Section 5. Section 6 gives a somewhat detailed description of an Eisenstein series originally introduced by Shimura as well as the results we'll need to prove the main theorem. Finally, we conclude by proving the main theorem in Section 7.

2 Modular Forms

In this section we recall the basics on modular forms and Siegel modular forms that will be needed throughout the rest of the paper.

2.1 Basic definitions

Given a ring R with identity, we write $\mathrm{Mat}_n(R)$ for the ring of n by n matrices with entries in R.

Set $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ and recall the degree n symplectic group is defined by

$$G_n = \mathrm{GSp}_{2n} = \left\{ g \in \mathrm{GL}_{2n} : {}^t g J_n g = \mu_n(g) J_n, \mu_n(g) \in \mathrm{GL}_1 \right\}.$$

We set $\mathrm{Sp}_{2n} = \ker(\mu_n)$. We denote $\mathrm{Sp}_{2n}(\mathbb{Z})$ by Γ_n to ease notation.

The Siegel upper half-space is given by

$$\mathfrak{h}_n = \{z \in \mathrm{Mat}_n(\mathbb{C}) : {}^t z = z, \mathrm{Im}(z) > 0\}.$$

We have an action of $G_n^+(\mathbb{R}) = \{g \in G_n(\mathbb{R}) : \mu_n(g) > 0\}$ on $\mathfrak{h}_n$ given by

$$gz = (a_g z + b_g)(c_g z + d_g)^{-1}$$

for $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$.

For $g \in G_n^+(\mathbb{R})$ and $z \in \mathfrak{h}_n$, we set

$$j(g, z) = \det(c_g z + d_g).$$

Let κ be a positive integer. Given $f : \mathfrak{h}_n \to \mathbb{C}$, we define the slash operator on f by

$$(f|_\kappa g)(z) = \mu_n(g)^{n\kappa/2} j(g, z)^{-\kappa} f(gz).$$

Let $\Gamma \subset \Gamma_n$ be a congruence subgroup. We say such an f is a genus n Siegel modular form of weight κ and level Γ if f is holomorphic and satisfies

$$(f|_\kappa \gamma)(z) = f(z)$$

for all $\gamma \in \Gamma$. If $n = 1$ we also require that f is holomorphic at the cusps so that we recover the theory of elliptic modular forms. We denote the space of genus n, level Γ, and weight κ modular forms by $M_\kappa(\Gamma)$.

Given $f \in M_\kappa(\Gamma)$, f has a Fourier expansion of the form

$$f(z) = \sum_{T \in \Lambda_n} a_f(T) e(\mathrm{Tr}(Tz))$$

where Λ_n is defined to be the set of n by n half integral positive semi-definite symmetric matrices and $e(w) := e^{2\pi i w}$. If $a_f(T) = 0$ unless $T > 0$ (i.e., T is positive definite), we say f is a cusp form. We write $S_\kappa(\Gamma)$ for the cusp forms in $M_\kappa(\Gamma)$. Given a ring R, we write $M_\kappa(\Gamma; R)$ for those modular forms whose Fourier coefficients all lie in R and likewise for the cuspforms.

Let $f_1, f_2 \in M_\kappa(\Gamma)$ with at least one of them a cusp form. The Petersson inner product of f_1 and f_2 is defined by

$$\langle f_1, f_2 \rangle_\Gamma = \int_{\Gamma \backslash \mathfrak{h}_n} f_1(z) \overline{f_2(z)} (\det y)^\kappa d\mu z,$$

where $z = x + iy$ with $x = (x_{\alpha,\beta})$, $y = (y_{\alpha,\beta}) \in \mathrm{Mat}_n(\mathbb{R})$,

$$d\mu z = (\det y)^{-(n+1)} \prod_{\alpha \leq \beta} dx_{\alpha,\beta} \prod_{\alpha \leq \beta} dy_{\alpha,\beta}$$

with $dx_{\alpha,\beta}$ and $dy_{\alpha,\beta}$ the usual Lebesgue measure on $\mathbb{R}$. We will use the following scaled definition that is independent of the congruence subgroup considered

$$\langle f_1, f_2 \rangle = \frac{1}{[\overline{\Gamma}_n : \overline{\Gamma}]} \langle f_1, f_2 \rangle_\Gamma$$

where $\overline{\Gamma}_n = \Gamma_n / \{\pm 1_{2n}\}$ and $\overline{\Gamma}$ is the image of Γ in $\overline{\Gamma}_n$.

2.2 Hecke algebras

Let $\Gamma \subset \Gamma_n$ be a congruence subgroup. Given $g \in G_n^+(\mathbb{Q})$, we write $T(g)$ to denote the double coset

$$\Gamma g \Gamma.$$

We define the usual action of $T(g)$ on Siegel modular forms by setting

$$T(g)f = \sum_i f|_\kappa g_i$$

where $\Gamma g \Gamma = \coprod_i \Gamma g_i$ and $f \in M_\kappa(\Gamma)$. Let p be prime and define

$$T^{(n)}(p) = T(\operatorname{diag}(1_n, p1_n))$$

and for $i = 1, \ldots, n$ set

$$T_i^{(n)}(p^2) = T(\operatorname{diag}(1_{n-i}, p1_i, p^2 1_{n-i}, p1_i)).$$

It is well known these generate the local Siegel Hecke algebra at p.

Let $\mathcal{H}_{\mathbb{Z}}^{(n)}$ denote the $\mathbb{Z}$-subalgebra of $\operatorname{End}_{\mathbb{C}}(S_\kappa(\Gamma))$ generated by $T^{(n)}(p)$ and $T_i^{(n)}(p^2)$ for $i = 1, \ldots, n$. Given any $\mathbb{Z}$-algebra A, we write $\mathcal{H}_A^{(n)}$ for $\mathcal{H}_{\mathbb{Z}}^{(n)} \otimes_{\mathbb{Z}} A$.

Let E be a finite extension of $\mathbb{Q}_\ell$ and $\mathcal{O}_E$ the ring of integers of E. Then we have $\mathcal{H}_{\mathcal{O}_E}^{(n)}$ is a semi-local complete finite $\mathcal{O}_E$-algebra. One has

$$\mathcal{H}_{\mathcal{O}_E}^{(n)} = \prod_{\mathfrak{m}} \mathcal{H}_{\mathfrak{m}}^{(n)}$$

where the product runs over all maximal ideals of $\mathcal{H}_{\mathcal{O}_E}^{(n)}$ and $\mathcal{H}_{\mathfrak{m}}^{(n)}$ denotes the localization of $\mathcal{H}_{\mathcal{O}_E}^{(n)}$ at $\mathfrak{m}$.

2.3 Congruences

Let $f, g \in M_\kappa(\Gamma)$. Let ℓ be an odd prime, and let $K/\mathbb{Q}_\ell$ be a finite extension containing all the Fourier coefficients of f and g. Let $\mathcal{O}$ be the ring of integers of K and λ a uniformizer of $\mathcal{O}$. We write

$$f \equiv g \pmod{\lambda^b}$$

to denote

$$\operatorname{val}_\lambda(a_f(T) - a_g(T)) \geq b$$

for all $T \in \Lambda_n$. We refer to this as a congruence of Fourier coefficients.

We will also use the notion of a congruence of eigenvalues. In this case if f and g are eigenforms for all $t \in \mathcal{H}_{\mathcal{O}}^{(n)}$ with eigenvalues $\lambda_f(t)$ and $\lambda_g(t)$ respectively, we write

$$f \equiv_{\text{ev}} g \pmod{\lambda^b}$$

to denote

$$\operatorname{val}_\lambda(\lambda_f(t) - \lambda_g(t)) \geq b$$

for all $t \in \mathcal{H}_{\mathcal{O}}^{(n)}$.

2.4 L-functions

In this section we introduce the L-functions that will be needed in this paper. In the case of the relevant L-functions attached to elliptic modular forms, we also introduce the appropriate canonical periods.

Given a general L-function $L(s)$ with an Euler product

$$L(s) = \prod_p L_p(s)$$

and a finite set of primes Σ, we write

$$L^{\Sigma}(s) = \prod_{p \notin \Sigma} L_p(s)$$

and

$$L_{\Sigma}(s) = \prod_{p \in \Sigma} L_p(s).$$

We begin with the case of an elliptic modular form $f \in S_\kappa(\Gamma_1)$. We assume f is a normalized eigenform with Fourier expansion

$$f(z) = \sum_{n \geq 1} a_f(n) e(nz).$$

Let $\pi_f = \otimes_p \pi_{f,p}$ be the automorphic representation associated to f. For each prime p there exists a character σ_p so that $\pi_{f,p} = \pi(\sigma_p, \sigma_p^{-1})$. The p-Satake parameter of f is given by $\alpha_0(p; f) = \sigma_p(p)$. We will drop the f from the notation when it is clear from context. The L-function of f is given by

$$\begin{aligned} L(s, f) &= \prod_p (1 - \alpha_0(p) p^{-s-(\kappa+1)/2})^{-1} (1 - \alpha_0(p)^{-1} p^{-s-(\kappa+1)/2})^{-1} \\ &= \prod_p (1 - a_f(p) p^{-s} + p^{\kappa-1-2s})^{-1} \\ &= \sum_{n \geq 1} a_f(n) n^{-s}. \end{aligned}$$

Given a Dirichlet character χ, we will also make use of the twisted L-function

$$L(s, f, \chi) = \sum_{n \geq 1} \chi(n) a_f(n) n^{-s}.$$

Let $\ell \geq \kappa$ be a prime and let K be a suitably large finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$. Let $f \in S_\kappa(\Gamma_1; \mathcal{O})$ be a normalized eigenform. Let $\rho_{f,\lambda}$ be the λ-adic Galois representation associated to f and assume the residual representation $\overline{\rho}_{f,\lambda}$ is irreducible. Then we have canonical complex periods $\Omega_f^{\pm}$ (determined up to ℓ-units) by work of Vatsal ([39]). Note, Vatsal shows that such periods exist for level greater than 3, but using arguments in [15] we can define $\Omega_f^{\pm}$ for arbitrary level. One can see [8] for more details. Using these periods we have the following theorem.

Theorem 1. *([29, 39]) Let $f \in S_\kappa(\Gamma_1; \mathcal{O})$ be as in the above discussion. There exists complex periods $\Omega_f^{\pm}$ such that for each integer m with $0 < m < \kappa$ and every Dirichlet character χ one has,*

$$\frac{L(m,f,\chi)}{\tau(\chi)(2\pi\sqrt{-1})^m} \in \begin{cases} \Omega_f^{+}\mathcal{O}_\chi & if \quad \chi(-1) = (-1)^m \\ \Omega_f^{-}\mathcal{O}_\chi & if \quad \chi(-1) = (-1)^{m-1}, \end{cases}$$

where $\tau(\chi)$ is the Gauss sum of χ and $\mathcal{O}_\chi$ is the extension of $\mathcal{O}$ generated by the values of χ.

With this theorem in mind we set the following notation for the algebraic part of $L(m,f,\chi)$ with $0 < m < \kappa$

$$L_{\mathrm{alg}}(m,f,\chi) := \frac{L(m,f,\chi)}{\tau(\chi)(2\pi\sqrt{-1})^m\Omega_f^{\pm}},$$

where the choice of period is from the theorem.

For Siegel modular forms of genus greater than 1 there are two relevant L-functions: the standard and spinor L-functions. Let $f \in S_\kappa(\Gamma_n)$ be an eigenform. Let $\alpha_0(p;f), \alpha_1(p;f), \ldots, \alpha_n(p;f)$ denote the p-Satake parameters of f normalized so that

$$\alpha_0(p;f)^2\alpha_1(p;f)\cdots\alpha_n(p;f) = 1.$$

One can see [4] for more information on the Satake parameters. We drop f and/or p in the notation for the Satake parameters when they are clear from context. Set $\widetilde{\alpha_0} = p^{\frac{2n\kappa - n(n+1)}{4}}\alpha_0$ and

$$L_p(X,f;\mathrm{spin}) = (1 - \widetilde{\alpha_0}X)\prod_{j=1}^{n}\prod_{1\le i_1\le\cdots\le i_j\le n}(1 - \widetilde{\alpha_0}\alpha_{i_1}\cdots\alpha_{i_j}X).$$

The spinor L-function associated to f is given by

$$L(s,f;\mathrm{spin}) = \prod_p L_p(p^{-s},f;\mathrm{spin})^{-1}.$$

One should note that in the case f is an elliptic modular form the spinor L-function is exactly $L(s,f)$ defined above.

Set

$$L_p(X,f;\mathrm{st}) = (1 - X)\prod_{i=1}^{n}(1 - \alpha_i(p)X)(1 - \alpha_i(p)^{-1}X).$$

Then, we define the standard L-function associated to f by

$$L(s,f;\mathrm{st}) = \prod_p L_p(p^{-s},f;\mathrm{st})^{-1}.$$

Given a Hecke character χ, the twisted standard L-function is given by

$$L(s,f,\chi;\mathrm{st}) = \prod_p L_p(\chi(p)p^{-s},f;\mathrm{st})^{-1}.$$

In the case that $f \in S_\kappa(\Gamma_1; \mathcal{O})$ is an elliptic modular form the standard L-function is usually denoted by $L(s, \mathrm{ad}^0 f)$, i.e., it is the adjoint L-function. In this case we have via [42] that

$$\frac{L(m, \mathrm{ad}^0 f)}{\pi^{2m+\kappa-1}\Omega_f^+\Omega_f^-} \in \overline{\mathbb{Q}}$$

for $m = 1, 3, \ldots, \kappa - 1$ and

$$\frac{L(m, \mathrm{ad}^0 f)}{\pi^{m+\kappa-1}\Omega_f^+\Omega_f^-} \in \overline{\mathbb{Q}}$$

for $m = 2-\kappa, 4-\kappa, \ldots, 0$. We will only be interested in the first case; we denote this algebraic value by $L_{\mathrm{alg}}(m, \mathrm{ad}^0 f)$.

3 The Ikeda lift

In this section we will present an introduction to the Ikeda lift. For the details the reader is referred to Kohnen's paper [23] or Ikeda's original paper [16]. The Ikeda lifting can be viewed as a composition of the Shintani map from the space of elliptic modular forms to the space of half-integral weight modular forms and a map from the space of half-integral weight forms to the correct space of Siegel modular forms.

Throughout we assume κ, n to be positive even integers. We note here that we begin with weight $2\kappa - n$ instead of 2κ as is used in [16, 23]. This normalization is more convenient for our purposes.

We begin by recalling the algebraic version of Shintani's lift that we require. One has the following result of Shintani ([33]).

Theorem 2. *There is a linear function*

$$\theta_{\kappa,n} : S_{2\kappa-n}(\Gamma_1) \to S^+_{\kappa-\frac{n}{2}+\frac{1}{2}}(\Gamma_0(4))$$

that is Hecke equivariant, i.e., given any prime p one has $\theta_{\kappa,n}(f|T(p)) = \theta_{\kappa,n}(f)|T(p^2)$.

We have the following result of Stevens that will be pivotal for the algebraic construction.

Proposition 3. *([36, Prop. 2.3.1]) Let $f \in S_{2\kappa-n}(\Gamma_1; \mathcal{O})$ be a Hecke eigenform where $\mathcal{O}$ is the ring of integers of a field that can be embedded in $\mathbb{C}$. Then there is a nonzero complex number $\Omega(f) \in \mathbb{C}^\times$ so that*

$$\frac{1}{\Omega(f)}\theta_{\kappa,n}(f) \in S^+_{\kappa-\frac{n}{2}+\frac{1}{2}}(\Gamma_0(4); \mathcal{O}).$$

Moreover, if $\mathcal{O}$ is a discrete valuation ring $\Omega(f)$ can be chosen so that at least one of the Fourier coefficients of $\frac{1}{\Omega(f)}\theta_{\kappa,n}(f)$ is a unit in $\mathcal{O}$.

From now on we write $\theta_{\kappa,n}^{\mathrm{alg}}(f)$ for $\frac{1}{\Omega(f)}\theta_{\kappa,n}(f)$ and will always choose the period so that $\theta_{\kappa,n}^{\mathrm{alg}}(f)$ has a unit Fourier coefficient in the case $\mathcal{O}$ is a discrete valuation ring. We write

$$\theta_{\kappa,n}^{\mathrm{alg}}(f)(z) = \sum_{\substack{m>0 \\ m\equiv 0,1 \pmod 4}} c(m)e(mz).$$

Let $T > 0$ be in Λ_n, i.e., T is an $n \times n$ half integral positive definite symmetric matrix. Set D_T to be the determinant of $2T$, Δ_T to be the absolute value of the discriminant of $\mathbb{Q}(\sqrt{D_T})$, χ_T to be the primitive Dirichlet character associated to $\mathbb{Q}(\sqrt{D_T})/\mathbb{Q}$, and $\mathfrak{f}_T$ to be the rational number satisfying $D_T = \Delta_T \mathfrak{f}_T^2$.

Let $S_n(R)$ denote the set of symmetric $n \times n$ matrices over a ring R. For a rational prime p, let $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ be the unique additive character given by

$$\psi_p(x) = \exp(-\left\{x\right\}_p),$$

where $\left\{x\right\}_p \in \mathbb{Z}[\frac{1}{p}]$ is the p-adic fractional part of x. The Siegel series for T is defined to be

$$b_p(T,s) := \sum_{S\in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} \psi_p(\mathrm{Tr}(TS))|\nu(S)|_p p^{-s}, \text{ for } \mathrm{Re}(s) >> 0,$$

where $\nu(S) := \det(S_1)\cdot \mathbb{Z}_p$, and S_1 is from the factorization $S = S_1^{-1}S_2$ for a symmetric coprime pair of matrices S_1, S_2. We have a factorization of the Siegel series given by

$$b_p(T,s) = \gamma_p(T,p^{-s})F_p(T,p^{-s}),$$

where

$$\gamma_p(T,X) = \frac{1-X}{1-p^{\frac{n}{2}}\chi_T(p)X}\prod_{i=1}^{n/2}(1-p^{2i}X^2),$$

and $F_p(T,X) \in \mathbb{Z}[X]$ has constant term 1 and $\deg(F_p(T,X)) = 2\,\mathrm{ord}_p(\mathfrak{f}_T)$. Using this polynomial $F_p(T,X)$ we define

$$\tilde{F}_p(T,X) := X^{-\mathrm{ord}_p(\mathfrak{f}_T)}F_p(T,p^{-\frac{n}{2}-\frac{1}{2}}X).$$

For each $T > 0$ in Λ_n, define

$$a(T) = c(|\Delta_T|)\mathfrak{f}_T^{\kappa-\frac{n+1}{2}}\prod_p \tilde{F}_p(T,\alpha_0(p;f)), \tag{1}$$

and form the following series

$$I_n(f)(z) = \sum_T a(T)e(\mathrm{Tr}(Tz))$$

where we recall $\alpha_0(p;f)$ is the p^{th} Satake parameter of f. We have the following theorem.

Theorem 4. *([16, Thm. 3.2, 3.3]) The series $I_n(f)(z)$, referred to as the Ikeda lift of f, is an eigenform in $S_\kappa(\Gamma_n)$ whose standard L-function factors as*

$$L(s, F; \mathrm{st}) = \zeta(s) \prod_{i=1}^{n} L(s + \kappa - i, f).$$

We will also need further information about the integrality of the Fourier coefficients of $I_n(f)$. In particular, the following result is essential to our applications.

Theorem 5. *[23, Thm. 1] Let $\theta^{\mathrm{alg}}_{\kappa,n}(f)$ be as above and let $a(T)$ be as in Equation 1. Then,*

$$a(T) = \sum_{d \mid \mathfrak{f}_T} d^{\kappa-1} \phi(d; T) c\left(|\Delta_T|(\mathfrak{f}_T/d)^2\right),$$

where $\phi(d; T)$ is an integer valued function.

Note, as an immediate consequence of this theorem and Proposition 3 we have the following corollary.

Corollary 6. *Let $f \in S_{2\kappa-n}(\Gamma_1; \mathcal{O})$ be a Hecke eigenform where $\mathcal{O}$ is the ring of integers of a field that can be embedded in $\mathbb{C}$. Then $I_n(f)$ has Fourier coefficients in $\mathcal{O}$.*

We will also make use of the following result of Katsurada.

Proposition 7. *[20, Prop. 4.6] Let $f \in S_{2\kappa-n}(\Gamma_1)$ be a normalized eigenform with Ikeda lift $I_n(f)$. Let $\mathcal{O}$ be the ring of integers of a field that can be embedded in $\mathbb{C}$ and let λ be a prime in $\mathcal{O}$. If there is a fundamental discriminant D so that the D^{th} Fourier coefficient of $\theta^{\mathrm{alg}}_{\kappa,n}(f)$ is not divisible by λ, then there is a Fourier coefficient of $I_n(f)$ that is not divisible by λ. In particular, if $\mathcal{O}$ is the ring of integers of some $K \subset \overline{\mathbb{Q}}_\ell$ with prime λ, then $I_n(f)$ has a Fourier coefficient that is a unit modulo λ.*

Proof. The only thing to prove is the last statement, but this follows immediately from our normalization of $\theta^{\mathrm{alg}}_{\kappa,n}$. $\square$

Let $f_1, \ldots, f_r$ be an orthogonal basis of $S_{2\kappa-n}(\Gamma_1)$ consisting of normalized eigenforms. We denote the span of $I_n(f_1), \ldots, I_n(f_r)$ in $S_\kappa(\Gamma_n)$ by $S^{\mathrm{Ik}}_\kappa(\Gamma_n)$. We denote the orthogonal complement of $S^{\mathrm{Ik}}_\kappa(\Gamma_n)$ in $S_\kappa(\Gamma_n)$ with respect to the Petersson product by $S^{\text{N-Ik}}_\kappa(\Gamma_n)$.

4 A conjecture on the Ikeda ideal

In this section we present a conjecture of Katsurada on the congruence primes of Ikeda lifts. We then introduce the Ikeda ideal and show how one can use the Ikeda ideal to study all the congruences between $I_n(f)$ and forms in $S^{\text{N-Ik}}_\kappa(\Gamma_n)$ at once. This allows us to prove a stronger congruence result under roughly the same conditions as given in [20].

4.1 A conjecture of Katsurada

Definition 8. Let $F \in S_\kappa(\Gamma_n; \mathcal{O})$ be an eigenform. Let $\lambda \subset \mathcal{O}$ be a prime of residue characteristic ℓ. We say λ is a congruence prime of F with respect to $V \subset (\mathbb{C}F)^\perp$ if there exists an eigenform $G \in V$ such that $F \equiv_{\mathrm{ev}} G \pmod{\lambda}$. (Note, in order for this congruence to make sense, we may need to extend $\mathcal{O}$ so that $G \in S_\kappa(\Gamma_n; \mathcal{O})$ as well.)

One should note the above definition can be extended to levels other than Γ_n, but we will have no need of such a definition in this paper.

Let $f \in S_{2\kappa-n}(\Gamma_1)$ be a normalized eigenform. Katsurada's conjecture states that all of the primes dividing certain special values of L-functions of f are congruence primes for the Ikeda lift $I_n(f)$ with respect to the space $S_\kappa^{\mathrm{Ik}}(\Gamma_n)^\perp$.

Conjecture 9. [20, Conj. A] Let $\kappa > n$ be integers and $f = f_1, f_2, \ldots, f_r \in S_{2\kappa-n}(\Gamma_1; \mathcal{O})$ be a basis of normalized eigenforms. Let $\lambda \subset \mathcal{O}$ be a prime not dividing $(2\kappa - 1)!$. Then λ is a congruence prime of $I_n(f)$ with respect to $S_\kappa^{\mathrm{Ik}}(\Gamma_n)^\perp$ if

$$\lambda | L_{\mathrm{alg}}(\kappa, f) \prod_{i=1}^{\frac{n}{2}-1} L_{\mathrm{alg}}(2i+1, \mathrm{ad}^0 f).$$

As evidence for this conjecture, Katsurada proves the following theorem.

Theorem 10. *[20, Thm. 4.7] Let $\mathcal{O}$, f, and λ be as in the conjecture with $\kappa > 2n + 4$. Then λ is a congruence prime for $I_n(f)$ with respect to $S_\kappa^{\mathrm{Ik}}(\Gamma_n)^\perp$ if the following are satisfied*

1. $$\lambda | L_{\mathrm{alg}}(\kappa, f) \prod_{i=1}^{\frac{n}{2}-1} L_{\mathrm{alg}}(2i+1, \mathrm{ad}^0 f);$$

2. *for some integer m satisfying $n/2 < m < \kappa/2 - n/2$ and some fundamental discriminant D satisfying $(-1)^{\frac{n}{2}} D > 0$,*

$$\lambda \nmid D\zeta_{\mathrm{alg}}(2m) L_{\mathrm{alg}}(\kappa - n/2, \chi_D) \prod_{i=1}^{n} L_{\mathrm{alg}}(2m + \kappa - i, f);$$

3. *for a constant $C_{\kappa,n} := \prod_{j \le \frac{2\kappa-n}{12}} (1 + j + \cdots + j^{n-1})$ if $n > 2$ and $C_{\kappa,2} = 1$,*

$$\lambda \nmid \frac{C_{\kappa,n} \langle f, f \rangle}{\Omega_f^+ \Omega_f^-}.$$

It is noted in [20] that the second condition allows freedom to choose m so it is reasonable to expect in many cases one can find an m with $\lambda \nmid \zeta_{\mathrm{alg}}(2m) \prod_{i=1}^{n} L_{\mathrm{alg}}(2m+\kappa - i, f)$.

4.2 The Ikeda ideal

The conjecture in the previous subsection gives conditions when one will have a congruence between an Ikeda lift $I_n(f)$ and a form in $S_\kappa^{\text{N-Ik}}(\Gamma_n)$. In this section we will introduce the Ikeda ideal associated to $I_n(f)$ that will capture this information as well. In fact, the ideal actually captures more information as it measures all congruences between $I_n(f)$ and forms in $S_\kappa^{\text{N-Ik}}(\Gamma_n)$.

Let f be a normalized eigenform in $S_{2\kappa-n}(\Gamma_1;\mathcal{O})$ and $I_n(f)$ the Ikeda lift of f for $\mathcal{O}$ the ring of integers in a suitably large finite extension of $\mathbb{Q}_\ell$. Recall the Hecke algebra over $\mathcal{O}$ acting on $S_\kappa(\Gamma_n)$ is denoted by $\mathcal{H}_\mathcal{O}^{(n)}$.

Let $X \subset S_\kappa^{\text{Ik}}(\Gamma_n)$ be a Hecke-stable subspace and let Y be the orthogonal complement in $S_\kappa(\Gamma_n)$ to X under the Petersson product. In particular, the examples we will be interested in are when $X = \mathbb{C}I_n(f)$ or $X = S_\kappa^{\text{Ik}}(\Gamma_n)$. Let $\mathcal{H}_\mathcal{O}^{(n),Y}$ denote the image of $\mathcal{H}_\mathcal{O}^{(n)}$ in $\text{End}_\mathbb{C}(Y)$ and let $\phi : \mathcal{H}_\mathcal{O}^{(n)} \to \mathcal{H}_\mathcal{O}^{(n),Y}$ denote the natural surjection.

We let $\text{Ann}(I_n(f))$ denote the annihilator of $I_n(f)$ in $\mathcal{H}_\mathcal{O}^{(n)}$. The semisimplicity of $\mathcal{H}_\mathcal{O}^{(n)}$ gives that there is an isomorphism

$$\mathcal{H}_\mathcal{O}^{(n)} / \text{Ann}(I_n(f)) \cong \mathcal{O}.$$

Using that ϕ is surjective we have that $\phi(\text{Ann}(I_n(f)))$ is an ideal in $\mathcal{H}_\mathcal{O}^{(n),Y}$. We refer to this ideal as the *Ikeda ideal associated to* $I_n(f)$ *with respect to* Y and denote it by $\mathcal{I}_n^Y(f)$. We will be interested in the index of this ideal. In particular, one has that there exists an integer m so that

$$\mathcal{H}_\mathcal{O}^{(n),Y} / \mathcal{I}_n^Y(f) \cong \mathcal{O}/\lambda^m\mathcal{O}.$$

We give here two elementary propositions to relate this index to Katsurada's conjecture.

Proposition 11. *With the notation as above, if there exists $G \in Y$, not necessarily an eigenform, so that*

$$I_n(f) \equiv G \pmod{\lambda^b},$$

then $m \geq b$.

Proof. Assume that $b > m$ and consider the following diagram

Note that each map in the diagram is an $\mathcal{O}$-algebra surjection. Let $t \in \phi^{-1}(\lambda^m) \subset \mathcal{H}_\mathcal{O}^{(n)}$. Then by definition we have $tG = \lambda^m G$. Moreover, by the commutativity of the diagram we see that $t \in \text{Ann}(I_n(f))$, so the assumed congruence gives

$$\lambda^m G \equiv 0 \pmod{\lambda^b},$$

i.e.,

$$G \equiv 0 \pmod{\lambda^{b-m}}.$$

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{O}}^{(n)} \arrow[r, "\phi"] \arrow[d] & \mathcal{H}_{\mathcal{O}}^{(n),Y} \arrow[d] \\
\mathcal{H}_{\mathcal{O}}^{(n)}/\operatorname{Ann}(I_n(f)) \arrow[r, "\phi"] \arrow[d, "\simeq"] & \mathcal{H}_{\mathcal{O}}^{(n),Y}/\mathcal{I}_n^Y(f) \arrow[d, "\simeq"] \\
\mathcal{O} \arrow[r] & \mathcal{O}/\lambda^m\mathcal{O}.
\end{tikzcd}$$

However, since we are assuming $b > m$, this gives

$$I_n(f) \equiv G \equiv 0 \pmod{\lambda}.$$

This contradicts Proposition 7 and so it must be that $b \leq m$. $\square$

Proposition 12. *With the notation as above, suppose $m \geq 1$. Then there exists an eigenform $G \in Y$ so that*

$$I_n(f) \equiv_{\mathrm{ev}} G \pmod{\lambda}.$$

Proof. Let $\mathcal{O}$ be a suitably large extension of $\mathbb{Z}_\ell$ so that $I_n(f) \in S_\kappa(\Gamma_n;\mathcal{O})$ and we have an orthogonal basis $F_1, \dots, F_r$ of Y defined over $\mathcal{O}$. Suppose there are no eigenforms $G \in Y$ eigenvalue congruent to $I_n(f)$.

Let $\mathcal{S}$ denote the $\mathbb{C}$-vector space spanned by $I_n(f), F_1, \dots, F_r$. Let $\mathcal{H}_{\mathcal{O}}^{(n),S}$ denote the image of the Hecke algebra $\mathcal{H}_{\mathcal{O}}^{(n)}$ in $\operatorname{End}_{\mathbb{C}}(\mathcal{S})$. For each eigenform $F \in \mathcal{S}$ defined over $\mathcal{O}$ we obtain a maximal ideal $\mathfrak{m}_F$ of $\mathcal{H}_{\mathcal{O}}^{(n),S}$ given as the kernel of the map $\mathcal{H}_{\mathcal{O}}^{(n),S} \to \mathcal{O}/\lambda : t \mapsto \lambda_F(t) \pmod{\lambda}$. We have that eigenforms F and G are eigenvalue congruent modulo λ if and only $\mathfrak{m}_F = \mathfrak{m}_G$.

We now use the fact that $I_n(f)$ is not congruent to any of $F_1, \dots, F_r$ to conclude that

$$\mathcal{H}_{\mathcal{O}}^{(n),S} = \mathcal{H}_{\mathfrak{m}_{I_n(f)}}^{(n),S} \times \prod_{\mathfrak{m}} \mathcal{H}_{\mathfrak{m}}^{(n),S}$$

where the product is over the maximal ideals corresponding to $F_1, \dots, F_r$. However, this gives that $\mathcal{I}_n^Y(f) = \prod_{\mathfrak{m}} \mathcal{H}_{\mathfrak{m}}^{(n),S}$, and this is exactly $\mathcal{H}_{\mathcal{O}}^{(n),Y}$. This contradicts the assumption that $m \geq 1$. Thus, it must be that there is a congruence as desired. $\square$

To match the previous results up with Katsurada's results, simply take $X = S_\kappa^{\mathrm{Ik}}(\Gamma_n)$ and $Y = S_\kappa^{\text{N-Ik}}(\Gamma_n)$. In fact, one has that the index of the Ikeda ideal measures all congruences between forms in Y and $I_n(f)$. This follows from the following result, rephrased for our situation. One should note that we use the fact that the space of Ikeda lifts satisfies multiplicity one ([18, Theorem 7.1]) in order to apply this result.

Proposition 13. *[5, Prop. 4.3] Let X and Y be as above and let $F_1, \dots, F_r$ be a basis of Y. For each $1 \le i \le r$, let m_i be the largest integer so that*

$$I_n(f) \equiv_{\mathrm{ev}} F_i \pmod{\lambda^{m_i}}.$$

Then one has

$$\frac{1}{e}\sum_{i=1}^{r} m_i \ge \mathrm{val}_\ell(\#\mathcal{H}_{\mathcal{O}}^{(n),Y}/\mathcal{I}_n^Y(f))$$

where e is the ramification index of $\mathcal{O}$ over $\mathbb{Z}_\ell$.

Thus, one can view results giving a lower bound on the Ikeda ideal as a strengthening of the results of [20] where one is only concerned with a congruence modulo λ to a single eigenform.

5 Main results

We now state the main result of this paper. The proof will be given in Section 7. After stating the theorem, we discuss the main hypotheses.

Theorem 14. *Let κ and n be positive even integers with $\kappa > n+1$ and let ℓ be a prime so that $\ell > 2\kappa - n$. Assume $\ell \nmid \prod_{p \le (2\kappa-n)/12}(1 + p + \cdots + p^{n-1})$. Let $f \in S_{2\kappa-n}(\Gamma_1)$ be a newform and let $\mathcal{O}$ be a suitably large finite extension of $\mathbb{Z}_\ell$ that contains all the eigenvalues of f with λ a uniformizer of $\mathcal{O}$. Furthermore, assume that $\overline{\rho}_{f,\lambda}$ is irreducible and $\mathrm{val}_\lambda\left(\frac{\langle f,f\rangle}{\Omega_f^+\Omega_f^-}\right) = 0$. We make the following assumptions.*

1. *There exists an integer $N > 1$ prime to ℓ and a Dirichlet character χ of conductor N with $\chi(-1) = (-1)^\kappa$ so that*

$$\mathrm{val}_\lambda\left(L^N(n-\kappa+1,\chi)\prod_{j=1}^{n} L^N_{\mathrm{alg}}(n+1-j,f,\chi)\right) = 0.$$

2. *There exists a fundamental discriminant D prime to ℓ so that $(-1)^{n/2}D > 0$, $\chi_D(-1) = -1$, and*

$$\mathrm{val}_\lambda(L_{\mathrm{alg}}(\kappa - n/2, f, \chi_D)) = 0.$$

3. *We have*

$$\mathrm{val}_\lambda\left(L_{\mathrm{alg}}(\kappa,f)\prod_{j=1}^{n/2-1} L_{\mathrm{alg}}(2j+1,\mathrm{ad}^0 f)\right) = b > 0.$$

If $F_1, \dots, F_r$ is a basis of eigenforms of $S_\kappa^{\text{N-Ik}}(\Gamma_n)$ defined over $\mathcal{O}$ and we let m_i be the largest integer so that

$$I_n(f) \equiv_{\text{ev}} F_i \pmod{\lambda^{m_i}},$$

then we have

$$\frac{1}{e}\sum_{i=1}^{r} m_i \geq b$$

where e is the ramification index of $\mathcal{O}$ over $\mathbb{Z}_\ell$.

The first hypothesis to make mention of is the condition that $\operatorname{val}_\lambda\left(\frac{\langle f,f\rangle}{\Omega_f^+\Omega_f^-}\right) = 0$. This condition is equivalent to assuming there are no other normalized eigenforms $g \in S_{2\kappa-n}(\Gamma_1;\mathcal{O})$ that are eigenvalue equivalent to f modulo λ. One can see [14, 27] for further discussion. For a particular f this condition can be easily checked using [7] or [28].

The two hypotheses we focus on are the ones concerning the λ-indivisibility of L-values. We begin with the assumption that $\operatorname{val}_\lambda(L_{\text{alg}}(\kappa - n/2, f, \chi_D)) = 0$. Note this is a central critical value since the weight of f is $2\kappa - n$. There have been several results on the λ-divisibility of this particular special value due to its relation with the Fourier coefficients of the half-integral weight modular form $\theta_{\kappa,n}^{\text{alg}}(f)$. For example, it is shown in [10, Corollary 3] that for non-exceptional primes ℓ there is a period Ω of f with the property that for infinitely many fundamental discriminants D prime to ℓ with $(-1)^{n/2}D > 0$ one has

$$\operatorname{ord}_\lambda\left(\frac{D^{\kappa-n/2-1/2}L_{\text{alg}}(\kappa-n/2, f, \chi_D)}{\Omega}\right) = 0.$$

Since we assume $\overline{\rho}_{f,\lambda}$ is irreducible, ℓ is automatically a non-exceptional prime for f (see [38, Corollary 2] for example.) However, we are unable to apply this result in our situation as the period Ω used is not the canonical period Ω_f^+ that we are using to normalize the L-value. We are unaware of any known period relation between Ω and Ω_f^+. However, this does reduce this consideration to another period ratio, and since we have already assumed above λ does not divide a period ratio, this assumption is a reasonable one as well.

We next consider $L(n-\kappa+1,\chi)$. It is well known that $L(n-\kappa+1,\chi) = -\frac{B_{\kappa-n,\chi}}{\kappa-n}$, which means that the λ-adic valuation of $L(n-\kappa+1,\chi)$ is given by the λ-adic valuation of $B_{\kappa-n,\chi}$ and so can be related to class numbers. For instance, let p be a prime with $p \neq \ell$, $m \geq 1$ and φ be a Dirichlet character. In this setting Washington proves ([40]) that for all but finitely many Dirichlet characters ψ of p-power conductor with $\varphi\psi(-1) = (-1)^m$ one has

$$\operatorname{ord}_\lambda(L(1-m,\varphi\psi)/2) = 0.$$

In our set-up we can take $m = \kappa - n$, $\chi = \varphi\psi$, and observe that $\chi(-1) = (-1)^\kappa = (-1)^{\kappa-n}$ to see there are infinitely many χ so that

$$\operatorname{ord}_\lambda(L(n-\kappa+1,\chi)) = 0.$$

If this were the only L-value controlled by χ we would be able to remove the hypothesis regarding this L-value. However, we also require that $\mathrm{val}_\lambda\left(\prod_{j=1}^n L_{\mathrm{alg}}^N(n+1-j,f,\chi)\right) = 0$. This means we must choose a χ so that all of these L-values are simultaneously λ-adic units. This is a much more delicate issue. We note here that we have a great deal of freedom in choosing χ, namely the only restrictions concern the parity of χ and that its conductor be prime to ℓ. Thus, we have infinitely many characters to choose from so it is reasonable to expect that one can often find such a χ. In the case $n=2$, i.e., when one considers Saito-Kurokawa lifts, one can find computational evidence supporting the existence of such a χ in [2]. One can use the same methods outlined there to produce computational evidence for $n>2$ as well.

6 Siegel Eisenstein Series

In this section we recall the definition of a Siegel Eisenstein series associated to a character. Following Shimura we then make a suitable choice of a section so that the Fourier coefficients of the Eisenstein series can be computed. Finally, we consider the pullback of our Siegel Eisenstein series and recall an inner product formula of Shimura. Throughout this section we assume κ and n are even integers with $\kappa > n+1$.

6.1 Siegel Eisenstein series - general set up

Let P_n be the Siegel parabolic subgroup of G_n given by $P_n = \{g \in G_n : c_g = 0\}$. We have that P_n factors as $P_n = N_{P_n} M_{P_n}$ where N_{P_n} is the unipotent radical

$$N_{P_n} = \left\{ n(x) = \begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix} : {}^t x = x, x \in \mathrm{Mat}_n \right\}$$

and M_{P_n} is the Levi subgroup

$$M_{P_n} = \left\{ \begin{pmatrix} A & 0_n \\ 0_n & \alpha({}^t A)^{-1} \end{pmatrix} : A \in \mathrm{GL}_n, \alpha \in \mathrm{GL}_1 \right\}.$$

Let $\mathbb{A}$ denote the rational adeles. Fix an idele class character χ and consider the induced representation

$$I(\chi) = \mathrm{Ind}_{P_n(\mathbb{A})}^{G_n(\mathbb{A})}(\chi) = \bigotimes_v I_v(\chi_v)$$

consisting of smooth functions $\mathfrak{f}$ on $G_n(\mathbb{A})$ that satisfy

$$\mathfrak{f}(pg) = \chi(\det(A_p))\mathfrak{f}(g)$$

for $p = \begin{pmatrix} A_p & B_p \\ 0 & D_p \end{pmatrix} \in P_n(\mathbb{A})$ and $g \in G_n(\mathbb{A})$. For $s \in \mathbb{C}$ and $\mathfrak{f} \in I(\chi)$ define

$$\mathfrak{f}(pg, s) = \chi(\det(A_p))|\det(A_p D_p^{-1})|^s \mathfrak{f}(g)$$

For v a place of $\mathbb{Q}$ we define $I_v(\chi_v)$ and $\mathfrak{f}_v(pg,s)$ analogously. We associate to such a section the Siegel Eisenstein series

$$E_{\mathbb{A}}(g,s;\mathfrak{f}) = \sum_{\gamma \in P_n(\mathbb{Q})\backslash G_n(\mathbb{Q})} \mathfrak{f}(\gamma g, s).$$

Observe that $E_{\mathbb{A}}(g,s;\mathfrak{f})$ converges absolutely and uniformly for (g,s) on compact subsets of $G_n(\mathbb{A}) \times \{s \in \mathbb{C} : \mathrm{Re}(s) > (n+1)/2\}$. Moreover, it defines an automorphic form on $G_n(\mathbb{A})$ and a holomorphic function on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$ with meromorphic continuation to $\mathbb{C}$ with at most finitely many poles. Furthermore, [24] gives a functional equation for $E_{\mathbb{A}}(g,s;\mathfrak{f})$ relating the value at $(n+1)/2 - s$ to the value at s.

6.2 A choice of section

For our applications we need to restrict the possible χ and pick a particular section $\mathfrak{f}$. Let $N > 1$ be an integer.

Let $\chi = \otimes_v \chi_v$ be an idele class character that satisfies

$$\chi_\infty(x) = \left(\frac{x}{|x|}\right)^\kappa$$
$$\chi_p(x) = 1 \text{ if } p \nmid \infty, x \in \mathbb{Z}_p^\times, \text{ and } x \equiv 1 \pmod{N}.$$

For each finite prime p, we set

$$K_{0,p}^{(n)}(N) = \{g \in G_n(\mathbb{Q}_p) : a_g, b_g, d_g \in \mathrm{Mat}_n(\mathbb{Z}_p), c_g \in \mathrm{Mat}_n(N\mathbb{Z}_p)\}.$$

From this definition it is immediate that if $p \nmid N$ we have

$$K_{0,p}^{(n)}(N) = G_n(\mathbb{Q}_p) \cap \mathrm{Mat}_{2n}(\mathbb{Z}_p).$$

At the infinite place we put

$$K_\infty^{(n)} = \{g \in \mathrm{Sp}_{2n}(\mathbb{R}) : g(i_n) = i_n\}.$$

Set

$$K_0^{(n)}(N) = \prod_p K_{0,p}^{(n)}(N).$$

We choose our section $\mathfrak{f} = \otimes_v \mathfrak{f}_v$ as follows:

1. We set $\mathfrak{f}_\infty$ to be the unique vector in $I_\infty(\chi_\infty, s)$ so that

   $$\mathfrak{f}_\infty(k,\kappa) = j(k,i)^{-\kappa}$$

   for all $k \in K_\infty^{(n)}$.

2. If $p \nmid N$ we set $\mathfrak{f}_p$ to be the unique $K_{0,p}^{(n)}(N)$-fixed vector so that

   $$\mathfrak{f}_p(1) = 1.$$

3. If $p \mid N$ we set $\mathfrak{f}_p$ to be the vector given by

$$\mathfrak{f}_p(k) = \chi_p(\det(a_k))$$

for all $k \in K_{0,p}^{(n)}(N)$ with $k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ and

$$\mathfrak{f}_p(g) = 0$$

for all $g \notin P_n(\mathbb{Q}_p)K_{0,p}^{(n)}(N)$.

This Eisenstein series is the Eisenstein series studied by Shimura in [31] and [32].

Define

$$\Lambda_n^N(s,\chi) = L^N(2s,\chi) \prod_{i=1}^{[n/2]} L^N(4s-2i,\chi^2)$$

and normalize $E_{\mathbb{A}}$ by setting

$$\mathbf{E}_{\mathbb{A}}(g,s;\mathfrak{f}) = \pi^{-n(n+2)/4}\Lambda_n^N(s,\chi)E_{\mathbb{A}}(gJ_n^{-1},s;\mathfrak{f}).$$

Set

$$G_\kappa^n(z;\mathfrak{f}) = \mathbf{E}_{\mathbb{A}}\left(\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}, \frac{n+1}{2} - \frac{\kappa}{2}; \mathfrak{f}\right) \tag{2}$$

We have $G_\kappa^n(z;\mathfrak{f})$ is a Siegel modular form of weight κ and level $\Gamma_0^{(n)}(N)$ ([30]) where

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n : C \equiv 0 \pmod{N} \right\}.$$

Write

$$G_\kappa^n(z;\mathfrak{f}) = \sum_{T\in\Lambda_n} a(T;\mathfrak{f})e(\mathrm{Tr}(Tz)).$$

The Fourier coefficients $a(T;\mathfrak{f})$ are well-known for this particular choice of section and normalization ([32, Chapters 18-19]).

Theorem 15. *Let $\ell \geq n+1$ be an odd prime with $\ell \nmid N$. Then*

$$G_\kappa^n(z;\mathfrak{f}) \in M_\kappa(\Gamma_0^{(n)}(N); \mathbb{Z}_\ell[\chi, i^{n\kappa}]).$$

Proof. One can see [1] or [8] for this fact. □

6.3 Pullbacks of Siegel Eisenstein series

Let $N > 1$ be an integer and $\ell > n+1$ a prime with $\ell \nmid N$.

Consider the diagonal embedding of $\mathfrak{h}^n \times \mathfrak{h}^n$ to $\mathfrak{h}^{2n}$ via the map

$$\mathfrak{h}^n \times \mathfrak{h}^n \hookrightarrow \mathfrak{h}^{2n}$$
$$(z,w) \mapsto \mathrm{diag}[z,w] = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}.$$

We also have an embedding of $\Gamma_n \times \Gamma_n$ into Γ_{2n} given by

$$\Gamma_n \times \Gamma_n \hookrightarrow \Gamma_{2n}$$
$$\left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \right) \mapsto \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

This allows us to view the natural action of $\Gamma_n \times \Gamma_n$ on $\mathfrak{h}^n \times \mathfrak{h}^n$ as a restriction of the action of Γ_{2n} on $\mathfrak{h}^{2n}$.

We will be interested in the restriction of the Eisenstein series $G_\kappa^{2n}(Z;\mathfrak{f})$ to $\mathfrak{h}^n \times \mathfrak{h}^n$. We refer to such a restriction as a pullback. These pullbacks have been considered in [6, 12, 13, 31, 32]. In general, if F is a modular form of degree $2n$, level $\Gamma_0^{(2n)}(N)$, and weight κ, then the restriction of F to $\mathfrak{h}^n \times \mathfrak{h}^n$ is a modular form of degree n, level $\Gamma_0^{(n)}(N)$, and weight κ when considered as a function of z or w.

Shimura calculates the following set of representatives for $P_{2n}\backslash G_{2n}/(G_n \times G_n)$.

Lemma 16. *[31, Lemma 4.2] For $0 \leq r \leq n$ let τ_r denote the element of G_{2n} given by*

$$\tau_r = \begin{pmatrix} 1_{2n} & 0 \\ \rho_r & 1_{2n} \end{pmatrix}, \quad \rho_r = \begin{pmatrix} 0_n & e_r \\ {}^t e_r & 0_n \end{pmatrix}, \quad e_r = \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the τ_r form a complete set of representatives for $P_{2n}\backslash G_{2n}/(G_n \times G_n)$.

We will make use of τ_n. Let $F \in S_\kappa(\Gamma_n)$ be an eigenform. We can specialize [31, Eqn. 6.17] to obtain

$$\langle (G_\kappa^{2n}|\tau_n)(\mathrm{diag}[z,w];\mathfrak{f}), F^c(w)\rangle = \mathcal{A}_{\kappa,n,N}\pi^{-\frac{n(n+1)}{2}} L(n+1-\kappa, F, \chi; \mathrm{st})F(z) \quad (3)$$

where we have used $F|J_n = F$ since F has level Γ_n and

$$\mathcal{A}_{\kappa,n,N} = \frac{2^{n(2\kappa-3n+2)/2}}{[\Gamma_n : \Gamma_0^{(n)}(N)]} \prod_{j=0}^{n-1} \frac{\Gamma((n-j)/2)}{\Gamma((2n+1-j)/2)}.$$

As it will be important in the next section, note that since $G_\kappa^{2n}(z;\mathfrak{f}) \in M_\kappa(\Gamma_0^{(2n)}(N); \mathbb{Z}_\ell[\chi])$, we have $(G_\kappa^{2n}|\tau_n)(z;\mathfrak{f}) \in M_\kappa(\tau_n^{-1}\Gamma_0^{(2n)}(N)\tau_n; \mathbb{Z}_\ell[\chi])$ by the q-expansion principle for Siegel modular forms ([11, Prop. 1.5]). The Fourier expansion of $(G_\kappa^{2n}|\tau_n)(\mathrm{diag}[z,w];\mathfrak{f})$ can be written as

$$(G_\kappa^{2n}|\tau_n)(\mathrm{diag}[z,w];\mathfrak{f}) = \sum_{T_1,T_2\in\Lambda_n} \left(\sum_{T\in\Lambda_{2n}(T_1,T_2)} a\left(T; G_\kappa^{2n}|\tau_n\right)\right) e(\mathrm{Tr}(T_1 z))e(\mathrm{Tr}(T_2 w))$$

where $a(T; G_\kappa^{2n}|\tau_n)$ is the Tth Fourier coefficient of $G_\kappa^{2n}|\tau_n$ and for $T_1, T_2 \in \Lambda_n$ we define

$$\Lambda_{2n}(T_1,T_2) = \left\{ T \in \Lambda_{2n} : T = \begin{pmatrix} T_1 & b \\ b & T_2 \end{pmatrix} \right\}.$$

This immediately gives that the Fourier coefficients of $(G_\kappa^{2n}|\tau_n)(\mathrm{diag}[z,w];\mathfrak{f})$ lie in $\mathbb{Z}_\ell[\chi]$ as well.

7 Constructing a congruence

In this section we prove Theorem 14. We work under the hypotheses listed in the theorem.

Our first step in constructing the congruence is to replace the Eisenstein series $(G_\kappa^{2n}|\tau_n)(\mathrm{diag}[z,w];\mathfrak{f})$ with a form of level $\Gamma_n \times \Gamma_n$. To do this, we take the trace:

$$\widetilde{G}_\kappa^{2n}(\mathrm{diag}[z,w];\mathfrak{f}) = \sum_{\gamma_1,\gamma_2} (G_\kappa^{2n}|\tau_n)(\mathrm{diag}[z,w];\mathfrak{f})|(\gamma_1 \times \gamma_2)$$

where the sum is over $(\Gamma_n \times \Gamma_n)/(\tau_n^{-1}\Gamma_0^{(n)}(N)\tau_n \times \tau_n^{-1}\Gamma_0^{(n)}(N)\tau_n)$. We note again that this has Fourier coefficients in $\mathbb{Z}_\ell[\chi]$ by the q-expansion principle. Moreover, we know that $\widetilde{G}_\kappa^{2n}$ is a cusp form in each variable via Section 3.2 of [9].

Let $F_0 = I_n(f), F_1, \ldots, F_r$ be an orthogonal basis of eigenforms for $S_\kappa(\Gamma_n)$. Note that $F_0^c, \ldots, F_r^c$ is also an orthogonal basis of eigenforms for $S_\kappa(\Gamma_n)$. Applying [31, Eqn. 7.7] we may write

$$\widetilde{G}_\kappa^{2n}(\mathrm{diag}[z,w];\mathfrak{f}) = \sum_{\substack{0\leq i\leq r\\ 0\leq j\leq r}} c_{i,j} F_i(z) F_j^c(w)$$

for some $c_{i,j} \in \mathbb{C}$. Furthermore, from [9, Lem. 5.1] we have that we can rewrite this as

$$\widetilde{G}_\kappa^{2n}(\mathrm{diag}[z,w];\mathfrak{f}) = c_0 I_n(f)(z) I_n(f)(w) + \sum_{0<j\leq r} c_j F_j(z) F_j^c(w) \tag{4}$$

where we write $c_j = c_{j,j}$ and we have used that since $f^c = f$, Corollary 6 gives $I_n(f)^c = I_n(f)$.

We now turn our attention to the constant c_0. Our goal is to show that we can write c_0 as a product of an element of $\mathcal{O}^\times$ and λ^{-m} for some $m > 0$.

Consider the inner product $\langle \widetilde{G}_\kappa^{2n}(\mathrm{diag}[z,w];\mathfrak{f}), I_n(f)(w)\rangle$. Note,

$$\langle \widetilde{G}_\kappa^{2n}(\mathrm{diag}[z,w];\mathfrak{f}), I_n(f)(w)\rangle = \langle (G_\kappa^{2n}|\tau_n)(\mathrm{diag}[z,w];\mathfrak{f}), I_n(f)(w)\rangle,$$

where we view the forms on the left hand side as being level Γ_n and on the right hand side as being level $\tau_n^{-1}\Gamma_0^{(n)}(N)\tau_n$. Taking the inner product of both sides of Equation 4 with $I_n(f)(w)$, applying Equation 3, and solving for c_0 we obtain

$$c_0 = \frac{\mathcal{A}_{k,n,N} L^N(n-\kappa+1, I_n(f), \chi; \mathrm{st})}{\pi^{\frac{n(n+1)}{2}} \langle I_n(f), I_n(f)\rangle}.$$

In [17], Ikeda made a conjecture relating $\langle I_n(f), I_n(f)\rangle$ to $\langle f, f\rangle$. We have the following theorem of Katsurada and Kawamura which proves Ikeda's conjecture assuming n is even. We rephrase their result to suit our purposes.

Theorem 17. *[21, Theorem 2.3] Let κ be a positive even integer and let $\ell > n+1$ be a prime. Let $f \in S_{2\kappa-n}(\Gamma_1;\mathcal{O})$ be a newform with $\mathcal{O}$ a suitably large finite extension of $\mathbb{Z}_\ell$. Assume $\mathrm{val}_\lambda\left(\frac{\langle f,f\rangle}{\Omega_f^+\Omega_f^-}\right) = 0$. Let D be a fundamental discriminant such that $(-1)^{\frac{n}{2}}D > 0$, $\chi_D(-1) = -1$ and assume $\ell \nmid D$. Then if $I_n(f)$ is the Ikeda lift of f as given above, we have*

$$\begin{aligned}\frac{\langle I_n(f), I_n(f)\rangle}{\langle f,f\rangle^{\frac{n}{2}}} &= u_1 \cdot \frac{\Gamma(\kappa)\prod_{j=1}^{\frac{n}{2}-1}\Gamma(2j+2\kappa-n)|c(|D|)|^2\prod_{j=1}^{n/2}\zeta_{\mathrm{alg}}(2j)}{\Gamma(\kappa-\frac{n}{2})} \\ &\times \quad \frac{L_{\mathrm{alg}}(\kappa,f)\prod_{j=1}^{n/2-1}L_{\mathrm{alg}}(2j+1,\mathrm{ad}^0 f)}{L_{\mathrm{alg}}(\kappa-\frac{n}{2},f,\chi_D)},\end{aligned}$$

where $\mathrm{val}_\lambda(u_1) = 0$, $c(|D|)$ is the $|D|^{th}$ Fourier coefficient of $\theta^{\mathrm{alg}}_{\kappa,n}(f)$ from above and we have used the assumption on $\frac{\langle f,f\rangle}{\Omega_f^+\Omega_f^-}$ to normalize the adjoint L-function to our conventions.

We now apply this result to remove the period $\langle I_n(f), I_n(f)\rangle$ in our expression for c_0 to obtain

$$c_0 = \frac{\mathcal{B}_{\kappa,n}}{|c(|D|)|^2}\cdot\frac{L^N(n-\kappa+1, I_n(f),\chi;\mathrm{st})L_{\mathrm{alg}}(\kappa-n/2,f,\chi_D)}{\pi^{\frac{n(n+1)}{2}}\langle f,f\rangle^{\frac{n}{2}}\zeta_{\mathrm{alg}}(n)\prod_{i=1}^{n/2-1}\zeta_{\mathrm{alg}}(2i)L_{\mathrm{alg}}(2i+1,\mathrm{ad}^0 f)L_{\mathrm{alg}}(\kappa,f)},$$

where

$$\mathcal{B}_{\kappa,n} = u_2\cdot\frac{\Gamma(\kappa-n/2)\prod_{j=1}^{n-1}\Gamma((n-j)/2)}{[\Gamma_n:\Gamma_0^{(n)}(N)]\Gamma(k)\prod_{j=1}^{n-1}\Gamma((2n+1-j)/2)\prod_{j=1}^{n/2-1}\Gamma(2i+2k-n)}$$

where u_2 satisfies $\mathrm{val}_\lambda(u_2) = 0$.

The following factorization is a direct consequence of Theorem 4:

$$L^N(n-k+1, I_n(f),\chi;\mathrm{st}) = L^N(n-k+1,\chi)\prod_{i=1}^{n}L^N(n+1-i,f,\chi). \tag{5}$$

Applying the assumption that $\mathrm{val}_\lambda\left(\frac{\langle f,f\rangle}{\Omega_f^+\Omega_f^-}\right) = 0$, we can replace $\langle f,f\rangle^{n/2}$ by $u_3(\Omega_f^+\Omega_f^-)^{n/2}$ for u_3 a λ-unit. Furthermore, note that if Ω_f^+ is the period associated to $L(n+1-i,f,\chi)$ as in Theorem 1, then Ω_f^- is the period associated to $L(n+1-(i+1),f,\chi)$, and vice versa. Using this we can rewrite our expression for c_0 as

$$c_0 = u_4\cdot\mathcal{B}_{\kappa,n}\cdot\mathcal{C}_{D,n,\chi}\cdot\mathcal{L}_{f,\chi,D}$$

where u_4 is a λ-unit, $\mathcal{B}_{\kappa,n}$ is defined as above,

$$\mathcal{C}_{D,n,\chi} = \frac{1}{|c_h(|D|)|^2\prod_{i=1}^{n/2}\zeta_{\mathrm{alg}}(2i)},$$

and

$$\mathcal{L}_{f,\chi,\chi_D} = \frac{L^N(n-\kappa+1,\chi)L_{\mathrm{alg}}(\kappa-\frac{n}{2},f,\chi_D)\prod_{j=1}^{n} L_{\mathrm{alg}}^N(n+1-j,f,\chi)}{L_{\mathrm{alg}}(\kappa,f)\prod_{j=1}^{n/2-1} L_{\mathrm{alg}}(2j+2,\mathrm{ad}^0 f)}.$$

Note, it was shown in Section 4.2 of [8] that $L^N(n-k+1,\chi) \in \mathbb{Z}_\ell[\chi]$. As $\mathcal{B}_{\kappa,n}, \mathcal{C}_{D,n,\chi}$, and $\mathcal{L}_{f,\chi,D}$ are algebraic, we may consider the λ divisibility of c_0. First, using that n is even and $\ell > n+1$ we have $\mathrm{val}_\lambda(\mathcal{B}_{k,n}) \leq 0$.

Next we turn our attention to $\mathcal{C}_{D,n,\chi}$. Our choice of $\theta_{\kappa,n}^{\mathrm{alg}}(f)$ given in Section 3, gives that $|c(|D|)| \in \mathcal{O}$, and so $\mathrm{val}_\lambda(|c(|D|)|^2) \geq 0$. Consider $\zeta_{\mathrm{alg}}(2j)$ for some $1 \leq j \leq n/2$. By a well-known identity we have $\zeta_{\mathrm{alg}}(2j) = (-1)^{j+1}\frac{B_{2j}2^{2j-1}}{(2j)!}$, where B_m is the m^{th} Bernoulli number. Then, it is an immediate consequence of the Von Staudt-Clausen Theorem, see for example [19, page 233], that $\zeta_{\mathrm{alg}}(2j)$ is λ-integral, and hence $\mathrm{val}_\lambda(\zeta_{\mathrm{alg}}(2j)) \geq 0$. Thus, we have $\mathrm{val}_\lambda(\mathcal{C}_{D,n,\chi}) \leq 0$.

By assumption we have $\mathrm{val}_\lambda(\mathcal{L}_{f,\chi,\chi_D}) < 0$, so under our assumptions we have $\mathrm{val}_\lambda(c_0) < 0$. We now show how this gives the desired congruence. Write $c_0 = \alpha\lambda^{-b}$ for some $b > 0$ and λ-unit α. Using this, we may rewrite Equation 4 as

$$\widetilde{G}_\kappa^{2n}(\mathrm{diag}[z,w];\mathfrak{f}) = \alpha\lambda^{-b}I_n(f)(z)I_n(f)(w) + \sum_{0<j\leq r} c_j F_j(z)F_j^c(w). \tag{6}$$

Note, by Proposition 7 there is a T_0 so that $\mathrm{val}_\lambda(a_{I_n(f)}(T_0)) = 0$. We expand Equation 6 in terms of z and equate the T_0^{th} Fourier coefficients to obtain

$$\sum_{T_2\in\Lambda_n}\left(\sum_{T\in\Lambda_{2n}(T_0,T_2)} a(T,G_\kappa^{2n}|\tau_n)\right)e(\mathrm{Tr}(T_2w)) = \alpha\lambda^{-b}a_{I_n(f)}(T_0)I_n(f)(w) + \sum_{0<j\leq r} c_j a_{F_j}(T_0)F_j^c(w).$$

We now multiply the equation through by λ^m and recall that $a(T,G_\kappa^{2n}|\tau_n) \in \mathcal{O}$ for all T to observe that

$$I_n(f)(w) \equiv -\frac{\lambda^b}{\alpha a_{I_n(f)}(T_0)}\sum_{0<j\leq r} c_j a_{F_j}(T_0)F_j^c(w) \pmod{\lambda^b}.$$

Note that since $a_{I_n(f)}(T_0)$ is a λ-unit, we obtain the form on the right hand side of the congruence cannot be trivial modulo λ^b, i.e., we have constructed a nontrivial congruence. Set

$$G(w) = -\frac{\lambda^b}{\alpha a_{I_n(f)}(T_0)}\sum_{0<j\leq r} c_j a_{I_n(f)}(T_0)F_j(w).$$

Our next claim is that $I_n(f)$ cannot be eigenvalue congruent modulo λ to $I_n(g)$ for any $g \in S_{2\kappa-n}(\Gamma_1;\mathcal{O})$. This is given in the proof of Theorem 4.7 in [20], but we reproduce his proof here for the convenience of the reader. First note that for any prime p, Katsurada constructs a Hecke operator $t(p) \in \mathcal{H}_{\mathcal{O}}^{(n)}$ so that

$$\lambda_{I_n(f)}(t(p)) = p^{(n-1)\kappa - n(n+1)/2}\lambda_f(p)\sum_{j=1}^{n} p^j.$$

Assume that there is such a g so that $I_n(f) \equiv_{\text{ev}} I_n(g) \pmod{\lambda}$. In particular, using the Hecke operator $t(p)$ for any prime p with $\ell \nmid p$ we have

$$(1 + p + \cdots + p^{n-1})\lambda_f(p) \equiv (1 + p + \cdots + p^{n-1})\lambda_g(p) \pmod{\lambda}.$$

By assumption we have $\ell \nmid (1 + p + \cdots + p^{n-1})$ for all $p \leq (2\kappa - n)/12$, so for such p we have

$$\lambda_f(p) \equiv \lambda_g(p) \pmod{\lambda}.$$

However, we can now apply the results in [37] to conclude that λ is a congruence prime of f. However, by our assumption that $\mathrm{val}_\lambda\left(\frac{\langle f,f\rangle}{\Omega_f^+\Omega_f^-}\right) = 0$ this cannot happen. Thus, we cannot have an eigenvalue congruence between $I_n(f)$ and another Ikeda lift.

We now return to the setting of Ikeda ideals. Let $X = \mathbb{C}I_n(f)$ and $Y = (\mathbb{C}I_n(f))^\perp$ where the notation follows that given in Section 4.2. Let $F_0 = I_n(f), F_1, \ldots, F_{r_1}$ be a basis of $S_\kappa^{\text{Ik}}(\Gamma_n)$ and $F_{r_1+1}, \cdots, F_r$ be a basis of $S_\kappa^{\text{N-Ik}}(\Gamma_n)$ defined over $\mathcal{O}$. We have constructed a congruence

$$I_n(f) \equiv G \pmod{\lambda^b}$$

for some $b \geq 1$ and $G \in Y$. We now apply Proposition 13 to conclude

$$\frac{1}{e}\sum_{i=1}^{r} m_i \geq b.$$

However, we know that $m_1 = \cdots = m_{r_1} = 0$, so in fact we obtain

$$\frac{1}{e}\sum_{i=r_1+1}^{r} m_i \geq b.$$

In short, though we do not obtain a lower bound on the Ikeda ideal associated to $I_n(f)$ with respect to $S_\kappa^{\text{N-Ik}}(\Gamma_n)$, we can still recover the same lower bound on the total number of congruences we desire, which concludes the proof of the main theorem.

References

[1] M. Agarwal. *p-adic L-functions for* GSp(4)×GL(2). PhD thesis, University of Michigan, Ann Arbor, MI, 2007.

[2] M. Agarwal and J. Brown. Computational evidence for the Bloch-Kato conjecture for elliptic modular forms of square-free level. `http://www.ces.clemson.edu/~jimlb/ResearchPapers/BlochKatoCompEvid.pdf`. Accessed July 17, 2013.

[3] M. Agarwal and J. Brown. Saito-Kurokawa lifts of square-free level and lower bounds on Selmer groups. *preprint*, pages 1–52, 2011.

[4] M. Asgari and R. Schmidt. Siegel modular forms and representations. *Manuscripta Math.*, 104:172–200, 2001.

[5] T. Berger, K. Klosin, and K. Kramer. On higher congruences between automorphic forms. *preprint*, pages 1–11, 2013.

[6] S. Böcherer. Über die Fourier-Jacobi-entwicklung Siegelscher Eisensteinreihen ii. *Math. Z.*, 189:81–110, 1985.

[7] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[8] J. Brown. Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture. *Comp. Math.*, 143(2):290–322, 2007.

[9] J. Brown. On the cuspidality of pullbacks of Siegel Eisenstein series and applications to the Bloch-Kato conjecture. *Int. Math. Res. Not.*, 7:1706–1756, 2011.

[10] J. Brunier and K. Ono. Coefficients of half-integral weight modular forms. *J. Number Theory*, 99:164–179, 2003.

[11] C. Chai and G. Faltings. *Degeneration of Abelian Varieties.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3.Folge Band 22, A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[12] P. Garrett. Pullbacks of Eisenstein series; applications. In *Automorphic forms of several variables (Katata, 1983)*, volume 46 of *Prog. in Math.*, pages 114–137, Boston, MA, 1984. Birkhauser.

[13] P. Garrett. On the arithmetic of Siegel-Hilbert cuspforms: Petersson inner products and Fourier coefficients. *Invent. Math.*, 107:453–481, 1992.

[14] H. Hida. Congruences of Cusp Forms and Special Values of Their Zeta Functions. *Invent. Math.*, 63:225–261, 1981.

[15] H. Hida. Theory of $p-$adic Hecke Algebras and Galois Representations. *Sugaku Expositions*, 2-3:75–102, 1989.

[16] T. Ikeda. On the lifting of elliptic modular forms to Siegel cusp forms of degree $2n$. *Ann. of Math.*, 154:641–681, 2001.

[17] T. Ikeda. Pullback of the lifting of elliptic cusp forms and Miyawaki's conjecture. *Duke Math. J.*, 131(3):469–497, 2006.

[18] T. Ikeda. On the lifting of automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ to $\mathrm{Sp}_{2n}(\mathbb{A})$ of $\widetilde{\mathrm{Sp}_{2n+1}}(\mathbb{A})$ over a totally real algebraic field. *preprint*, pages 1–58, 2013.

[19] K. Ireland and M. Rosen. *A classical introduction to modern number theory*, volume 84 of *Graduate Texts in Mathematics*. Springer, New York, 1990.

[20] H. Katsurada. Congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts. *preprint*.

[21] H. Katsurada and H. Kawamura. Ikeda's conjecture on the period of Duke-Imamoḡlu-Ikeda lift. *Submitted*.

[22] K. Klosin. Congruences among modular forms on $U(2,2)$ and the Bloch-Kato conjecture. *Ann. Inst. Fourier*, 59(1):81–166, 2009.

[23] W. Kohnen. Lifting modular forms of half-integral weight to Siegel modular forms of even genus. *Math. Ann.*, 322:787–809, 2002.

[24] R. Langlands. *On the functional equations satisfied by Eisenstein series*, volume 544 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1976.

[25] B. Mazur. How can we construct abelian Galois extensions of basic number fields? *Bull. Amer. Math. Soc.*, 48:155–209, 2011.

[26] K. Ribet. A modular construction of unramified p-extensions of $\mathbb{Q}(\mu_p)$. *Invent. Math.*, 34:151–162, 1976.

[27] K. Ribet. Mod p Hecke operators and congruences between modular forms. *Invent. Math.*, 71:193–205, 1983.

[28] SAGE Mathematics Software Version 2.6. http://www.sagemath.org/.

[29] G. Shimura. On the periods of modular forms. *Math. Ann.*, 229:211–221, 1977.

[30] G. Shimura. On Eisenstein Series. *Duke Math. J.*, 50:417–476, 1983.

[31] G. Shimura. Eisenstein series and zeta functions on symplectic groups. *Invent. Math.*, 119:539–584, 1995.

[32] G. Shimura. *Euler products and Eisenstein series*, volume 93 of *CBMS, Regional Conference Series in Mathematics*. AMS, Providence, 1997.

[33] T. Shintani. On construction of holomorphic cusp forms of half integral weight. *Nagoya Math. J.*, 58:83–126, 1975.

[34] C. Skinner and E. Urban. Sur les déformations p-adiques de certaines représentations automorphes. *J. Inst. Math. Jussieu*, 5:629–698, 2006.

[35] C. Skinner and E. Urban. The Iwasawa main conjectures for GL_2. *Invent. Math.*, 2013.

[36] G. Stevens. Λ-adic modular forms of half-integral weight and a Λ-adic Shintani lifting. *Contemp. Math.*, 174:129–151, 1994.

[37] Jacob Sturm. *On the congruence of modular forms*, volume 1240 of *Lecture Notes in Math.* Springer, Berlin, 1987.

[38] H.P.F. Swinnerton-Dyer. *On ℓ-adic representations and congruences for coefficients of modular forms*, volume 350 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1973.

[39] V. Vatsal. Canonical periods and congruence formulae. *Duke Math. J.*, 98(2):397–419, 1999.

[40] L. Washington. The non-p-part of the class number in a cyclotomic $\mathbb{Z}_p$-extension. *Invent. Math.*, 49:87–97, 1978.

[41] A. Wiles. The Iwasawa conjecture for totally real fields. *Ann. of Math.*, 131(3):493–540, 1990.

[42] D. Zagier. *Modular forms whose Fourier coefficients involve zeta-functions of qua- dratic fields*, volume 627 of *Lect. Notes in Math.* Springer, 1977.