Domination Cover Rubbling

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Abstract

Let G be a connected simple graph with vertex set V and a distribution of pebbles on V. The domination cover rubbling number of G is the minimum number of pebbles, so that no matter how they are distributed, it is possible that after a sequence of pebbling and rubbling moves, the set of vertices with pebbles is a dominating set of G. We begin by characterizing the graphs having small domination cover rubbling numbers and determining the domination cover rubbling number of several common graph families. We then give a bound for the domination cover rubbling number of trees and characterize the extremal trees. Finally, we give bounds for the domination cover rubbling number of graphs in terms of their domination number and characterize a family of the graphs attaining this bound.

Keywords: Graph pebbling; Graph rubbling; Domination cover pebbling; Domination cover rubbling

1 Introduction

Let G be a connected simple graph with vertex set V and a distribution of pebbles on the vertices of V. A placement of pebbles on the vertices such that each vertex of V is assigned a non-negative integer number of pebbles is called a *pebble distribution* on G. Two moves, namely a *pebbling move* and a *rubbling move*, are defined as follows. Let f be a pebble distribution on a graph G such that $f(u) \geq 2$ for some vertex $u \in V$, and let v be adjacent to u. Then a pebbling move, denoted $p(u \to v)$, removes two pebbles from u and places one on v. This defines a new pebble distribution, f' such that: $f'(u) = f(u) - 2$, $f'(v) = f(v) + 1$, and $f'(z) = f(z)$ for $z \in V - \{u, v\}$. Let w be a vertex of G, and let v and x be distinct vertices adjacent to w. Let f be a pebble distribution such that $f(v) \geq 1$ and $f(x) \geq 1$. Then a rubbling move, denoted $r(v, x \to w)$, removes one pebble from each of v and x and places one pebble on w. This defines a new pebble distribution f' such that: $f'(v) = f(v) - 1$, $f'(x) = f(x) - 1$, $f'(w) = f(w) + 1$, and $f'(z) = f(z)$ for $z \in V - \{v, w, x\}$. A vertex v is *reachable* if there is a way to place a pebble on v using a sequence of pebbling and rubbling moves.

In graph pebbling, which preceded graph rubbling, only the pebbling move is allowed; while in graph rubbling both pebbling and rubbling moves are available. According to Hurlbert [**?**], Lagarias and Saks first introduced graph pebbling as an approach to a problem in number theory. In 1989, Chung's [**?**] results on pebbling the n-cube were used to give a proof to the theorem originally proposed by Lagarias and Saks. Graph rubbling was introduced in [**?**] and studied for example in [**?**, **?**].

Crull et al.[**?**] introduced the concept of cover pebbling. The *cover pebbling number* of a graph G is the minimum number of pebbles needed so that from any initial pebble distribution, after a series of pebbling moves, it is possible to have at least one pebble on every vertex of G. Gardner et al. [**?**] considered a version of cover pebbling where it is not necessary to "cover" every vertex of G, just the vertices in a dominating set. A set S is a *dominating set* of G if every vertex in $V \setminus S$ is adjacent to a vertex in S. A *domination cover* is a pebble distribution such that the set of vertices with pebbles is a dominating set. The *domination cover pebbling number* $\psi(G)$ is the minimum number of pebbles, so that no matter how they are distributed, it is possible that after a sequence of pebbling moves to obtain a domination cover of G. Domination cover pebbling was introduced by Gardner et al. [**?**] and studied in [**?**, **?**].

In this paper, we consider the analog for rubbling as follows. The *domination cover rubbling number* $\psi_R(G)$ of a graph G is the minimum number of pebbles, so that no matter how they are distributed, it is possible to obtain a domination cover after a sequence of pebbling and rubbling moves.

We shall use the following terminology and notation. Let G be a graph with vertex set V and edge set E. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in$

$V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is $|N(v)|$. A *universal vertex* has degree $|V| - 1$. A vertex of degree one is called a *leaf*, and its neighbor is a *support vertex*. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of cardinality $\gamma(G)$ is called a γ*-set* of G.

The *distance* between two vertices u and v in a connected graph G, denoted by $d(u, v)$, is the length of a shortest (u, v)-path in G. The eccentricity of a vertex v is the maximum distance from v to any other vertex in G, and the maximum eccentricity is the *diameter* of G, denoted $\operatorname{diam}(G)$. A *peripheral vertex* of G has eccentricity equal to $\operatorname{diam}(G)$. The prism $G\Box P_2$ is the graph obtained from two copies of the graph G, say G_1 and G_2, with the same vertex labelings by adding edges such that each vertex of G_1 is adjacent to the vertex of G_2 which has the same label.

2 Small Values and Examples

We begin by characterizing the graphs G with small domination cover rubbling numbers, namely, $\psi_R(G) \in \{1, 2, 3\}$. An edge uv is called a *dominating edge* if $\{u, v\}$ is a dominating set of G, otherwise, it is a *non-dominating edge*. More generally, a subgraph of G whose vertices do not dominate G is called a *non-dominating subgraph*. In particular, a path P_3 whose vertices do not dominate G is called a *non-dominating* P_3.

Theorem 1. *Let G be a graph. Then*

1. $\psi_R(G) = 1$ *if and only if G is a complete graph.*
2. $\psi_R(G) = 2$ *if and only if G has a universal vertex and G is not a complete graph.*
3. $\psi_R(G) = 3$ *if and only if all of the following conditions hold:*
   (a) *G has no universal vertex.*
   (b) *Every vertex of G is incident to a dominating edge.*
   (c) *For every non-dominating edge uv, there exists a vertex $x \in N(u)$ such that $\{v, x\}$ dominates G or there exists an $x \in N(u) \cap N(v)$ such that $\{u, x\}$ dominates G.*
   (d) *For every non-dominating path $P_3 = (u, v, w)$, there exists a vertex x such that for some $y \in \{u, v, w\}$, $\{x, y\}$ is a dominating set of G and x is adjacent to both vertices of $\{u, v, w\} \setminus \{y\}$.*

Proof. The proof of (1) is clear. To prove (2), suppose that $G \neq K_n$ has a universal vertex v. By (1), we have $\psi_R(G) > 1$. Now, we will consider all possible placements of two pebbles. If either or both of the pebbles are placed on v, then we are finished as $\{v\}$ is a dominating set. Thus, we may assume that the two pebbles are placed in $N(v)$. But then either a pebbling move or a rubbling move will place a pebble on v. Thus, $\psi_R(G) = 2$.

Suppose $\psi_R(G) = 2$. By (1) we have that G is not a complete graph, so G has at least one vertex, say v, that is not a universal vertex. Consider placing two pebbles on v. Since $\psi_R(G) = 2$, we can perform a pebbling move $p(v \to w)$ such that $\{w\}$ is a domination cover. Since w dominates G, w is a universal vertex.

To prove (3), we assume that $\psi_R(G) = 3$. By (1) and (2), G has no universal vertex, so (a) holds. Suppose that three pebbles are placed on one vertex v. Since v is not a universal vertex, $\{v\}$ does not dominate G. Thus, exactly one pebbling move must reach a domination cover. That is, a pebbling move $p(v \to w)$ for some $w \in N(v)$ results in a domination cover $\{v, w\}$, so vw is a dominating edge. Since v is an arbitrary vertex, every vertex of G is incident to a dominating edge. Hence, (b) holds.

Suppose uv is a non-dominating edge of G, and consider the placement of two pebbles on u and one pebble on v. Then we must be able to reach a domination cover after exactly one move. Thus, either the pebbling move $p(u \to x)$ results in a domination cover $\{v, x\}$ where $x \in N(u)$, or the rubbling move $r(u, v \to x)$ results in a domination cover $\{u, x\}$ where $x \in N(u) \cap N(v)$. Hence, (c) holds.

Now suppose that G has a non-dominating $P_3 = (u, v, w)$ subgraph, and consider the placement of a pebble on each of three vertices u, v, and w. Since no single vertex dominates G, we must be able to reach a domination cover of two vertices in a single rubbling move. Thus, there exists a vertex x in $V \setminus \{u, v, w\}$ such that $\{x, y\}$ is a domination cover of G for some $y \in \{u, v, w\}$ and x is adjacent to both vertices of $\{u, v, w\} \setminus \{y\}$. Hence, (d) holds.

For the converse, assume that G satisfies Conditions (a) - (d). Since G has no universal vertex, (1) and (2) imply that $\psi_R(G) \geq 3$. We show that $\psi_R(G) = 3$ by showing that any placement of three pebbles can reach a domination cover.

Suppose that three pebbles are placed on one vertex v. Since every vertex is incident to a dominating edge, there is a vertex $v' \in N(v)$ such that $\{v, v'\}$ dominates G. Thus, the pebbling move $p(v \to v')$ gives a domination cover.

Consider the placement of two pebbles on u and one pebble on v. If $\{u, v\}$ is a dominating set of G, then we are finished. Assume that $\{u, v\}$ does not dominate G. If u and v are not adjacent, then since every vertex is incident to a dominating edge, there is a vertex $v' \in N(v)$ such that $\{v, v'\}$ dominates G. Hence, $uv' \in E(G)$. But then a pebbling move from u to v' gives a domination cover. Hence, we may assume that u and v are adjacent, i.e., that uv is a non-dominating edge. By Condition (c), there is a vertex $x \in N(u)$ such that $\{v, x, \}$ is a dominating set of G or there is a vertex $x \in N(u) \cap N(v)$ such that ux is a dominating edge of G. If $x \in N(u)$ and $\{v, x\}$ is a dominating set of G, then the pebbling move $p(u \to x)$ gives a domination cover. If $x \in N(u) \cap N(v)$ such that ux is a dominating edge of G, then the rubbling move $r(u, v \to x)$ results in pebbles on the domination cover $\{u, x\}$.

Finally assume that a pebble is placed on each of three vertices u, v, and w. If $\{u, v, w\}$ dominates G, then we have a domination cover. Hence, assume that $\{u, v, w\}$ is not a dominating set of G. Suppose that the subgraph induced by $\{u, v, w\}$ has at most one edge, say uv, if one. By Condition (b), every vertex is incident to a dominating edge, so there is a vertex $w' \in N(w)$ such that $\{w, w'\}$ is a dominating set of G. Thus, w' is adjacent to both u and v, implying that the rubbling move $r(u, v \to w')$ results in a domination cover. Hence, we may assume that the subgraph induced by $\{u, v, w\}$ has at least two edges and that (u, v, w) is a non-dominating P_3 subgraph of G. By Condition (d), there exists a vertex x in $V \setminus \{u, v, w\}$ such that $\{x, y\}$ is a dominating set of G for some $y \in \{u, v, w\}$ and x is adjacent to the two vertices of $\{u, v, w\} \setminus \{y\}$. But then a rubbling move from the two vertices in $\{u, v, w\} \setminus \{y\}$ to x gives a domination cover. $\square$

From Theorem **??**, we obtain several immediate results for specific graph families. Recall that the wheel W_n is the graph obtained from a cycle C_n by adding a new vertex adjacent to every vertex on the cycle. Since stars $K_{1,n}$ and wheels W_n have a universal vertex and are not complete graphs, we have the following corollary.

Corollary 2. *For $n \geq 3$, $\psi_R(K_{1,n}) = \psi_R(W_n) = 2$.*

While the domination cover rubbling number for the star $K_{1,n}$ is two, its domination cover pebbling number is n. To see that n pebbles are necessary, consider an initial pebbling distribution that places a single pebble on $n-1$ of the leaves. The set of $n-1$ leaves is not a domination cover and no pebbling move is possible. Hence, the domination cover pebbling number and the domination cover rubbling number can differ by an arbitrary amount.

The complete bipartite graph $K_{r,s}$ with $2 \leq r \leq s$ and the prism $K_3 \square P_2$ are examples of graphs G having $\psi_R(G) = 3$. The join of two graphs G_1 and G_2, denoted $G_1 + G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$.

Corollary 3. *Let G_1 and G_2 be graphs such that $G_1 + G_2 \neq K_n$. If $G_1 + G_2$ has a universal vertex, then $\psi_R(G_1 + G_2) = 2$, otherwise $\psi_R(G_1 + G_2) = 3$.*

Corollary 4. *Let $G = K_{s_1,\ldots,s_r}$ be the complete r-partite graph with $1 \leq s_1 \leq \ldots \leq s_r$. Then* $\psi_R(G) = \begin{cases} 1 & \textit{if } s_r = 1, \\ 2 & \textit{if } s_1 = 1,\ s_r > 1, \\ 3 & \textit{otherwise.} \end{cases}$

3 Paths and Cycles

Next, we present the domination cover rubbling number for paths and cycles. We will accomplish this in part by referring to results from [**?**]. In order to use these results, we need the following lemma.

Lemma 5. *Let G be either a path or a cycle. Suppose that a domination cover is reachable via pebbling and rubbling moves from some distribution of pebbles on G. Then, a domination cover is reachable from this same distribution using only pebbling moves.*

Proof. We begin by labeling the vertices of G by $(v_0, \ldots, v_{n-1})$.

Suppose that a domination cover is reachable from a given distribution via the ordered sequence of moves $\{m_1, \ldots, m_t\}$. Furthermore, we suppose that this sequence contains a minimum number of rubbling moves. If the sequence contains no rubbling moves, then the result holds. Now, suppose that $1 \leq t' \leq t$ is the largest integer such that $m_{t'}$ is a rubbling move, say $m_{t'} = r(v_{j-1}, v_{j+1} \to v_j)$, where we understand that the indices of the vertices are taken modulo n.

Case 1: For each t'' satisfying $t' < t'' \leq t$, we have $m_{t''} \neq p(v_j \to v_{j+1})$ and $m_{t''} \neq p(v_j \to v_{j-1})$.

In this case, we may simply delete $m_{t'}$ from the list of moves. To see this, note that if we delete $m_{t'}$ and perform the remaining moves, then we have at least one pebble remaining on each of v_{j-1} and v_{j+1}. Hence, $N[v_{j-1}] \cup N[v_{j+1}]$ is dominated, and since G is a path or cycle, we have that $N[v_j] \subseteq N[v_{j-1}] \cup N[v_{j+1}]$. Thus, the resulting list of moves reaches a domination cover in this case.

Case 2: For some t'' satisfying $t' < t'' \leq t$ we have $m_{t''} = p(v_j \to v_{j+1})$ or $m_{t''} = p(v_j \to v_{j-1})$.

Assume, without loss of generality, that $m_{t'}$ is followed by a pebbling move of the form $m_{t''} = p(v_j \to v_{j-1})$. Notice, the effect of applying $m_{t'}$ and then $m_{t''}$, regardless of any other moves, is that the number of pebbles on v_j and v_{j+1} both decrease by one and the number of pebbles on v_{j-1} remains the same. Hence, by deleting $m_{t'}$ and $m_{t''}$ from our list of moves, we see that all other moves will remain valid. Moreover, the final distribution after deleting these moves will have one more pebble on v_j and v_{j+1}, as well as the same number of pebbles on v_{j-1}, which is enough to see that the resulting distribution is also a domination cover. Thus, the resulting list of moves reaches a domination cover in this case.

In both cases, we have contradicted the minimality condition on the list of moves needed to reach a domination cover. Thus, there is no rubbling move in the set $\{m_1, \ldots, m_t\}$. $\square$

We note that applying Lemma **??**, we are able to eliminate the need for rubbling moves to obtain a dominating cover on a path or a cycle. Hence, the domination cover rubbling number equals the domination cover pebbling number for paths and cycles. The values for the domination cover pebbling numbers of paths and cycles were determined in [**?**]. Thus for paths, we simply state the result below. However, there is a minor error in the proof of the cycle theorem given in [**?**], and its correction changes the value given in [**?**] for the domination cover pebbling number of C_n if $n \equiv 0 \pmod 6$. We correct this value and for completion give a proof for all values of n.

Theorem 6. *For the path P_n with $n \geq 3$,*

$$\psi_R(P_n) = \begin{cases} 2^{n+1}\left(\frac{1-8^{-\left\lfloor \frac{n+1}{3} \right\rfloor}}{7}\right) + 1 & \text{if } n \equiv 1 \pmod 3, \\ 2^{n+1}\left(\frac{1-8^{-\left\lfloor \frac{n+1}{3} \right\rfloor}}{7}\right) & \text{otherwise.} \end{cases}$$

To prove the result for cycles, we will need the following lemma from [**?**].

Lemma 7. [**?**] *The value of the domination cover pebbling number for cycles is attained when the original configuration consists of placing all pebbles on a single vertex.*

We are now prepared to give the analogous result for cycles.

Theorem 8. *Let C_n be a cycle on n vertices.*

1. *If n is odd, then writing $n = 2m-1$ with $m \geq 2$,*

$$\psi_R(C_n) = 2^{m+2}\left(\frac{1-8^{-\left\lfloor \frac{m+1}{3} \right\rfloor}}{7}\right) + \phi(m),$$

where

$$\phi(m) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod 3, \\ 1 & \text{if } m \equiv 1 \pmod 3, \\ -1 & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

2. *If n is even, then writing $n = 2m-2$ with $m \geq 3$,*

$$\psi_R(C_n) = 2^{m+1}\left(\frac{1-8^{-\left\lfloor \frac{m+1}{3} \right\rfloor}}{7}\right) + 2^{m}\left(\frac{1-8^{-\left\lfloor \frac{m}{3} \right\rfloor}}{7}\right).$$

Proof. Let $C_n = (v_1, \ldots, v_n)$. For the upper bound, we show that $\psi_R(C_n) \leq r$, where r is the value of $\psi_R(C_n)$ stated in the theorem. That is, we show that any configuration of r pebbles can reach a domination cover. Lemmas **??** and **??** imply that we need only consider the configuration that initially places all pebbles on one vertex, say v_1.

We consider two cases based on the parity of n.

Case 1. n is odd. We write $n = 2m-1$ for $m \geq 2$. Then $r = 2^{m+2}\left(\frac{1-8^{-\lfloor \frac{m+1}{3} \rfloor}}{7}\right) + \phi(m)$ pebbles are initially placed on v_1, where $\phi(m)$ is defined in the statement of the theorem. Let $P = (v_1, v_2, \ldots, v_m)$ and $P' = (v_1, v_n, v_{n-1}, \ldots, v_{n-m+2} = v_{m+1})$ be the two P_m paths of the cycle sharing the vertex v_1.

Assume first that $m \equiv 0 \pmod 3$. Then $r = 2\psi_R(P_m)$. To reach a domination cover of P from v_1 using $\psi_R(P_m)$ pebbles, no pebble from the $\psi_R(P_m)$ pebbles is left on v_1 and exactly one pebble is moved to each vertex in the set $\{v_2, v_5, \ldots, v_{m-4}, v_{m-1}\}$. Then there are $r - \psi_R(P_m) = \psi_R(P_m)$ pebbles remaining on v_1, which is sufficient to reach a domination cover of P'. Next assume that $m \equiv 1, 2 \pmod 3$. Then $r = 2\psi_R(P_m) - 1$. To reach a domination cover of P from v_1 using $\psi_R(P_m)$ pebbles, exactly one pebble from the $\psi_R(P_m)$ pebbles is left on v_1 and exactly one pebble is left on each vertex in the set $\{v_3, v_6, \ldots, v_{m-4}, v_{m-1}\}$ if $m \equiv 1 \pmod 3$ (respectively, the set $\{v_4, \ldots, v_{m-4}, v_{m-1}\}$ if $m \equiv 2 \pmod 3$). After moving to a domination cover of P, there are $r - \psi_R(P_m) + 1 = 2\psi_R(P_m) - 1 - \psi_R(P_m) + 1 = \psi_R(P_m)$ pebbles remaining on v_1, which is sufficient to reach a domination cover of P'. Hence, $\psi_R(C_n) \leq r$ in this case.

Case 2. n is even. We write $n = 2m-2$ with $m \geq 3$. Then $r = 2^{m+1}\left(\frac{1-8^{-\lfloor \frac{m+1}{3} \rfloor}}{7}\right) + 2^{m}\left(\frac{1-8^{-\lfloor \frac{m}{3} \rfloor}}{7}\right)$ pebbles are initially placed on v_1. Note that $r = \psi_R(P_m) + \psi_R(P_{m-1})$ if $m \equiv 0 \pmod 3$ and $r = \psi_R(P_m) + \psi_R(P_{m-1}) - 1$, otherwise. Let P denote the path $P_m = (v_1, v_2, \ldots, v_m)$ and P' denote the path $P_{m-1} = (v_1, v_n, v_{n-1}, \ldots, v_{n-m+3} = v_{m+1})$ on the cycle.

Assume first that $m \equiv 0 \pmod 3$, and so, $(m-1) \equiv 2 \pmod 3$ and $r = \psi_R(P_m) + \psi_R(P_{m-1})$. To reach a domination cover of P from v_1 using $\psi_R(P_m)$ pebbles, no pebble from the $\psi_R(P_m)$ pebbles is left on v_1 and exactly one pebble is moved to each vertex in the set $\{v_2, v_5, \ldots, v_{n-4}, v_{n-1}\}$. Then there are $r - \psi_R(P_m) = \psi_R(P_{m-1})$ pebbles remaining on v_1, which is sufficient to reach a domination cover of P'.

Next assume that $m \equiv 1, 2 \pmod 3$. Then $r = \psi_R(P_m) + \psi_R(P_{m-1}) - 1$. To reach a domination cover of P from v_1 using $\psi_R(P_m)$ pebbles, exactly one pebble from the $\psi_R(P_m)$ pebbles is left on v_1 and exactly one pebble is left on each vertex in the set $\{v_3, v_6, \ldots, v_{n-4}, v_{n-1}\}$ if $m \equiv 1 \pmod 3$ (respectively, the set $\{v_4, \ldots, v_{n-4}, v_{n-1}\}$ if $m \equiv 2 \pmod 3$). After moving to a domination cover of P, there are $r - \psi_R(P_m) + 1 = \psi_R(P_{m-1}) - 1 + 1 = \psi_R(P_{m-1})$ pebbles remaining on v_1, which is sufficient to reach a domination cover of P'. Again, $\psi_R(C_n) \leq r$.

In both cases, we have the desired upper bound. To prove the lower bound, we show that at least r pebbles are necessary in any configuration initially placing all pebbles on a single vertex, say v_1. Notice that reaching a domination cover for C_n from the pebbles on v_1 is equivalent to reaching a domination cover for two paths $P_m = (v_1, v_2, ..., v_m)$ and $P_{n-m+1} = (v_1, v_n, ..., v_{n-m+1})$ simultaneously. We note that m could equal $n-1$, that is, we dominate the cycle by dominating one path. This implies that $\psi_R(C_n)$ is the minimim sum $\psi_R(P_m)+\psi_R(P_{n-m+1})$ among all possible values of m. Furthermore, this sum is minimized when the paths are balanced, that is, for odd n, $m = n-m+1 = \mathrm{diam}(C_n)+1$, and for even n, $m = n-m+2 = \mathrm{diam}(C_n)+1$. These are precisely the paths P and P' given to reach the domination covers establishing the upper bounds in Cases 1 and 2. From our previous comments, we see that all of the r pebbles are necessary to reach a domination cover of P and P'. That is, if $r-1$ pebbles are initially placed on v_1, then after reaching a domination cover of the path P, the number of pebbles remaining on v_1 is insufficent to reach a domination cover of P'. Hence, $\psi_R(C_n) \geq r$ in all cases and equality follows. This completes the proof. □

4 Stacking in Trees

In various types of graph pebbling and graph rubbling, a common question is when is it sufficient to consider the placement of all pebbles on a single vertex? For example, in [**?**], it is proven that to compute the cover pebbling number of a graph, it is sufficient to consider only the case in which pebbles are initially placed on a single vertex. This result is commonly referred to as the Stacking Theorem. Similarly, one can consider the situation when rubbling is allowed. Indeed, it has been conjectured by Sieben [**?**] that the Stacking Theorem holds in the area of cover rubbling. The analogous question which suggests itself is, can we determine the domination cover rubbling number of a graph by only considering pebble distributions which initially place all pebbles on a single vertex?

To answer to this question, we will consider the prism $K_n \Box P_2$, where $n \geq 4$. We will show that placing three pebbles on any single vertex is sufficient for reaching a domination cover but $\psi_R(K_n \Box P_2) = 4$. Hence, the Stacking Theorem does not hold for domination cover rubbling.

Theorem 9. *Let $G = K_n \Box P_2$ for $n \geq 4$. Then $\psi_R(G) = 4$.*

Proof. Let $G = K_n \Box P_2$ for $n \geq 4$, and let $v_1, \ldots, v_n$ denote the vertices of one copy of K_n and $w_1, \ldots, w_n$ the vertices of the other such that $v_i w_i$ for $1 \leq i \leq n$ are the edges between the corresponding vertices.

By Parts (1) and (2) of Theorem **??**, $\psi_R(G) \geq 3$. We use Part (3) of Theorem **??** to show that $\psi_R(G) > 3$. In particular, we show that Condition (d) does not hold for G.

Let (u, v, w) be a non-dominating path in G. Note, either $\{u, v, w\} \in \{v_1, \ldots, v_n\}$ or $\{u, v, w\} \in \{w_1, \ldots, w_n\}$. Assume, without loss of generality, that $\{u, v, w\} \in \{v_1, \ldots, v_n\}$. For any two element subset $S \subset \{u, v, w\}$, we have that if a vertex is adjacent to both elements of S, then it must lie in the set $\{v_1, \ldots, v_n\}$. Furthermore, no two element subset of $\{v_1, \ldots, v_n\}$ dominates G, i.e., Condition (d) does not hold, and we conclude that $\psi_R(G) \geq 4$.

Consider placing four pebbles on G. If there is at least one pebble on a vertex in $\{v_1, \ldots, v_n\}$ and at least one pebble on a vertex in $\{w_1, \ldots, w_n\}$, then the two pebbles form a domination cover. Hence, assume, without loss of generality, that all four pebbles are placed in $\{v_1, \ldots, v_n\}$.

If at least three pebbles are placed on a single vertex, say v_i, then the pebbling move $p(v_i \to w_i)$ gives a domination cover $\{v_i, w_i\}$. If at least two pebbles are placed on a vertex v_i and another pebble is placed on a different vertex v_j, then the pebbling move $p(v_i \to w_i)$ results in the domination cover $\{v_j, w_i\}$. If the four pebbles are all placed on distinct vertices, say v_i, v_j, v_k, v_ℓ, then the rubbling move $r(v_k, v_\ell \to v_i)$ followed by the pebbling move $p(v_i \to w_i)$ obtains the domination cover $\{v_j, w_i\}$. We conclude that $\psi_R(G) = 4$. $\square$

Using the notation from the previous proof, we see that if three pebbles are placed on any vertex v_i (resp. w_i), then the pebbling move $p(v_i \to w_i)$ (resp. $p(w_i \to v_i)$) results in a domination cover of $K_n \square P_2$. Hence, to determine $\psi_R(G)$ for a graph G, it is not sufficient to simply consider distributions which place all pebbles on a single vertex of G, i.e., the Stacking Theorem does not hold in the setting of domination cover rubbling on general graphs. However, our next result suggests that there may be one for trees. Recall that the periphery of a graph is the set of all vertices whose eccentricity equals the diameter. For trees, all peripheral vertices are leaves.

Theorem 10. *If T is a non-trivial tree with diameter d and domination number γ, then*

$$\psi_R(T) \leq 2^{d-1}\gamma - 2^{d-1} + 2.$$

Proof. Assume that T is a non-trivial tree with domination number γ and diameter d. If $T = K_2$, then $\psi_R(T) = 1 < 2 = 2^{d-1}\gamma - 2^{d-1} + 2$. Thus, we may assume that T has at least three vertices and that $d \geq 2$. Let $S = \{v_1, ..., v_\gamma\}$ be a γ-set of T such that S contains no leaves. This is possible since either a leaf or its support vertex must be in any γ-set and a support vertex dominates at least as many vertices as its adjacent leaves do. Let f be a pebble distribution that reaches a domination cover containing S, and let $P = \{u_1, ..., u_k\}$ be the set of vertices of T that are initially assigned at least one pebble under f. If P is a domination cover, then we are finished. Hence, we may assume that P is not a domination cover and that the vertices of P are assigned at least $\gamma + 1$ pebbles.

By the Pigeonhole Principle, since $\sum_{i=1}^{k} f(u_i) \geq \gamma + 1$, there is an element of S, say v_1, such that $\sum_{u \in N[v_1]} f(u) \geq 2$. Thus, a pebble can be placed on v_1 using at most one pebbling or rubbling move. Now there is a sequence of moves that brings pebbles from P to S. For an arbitrary, but fixed, sequence of moves, suppose that u_i contributes $n_{i,j}$

pebbles so that v_j can have one. Since the diameter of T is d and no element of S is a leaf, for all $i = 1, ..., k$ and $j = 2, ..., \gamma$, the distance between u_i and v_j is $d(u_i, v_j) \leq d - 1$.

Each of the $n_{i,j}$ pebbles that u_i contributes to v_j results in $2^{-d(u_i,v_j)}$ pebbles on v_j. Since v_j ends with a pebble, we have that

$$\sum_{i=1}^{k} n_{i,j} 2^{-d(u_i,v_j)} = 1.$$

Since $d(u_i, v_j) \leq d-1$ for all i and j, we have that $2^{d(u_i,v_j)} \leq 2^{d-1}$ or equivalently, $2^{-d(u_i,v_j)} \geq 2^{-(d-1)}$. Ergo,

$$1 = \sum_{i=1}^{k} n_{i,j} 2^{-d(u_i,v_j)} \geq \sum_{i=1}^{k} n_{i,j} 2^{-(d-1)} = 2^{-(d-1)} \sum_{i=1}^{k} n_{i,j}$$

$$\Rightarrow \sum_{i=1}^{k} n_{i,j} \leq 2^{d-1}.$$

Thus, for $j = 2, ..., \gamma$, the number of pebbles needed to cover v_j is at most 2^{d-1}. Hence, at most $2^{d-1}(\gamma - 1)$ pebbles are needed for v_2,...,v_γ and at most two pebbles are needed for v_1. Thus,

$$\psi_R(T) \leq \sum_{i=1}^{k} f(u_i) = 2^{d-1}\gamma - 2^{d-1} + 2.$$

□

The proof of Theorem **??** implies that the worst case scenario on trees is to place all of the pebbles on a single peripheral vertex whose support is distance $d - 2$ from any other support. We sum this up in the following conjecture.

Conjecture 11. (Stacking Theorem for Trees) *In order to determine the domination cover rubbling number of a tree, it is sufficient to consider only pebble distributions which place all pebbles on a single peripheral vertex.*

As evidence for Conjecture **??**, we present results on the domination cover rubbling number for trees of diameter three (i.e., double stars) and trees of diameter four. In both of these cases, the worst case is obtained by stacking on a single peripheral vertex. The double star $S_{r,s}$ is the tree with exactly two non-leaf vertices x and y where x is adjacent to $r \geq 1$ leaves and y is adjacent to $s \geq 1$ leaves.

Corollary 12. *For the double star $S_{r,s}$, $\psi_R(S_{r,s}) = \begin{cases} 5 & \text{if } r = s = 1, \\ 6 & \text{otherwise.} \end{cases}$*

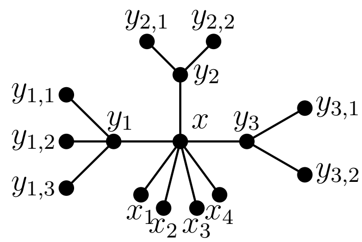

Figure 1: The graph $K_{1,3}(4;3,2,2)$

Proof. If $r = s = 1$, then $S_{r,s}$ is the path P_4, and from Theorem **??**, $\psi_R(P_4) = 5$.

Hence, we may assume at least one support vertex, say x, of $S_{r,s}$ is adjacent to two or more leaves. Since $\gamma(S_{r,s}) = 2$ and $\mathrm{diam}(S_{r,s}) = 3$, it follows from Theorem **??** that $\psi_R(S_{r,s}) \leq 6$. We now show that five pebbles are not sufficient. Suppose that all five pebbles are placed on a leaf x_1 adjacent to x. Note that since x has at least two leaf neighbors, either a pebble must be moved to x or to every one of its leaf neighbors. Similarly, to dominate the leaves adjacent to y, at least one pebble must be moved to a vertex in $N[y]$. Moving a pebble to each of x and y accomplishes this with the minimum number of pebbles. However, at most two pebbles can be moved to x from x_1, so at least one of x and y cannot receive a pebble. Hence, five pebbles will not guarantee a domination cover, and so, $\psi_R(S_{r,s}) = 6$. □

Note that we have established the domination cover rubbling number for trees of diameter two (namely stars) and trees of diameter three (namely double stars). Thus, a natural next step would be to determine the domination cover rubbling number for trees of diameter four. Any tree of diameter four can be obtained by appending pendant vertices to the existing vertices of $K_{1,n}$ for $n \geq 2$. Label the center of the star as x and its leaves as $y_1, ..., y_n$. Suppose that we append $c \geq 0$ pendant vertices to x, namely $x_1, ..., x_c$, and $a_i \geq 1$ pendant vertices to y_i, namely $y_{i,1}, ..., y_{i,a_i}$ for $1 \leq i \leq n$. Note that for $i \neq j$ and for any ℓ and m, the vertices $y_{i,\ell}$, y_i, x, y_j, and $y_{j,m}$ induce a path of length four, and this construction gives all trees of diameter four. The resulting graph will be denoted $K_{1,n}(c; a_1, ..., a_n)$, and without loss of generality, we will assume that $a_1 \geq a_2 \geq ... \geq a_n$. An example is shown in Figure **??**.

Corollary 13. *If $G = K_{1,n}(c; a_1, ..., a_n)$, where $a_1 \geq a_2 \geq \ldots \geq a_n$, is a tree of diameter four, then*

$$\psi_R(G) = \begin{cases} 8n-7 & \text{if } c = 0 \text{ and } a_1 = 1, \\ 8n-6 & \text{if } c = 0 \text{ and } a_1 \geq 2, \\ 8n-3 & \text{if } c \geq 1 \text{ and } a_1 = 1, \\ 8n-2 & \text{if } c \geq 1 \text{ and } a_1 \geq 2. \end{cases}$$

Proof. If $c = 0$ and $a_1 = 1$, then G is the star $K_{1,n}$ with each edge subdivided exactly once. We first show that $\psi_R(G) \geq 8n - 7$. Suppose that all $8n - 8$ pebbles are placed on $y_{1,1}$. Note that $\{y_{1,1}, y_2, y_3, ..., y_n\}$ is a dominating set of G whose distance from $y_{1,1}$ is minimum. Using pebbling moves, we can move at most $4n - 5$ pebbles to y_1 (leaving two). We can then move at most $2n - 3$ pebbles to x using pebbling moves. This allows us to distribute at most $n - 2$ pebbles amongst y_2,...,y_n. Using a similar argument, it is straightforward to check that $8n - 7$ pebbles placed on $y_{1,1}$ will yield a domination cover. We now show that $\psi_R(G) \leq 8n - 7$. Our strategy is to look at how different pebble placements reduce the number of pebbles needed on $y_{1,1}$. These reductions are based on the distance of the vertex to a minimal dominating set. Note that each additional pebble placed on y_1 reduces the amount of pebbles needed on $y_{1,1}$ by two. Similarly, each additional pebble placed on x will reduce the amount of pebbles needed on $y_{1,1}$ by four. The first pebble placed on y_i (where $i = 2, ..., n$) will reduce number of pebbles needed on $y_{1,1}$ by eight, each subsequent pebble will reduce it by four. If y_i is not assigned an initial pebble, then the first pebble placed on $y_{i,1}$ will reduce the number needed on $y_{1,1}$ by eight. However, each additional pebble on $y_{i,1}$, will neither increase nor decrease the number of required pebbles. In any case, this reduces the number of pebbles needed initially. Hence $\psi_R(G) = 8n - 7$.

Now assume that $c = 0$ and $a_1 \geq 2$. Then $\gamma(K_{1,n}(0; a_1, ..., a_n)) = n$, and by Theorem **??**, $\psi_R(G) \leq 8n - 6$. To show that we can do no better, suppose that all $8n - 7$ pebbles are initially placed on $y_{1,1}$. Note that $\{y_1, ..., y_n\}$ is a dominating set of G whose distance from $y_{1,1}$ is minimum. Using pebbling moves, we can move at most $4n - 4$ pebbles to y_1. We then remove $4n - 6$ pebbles from y_1 (leaving two) using pebbling moves and place $2n - 3$ on x. But to reach each y_i for $2 \leq i \leq n$ from x requires $2(n - 1) = 2n - 2 > 2n - 3$ pebbles. Thus, a domination cover cannot be reached. Hence, $\psi_R(G) \geq 8n - 6$, and so $\psi_R(G) = 8n - 6$.

Next assume that $c \geq 1$ and $a_1 = 1$. We show that $8n - 4$ pebbles will not guarantee a domination cover. Suppose that $8n - 4$ pebbles are placed on the vertex $y_{1,1}$. Note that either $y_{1,1}$ or y_1 must be in any dominating set. Thus, we can move at most $2n-2$ pebbles to x. Since a pebble must be left on x or a leaf neighbor of x, there are at most $2n - 4$ pebbles on x that can be moved. But again at least $2n - 2$ pebbles are necessary to cover each y_i for $2 \leq i \leq n$ from x. Hence, this does not yield a domination cover, and so, $\psi_R(G) \geq 8n-3$. Note that if $8n-3$ pebbles are placed on $y_{1,1}$, then using pebbling moves, we remove $8n - 4$ pebbles from $y_{1,1}$ (leaving one), and place $4n - 2$ pebbles on y_1. We remove these pebbles and place $2n-1$ on x. We then use pebbling moves to remove $2n-2$ pebbles from x (leaving one) and place one on each of y_2,...,y_n. This gives a domination cover. To show that we can do better, we use a similar pebble reduction argument as above to show that $\psi_R(G) = 8n - 6$. As this argument is nearly identical to the earlier one, we omit it.

Finally, assume that $c \geq 1$ and $a_1 \geq 2$. We first show that $8n-3$ pebbles will not guarantee a domination cover. Again, we assume that $8n-3$ pebbles are placed on the vertex $y_{1,1}$. Note that $\{x, y_1, ..., y_n\}$ is a dominating set of minimum distance from $y_{1,1}$. As before, we can remove at most $8n-4$ pebbles from $y_{1,1}$ (leaving one) and place $4n-2$ on y_1. We can then remove at most $4n-4$ pebbles from y_1 (leaving two) and place $2n-2$ pebbles on x. Since a pebble must be left on x or a leaf neighbor of x, there are at most $2n-4$ pebbles on x that can be moved. But again at least $2n-2$ pebbles are necessary to cover each y_i for $2 \leq i \leq n$ from x. Hence, this does not yield a domination cover, and so, $\psi_R(G) \geq 8n-3$. Note that if $8n-2$ pebbles are placed on $y_{1,1}$, then using pebbling moves, we remove all $8n-2$ pebbles from $y_{1,1}$ and place $4n-1$ pebbles on y_1. We remove $4n-2$ pebbles from y_1 (leaving one) and place $2n-1$ pebbles on x. We then use pebbling moves to remove $2n-2$ pebbles from x (leaving one) and place one on each of y_2,...,y_n to form a domination cover. To show that we can do better, we use a similar pebble reduction argument as above to show that $\psi_R(G) = 8n-2$. As this argument is nearly identical to the earlier one, we omit it. $\square$

Next we characterize the trees attaining the upper bound of Theorem **??**. For this purpose, we define a family of trees $\mathcal{T}$. A tree T is in $\mathcal{T}$ if and only if T is a star $K_{1,n}$ for $n \geq 2$, a double star $S_{r,s}$ for $(r,s) \neq (1,1)$, the diameter four tree $K_{1,n}(0; a_1, ..., a_n)$ for $n \geq 2$ and $a_1 \geq 2$, or T can be obtained from a double star $S_{1,s}$ for $s \geq 1$ by adding at least one pendant vertex adjacent to each leaf of $S_{1,s}$. Note that the tree T obtained from a double star in this manner has diameter five.

Theorem 14. *A non-trivial tree T with $\gamma(T) = \gamma$ and $\operatorname{diam}(T) = d$ has $\psi_R(T) = 2^{d-1}\gamma - 2^{d-1} + 2$ if and only if $T \in \mathcal{T}$.*

Proof. First assume that $T \in \mathcal{T}$. If T is the star $K_{1,n}$ for $n \geq 2$, then $\gamma(T) = 1$, $\operatorname{diam}(T) = 2$, and by Corollary **??**, $\psi_R(T) = 2 = 2^{d-1}\gamma - 2^{d-1} + 2$. If T is a double star $S_{r,s}$ for $(r,s) \neq (1,1)$, the $\gamma(T) = 2$, $\operatorname{diam}(T) = 3$, and by Corollary **??**, $\psi_R(T) = 6 = 2^{d-1}\gamma - 2^{d-1} + 2$. If T is the tree $K_{1,n}(0; a_1, ..., a_n)$ for $n \geq 2$ and $a_1 \geq 2$, then $\gamma(T) = n$, $\operatorname{diam}(T) = 4$, and by Corollary **??**, $\psi_R(T) = 8n - 6 = 2^{d-1}\gamma - 2^{d-1} + 2$. If T is obtained from a double star $S_{1,s}$ for $s \geq 1$ by adding at least one pendant vertex adjacent to each leaf of $S_{1,s}$, then $\gamma(T) = s+1$ and $\operatorname{diam}(T) = 5$. By Theorem **??**, $\psi_R(T) \leq 16s + 2$. Next we show that $\psi_R(T) \geq 16s + 2$. Suppose T is formed from the double star $S_{1,s}$ with centers x and y where x is adjacent to one leaf z and y is adjacent to $s \geq 1$ leaves. Let z' be a pendant vertex adjacent to z, and consider the pebbling distribution that places $16s+1$ pebbles on z'. Now at most $8s$ pebbles can be to z (leaving one on z'). But then at least one of z, x, and y must have a pebble to dominate x. Hence, at most $2s-1$ pebbles can be moved to y. But to reach a domination cover, at least one pebble must be moved from y to each of the s support vertices adjacent to y, and so, $2s$ pebbles are required on y. Hence, a domination cover cannot be reached from the pebbling distribution that places $16s + 1$ pebbles on z', and so $\psi_R(T) = 16s + 2 = 2^{d-1}\gamma - 2^{d-1} + 2$.

Next assume that T is a non-trivial tree with domination number γ and diameter d such that $\psi_R(T) = 2^{d-1}\gamma - 2^{d-1} + 2$. The proof of Theorem **??** implies that for equality to hold, $d \geq 2$. Moreover, the proof implies that for sharpness in the bound to occur, the pebbling distribution placing all $\psi_R(T)$ pebbles on a periphal leaf u' with support neighbor u, at least one of u and u' is in some domination cover D and every vertex in $D \setminus \{u\}$ is at distance exactly $d-2$ from u. Furthermore, $d - 2 \leq 3$ for otherwise at least one vertex on a shortest path between u and a vertex in D is not dominated by the vertices of D. Hence, $d \leq 5$.

If $d = 2$, then T is a star $K_{1,n}$ for $n \geq 2$ and $\psi_R(T) = 2 = 2^{d-1}\gamma - 2^{d-1} + 2$. And $T \in \mathcal{T}$, as desired. If $d = 3$, then T is the double star. By Corollary **??**, $\psi_R(T) = 6 = 2^{d-1}\gamma - 2^{d-1} + 2$ if and only if $T = S_{r,s}$ for $(r, s) \neq (1, 1)$, so $T \in \mathcal{T}$. If $d = 4$, then the proof of Corollary **??** implies that $\psi_R(T) = 2^{d-1}\gamma - 2^{d-1} + 2$ if and only if T is the tree $K_{1,n}(0; a_1, ..., a_n)$ with $a_1 \geq 2$. Again, $T \in \mathcal{T}$.

If $d = 5$, then T must have a support vertex u that is distance three from every other vertex in the domination cover. By the proof of Theorem **??**, we may assume that every support vertex with the possible exception of u is in the domination cover. Note that the leaf neighbors of u are peripheral vertices. Let w be the non-leaf neighbor of u. It follows that no neighbor of w is a support or a leaf vertex, and that w has degree 2. Let x be the other neighbor of w. Furthermore, our diameter condition implies that every support vertex of T is adjacent to x and w is the only neighbor of x that is not a support vertex. Hence, T can be obtained from a double star $S_{1,s}$ with centers w and x for $s \geq 1$ by adding at least one pendant vertex adjacent to each leaf of $S_{1,s}$. Again, $T \in \mathcal{T}$, as desired. □

Note that the difference between $\psi_R(T)$ and $2^{d-1}\gamma - 2^{d-1} + 2$ can be arbitrarily large. To see this, take the path on n vertices (see Theorem **??**).

5 Domination Cover Rubbling and Domination

We begin with the following straightforward observation relating $\gamma(G)$ and $\psi_R(G)$.

Observation 15. *For any graph G, $\gamma(G) \leq \left\lceil \frac{\psi_R(G)}{2} \right\rceil$.*

Proof. By definition, if we place $\psi_R(G)$ pebbles on any single vertex of G, then we can reach a domination cover. From this, we simply note that the largest number of vertices which can receive pebbles from this distribution after pebbling and rubbling moves is $\left\lceil \frac{\psi_R(G)}{2} \right\rceil$, which proves the result. □

Using Theorem **??**, we give an upper bound on the domination cover rubbling number of connected graphs. We will use the following well-known bound on the domination number.

Theorem 16. *[?, Thm. 2.24] For any graph G, $\gamma(G) \geq \left\lceil \frac{\operatorname{diam}(G)+1}{3} \right\rceil$.*

Theorem 17. *If G is a connected graph with domination number $\gamma(G) = \gamma$, then*

$$\psi_R(G) \leq 2^{3\gamma-2}(\gamma - 1) + 2.$$

Proof. Let G be a connected graph, and let S be a γ-set of G. Consider a spanning tree T of G formed by removing an edge from each cycle C of G as follows. If there is an edge uv on C where both u and v are in S or both u and v are in $V \setminus S$, then remove uv. If every edge on C is between the vertices of S and the vertices of $V \setminus S$, then, without loss of generality, $u \in S$ and $v \in V \setminus S$ and v has another neighbor in S. Hence, after removing uv to rid G of cycle C, the set S is still a dominating set of T, and so, the resulting graph T is a tree dominated by S. Since removing edges cannot decrease the domination number, $|S| = \gamma(G) \leq \gamma(T) \leq |S|$, so $\gamma(T) = \gamma(G) = \gamma$ and S is a γ-set of T.

Let $d = \operatorname{diam}(T)$. Theorem **??** implies that $d \leq 3\gamma(T) - 1$. By Theorem **??**, $\psi_R(T) \leq 2^{d-1}\gamma - 2^{d-1} + 2 \leq 2^{3\gamma-2}\gamma - 2^{3\gamma-2} + 2$.

Observe that adding edges to T cannot increase the domination cover rubbling number and since $\gamma(T) = \gamma(G) = \gamma$, $\psi_R(G) \leq \psi_R(T) \leq 2^{3\gamma-2}\gamma - 2^{3\gamma-2} + 2$. □

Next we consider the bound of Theorem **??** for graphs with domination number two. That is, if $\gamma(G) = 2$, then by Theorem **??**, $\psi_R(G) \leq 18$. We show that if $\gamma(G) = 2$ and $\psi_R(G) = 18$, then $\operatorname{diam}(G) \in \{4, 5\}$, and we characterize the extremal graphs having diameter five.

For two sets of vertices X and Y, let $[X, Y]$ denote the edges joining a vertex in X and a vertex in Y. If all possible edges in $[X, Y]$ are present, we say $[X, Y]$ is *full*; and if there are no edges in $[X, Y]$, then we say $[X, Y]$ is *empty*. A set X *dominates* a set Y if every vertex in Y is adjacent to a vertex in X. A graph with at least one vertex is *non-null*. For non-null graphs $G_{A1}, K_{B1}, G_{C1}, G_{C2}, K_{B2}$, and G_{A2}, let the vertex set be A_1, B_1, C_1, C_2, B_2, and A_2, respectively. We define a family of graphs $\mathcal{F}$ as follows.

Graph G is in $\mathcal{F}$ if G can be obtained from the disjoint union of non-null graphs G_{A1}, K_{B1}, G_{C1}, G_{C2}, K_{B2}, and G_{A2}, where K_{B1} and K_{B2} are complete graphs, by adding edges such that

1. $[A_i, B_i]$ and $[B_i, C_i]$ are full for $i \in \{1, 2\}$,

2. zero or more edges are added between vertices of A_i and vertices of C_i provided at least one vertex in A_i has no neighbor in C_i and no vertex in A_i dominates C_i for $i \in \{1, 2\}$, and

3. every vertex of C_1 is adjacent to at least one vertex of C_2 and every vertex of C_2 is adjacent to at least one vertex of C_1.

Note that $[A_1, B_2]$, $[A_1, C_2]$, $[A_1, A_2]$, $[B_1, C_2]$, $[B_1, B_2]$, $[B_1, A_2]$, $[C_1, B_2]$, and $[C_1, A_2]$ are empty. The path P_6 is an example of a graph in $\mathcal{F}$. As an example of a family of graphs in $\mathcal{F}$, consider the *blow-up* of a path $P_6 = (v_1, v_2, \ldots, v_6)$ which is formed by replacing each v_i, for $1 \leq i \leq 6$, with a non-null clique V_i, and adding all possible edges between the vertices of V_i and V_{i+1} for $1 \leq i \leq 5$.

We say that two vertices u and v are *twins* if they have the same closed neighborhoods, that is, $N[u] = N[v]$. Note that with this definition of twins, u and v are necessarily adjacent. For a vertex u, let $T[u]$ denote the set of twins of u. Note that $u \in T[u]$. We note that if $G \in \mathcal{F}$, then $\mathrm{diam}(G) = 5$, $\gamma(G) = 2$, and every γ-set contains a vertex of B_1 and a vertex of B_2. Moreover, if $u \in B_i$, then $B_i = T[u]$ for $i \in \{1, 2\}$.

Theorem 18. *Let G be a graph with $\gamma(G) = 2$. If $\psi_R(G) = 18$, then* $\mathrm{diam}(G) \in \{4, 5\}$. *Moreover, $\psi_R(G) = 18$ and* $\mathrm{diam}(G) = 5$ *if and only if $G \in \mathcal{F}$.*

Proof. Let G be a graph with $\gamma(G) = 2$ attaining the upper bound of Theorem **??**, that is, $\psi_R(G) = 18$. Theorem **??** implies that $\mathrm{diam}(G) \leq 5$. To show that $\mathrm{diam}(G) \geq 4$, we first prove a claim about the vertices in any arbitrary γ-set of G.

Claim 1. *If $\{u, v\}$ is a γ-set of G, then $d(u, v) = 3$.*

Proof. By assumption, $\gamma(G) = 2$. Let $\{u, v\}$ be a γ-set of G. Since $\{u, v\}$ is a dominating set and G is connected, it follows that $d(u, v) \leq 3$. Suppose, for contradiction that $d(u, v) \leq 2$. First assume that u and v are adjacent. We show that any pebble distribution of six pebbles on G yields a domination cover, contradicting the fact that $\psi_R(G) = 18$. Clearly, three pebbles placed on $\{u, v\}$ will yield a domination cover after at most one pebbling move. Thus, we may assume that at most two pebbles are placed on exactly one of u and v, say u, and no pebble is placed on v. If four pebbles are assigned such that two are in $N[u]$ and two pebbles are in $N[v]$, then pebbling or rubbling moves will reach both u and v, giving a domination cover using four pebbles. Thus, without loss of generality, we may assume that at most one pebble is placed on the vertices of $N(v) \setminus \{u\}$. Suppose first that no pebble is placed in $N(v) \setminus \{u\}$, that is, six pebbles are placed in $N[u]$. No matter how these pebbles are arranged in $N[u]$, a series of pebbling and rubbling moves can result in at least three pebbles on u. Then a pebbling move will reach v while leaving a pebble on u to form a domination cover. Next suppose that $N(v) \setminus \{u\}$ contains one pebble, leaving five pebbles in $N[u]$. No matter how the five pebbles are placed in $N[u]$, it is possible to use pebbling and rubbling moves to result in at least two pebbles on u. But then at most one rubbling move places a pebble on v while leaving a pebble on u, giving a domination cover.

Next assume that $d(u,v) = 2$. For this case, we show that any pebble distribution of ten pebbles on G yields a domination cover, and again have a contradiction. Clearly, placing a pebble on each of u and v forms a domination cover using two pebbles. If four pebbles are assigned such that two are in $N[u]$ and two pebbles are in $N[v]$, then pebbling or rubbling moves will reach both u and v, giving a domination cover using four pebbles. Hence, without loss of generality, we may assume that at most one pebble is placed in $N[v]$, else we are finished. If no pebble is placed in $N[v]$, then no matter how ten pebbles are placed in $N[u] \setminus N[v]$, pebbling and rubbling moves can result in five pebbles on u. We can then remove four pebbles from u and place two pebbles on a vertex in $N(u) \cap N(v)$ using pebbling moves. Finally, we can move the two pebbles from $N(u) \cap N(v)$ and place a pebble on v using a pebbling move. This results in a domination cover.

If one pebble is placed in $N[v]$ and nine pebbles in $N[u] \setminus N[v]$, then pebbling and rubbling moves can result in at least four pebbles on u. Since u and v have a common neighbor x, at least one pebble can be moved from u to x, while leaving two pebbles on u. Then, either there is a pebble on v or a rubbling move from x and the other vertex in $N(v)$ with a pebble can place a pebble on v, forming a domination cover. This completes the proof of Claim **??**. (□)

Claim **??** implies that for any γ-set $S = \{u, v\}$ of G, $\{N[u], N[v]\}$ is a partition of the vertex set of G. Our next claim shows that $\mathrm{diam}(G) \geq 4$.

Claim 2. *For every γ-set S, there exists a vertex at distance four from a vertex in S.*

Proof. Let $S = \{u, v\}$ be a γ-set of G. By Claim **??**, $d(u,v) = 3$. Thus, every vertex in $N[u]$ (respectively, $N[v]$) is at distance at most four from v (respectively, u). Suppose, to the contrary, that $d(u,w) \leq 3$ and $d(v,w) \leq 3$ for every $w \in V$. To reach a contradiction, we show that any pebble distribution of seventeen pebbles can reach a domination cover.

As before, we may assume that at most one pebble is placed in $N[v]$ or at most one is placed in $N[u]$, else we are finished. Without loss of generality, suppose that $N[v]$ contains at most one pebble. If no pebble is placed in $N[v]$, then no matter how seventeen pebbles are placed in $N[u]$, it is possible to use pebbling and rubbling moves that result in at least eight pebbles on u, and one pebble on a vertex w in $N(u)$. If $d(w,v) = 2$, then pebbling moves from u can place an additional three pebbles on w while leaving at least two pebbles on u. Then two pebbles can be moved from w to a neighbor, say x, of v. Finally, the pebbling move $p(x \to v)$ creates a domination cover. If $d(w,v) = 3$, then w has a neighbor $w' \in N(u)$, such that w' is adjacent to a vertex $x \in N(v)$. But then the rubbling move $r(u, w \to w')$ followed by three pebbling moves from u can place four pebbles on w' while leaving a pebble on u. Then two pebbles can be moved from w' to x, and the pebbling move $p(x \to v)$ creates a domination cover.

Suppose next that there is one pebble in $N[v]$ and sixteen pebbles are placed in $N[u]$. No matter how the sixteen pebbles are placed in $N[u]$, pebbling and rubbling moves can result in at least eight pebbles on u. Since $d(u, v) = 3$, pebbling moves can leave two pebbles on u and reach a vertex in $N(v)$. Now either v has a pebble or v can be reached by a pebbling or rubbling move. Thus, we have a domination cover and a contradiction. This completes the proof of Claim **??**. (□)

By Claim **??** and previous remarks, we have that $\text{diam}(G) \in \{4, 5\}$. To complete the proof of the theorem, we characterize the extremal graphs G having $\text{diam}(G) = 5$. That is, for a graph G with $\gamma(G) = 2$, we show that $\psi_R(G) = 18$ and $\text{diam}(G) = 5$ if and only if $G \in \mathcal{F}$.

Let $G \in \mathcal{F}$. Then $\gamma(G) = 2$ and $\text{diam}(G) = 5$. To show that $\psi_R(G) = 18$, it suffices to show that there is a pebble distribution of seventeen pebbles that cannot reach a domination cover. Let w be a vertex in A_1 that has no neighbor in C_1, and consider a placement of seventeen pebbles on w. Since there is a vertex in A_2 whose neighborhood is contained in $B_2 \cup A_2$, to reach a domination cover at least one pebble must be moved to the vertices of $B_2 \cup A_2$. From w, it requires sixteen pebbles to move a single pebble along a path to a vertex in B_2, but this leaves only one pebble on w and neither w nor any vertex in B_2 dominates the vertices of C_1. Hence, a domination cover cannot be reached from this pebble distribution. Thus, $\psi_R(G) = 18$.

Next assume that $\text{diam}(G) = 5$ and $\psi_R(G) = 18$. Let $\{u, v\}$ be a γ-set of G. Necessarily, $d(u, v) = 3$ and $\{N[u], N[v]\}$ is a partition of $V(G)$. Let $B_1 = T[u]$ and $B_2 = T[v]$. Let C_1 be the set of vertices in $N(u)$ are adjacent to a vertex in $N(v)$, and let C_2 be the set of vertices in $N(v)$ with a neighbor in C_1. Let $A_1 = N(u)\setminus(B_1 \cup C_1)$ and $A_2 = N(v)\setminus(B_2 \cup C_2)$. Since $u \in B_1$ and $v \in B_2$, $B_i \neq \emptyset$. By Claim **??**, $d(u, v) = 3$ for any $u \in B_1$ and $v \in B_2$, implying that $C_i \neq \emptyset$ for $i \in \{1, 2\}$. Note that for $i \in \{1, 2\}$, B_i induces a complete graph, $B_i \cap C_i = \emptyset$, and $[B_i, C_i]$ is full. Morever, if $A_i \neq \emptyset$, then $[A_i, B_i]$ is full for $i \in \{1, 2\}$. Also, $[A_1, B_2]$, $[A_1, C_2]$, $[A_1, A_2]$, $[B_1, C_2]$, $[B_1, B_2]$, $[B_1, A_2]$, $[C_1, B_2]$, and $[C_1, A_2]$ are empty.

Since $\text{diam}(G) = 5$, there is a vertex in A_1 that has no neighbor in C_1 and a vertex in A_2 that has no neighbor in C_2. Further note that no vertex in A_i dominates C_i, because such a vertex would be in B_i. Hence, $G \in \mathcal{F}$. This completes the proof of the theorem. □

We note that there are extremal graphs G having $\gamma(G) = 2$, $\psi_R(G) = 18$, and $\text{diam}(G) = 4$. For an example of such a graph, see Figure **??**.

Although Theorem **??** shows that the bound in Theorem **??** is sharp for graphs G with $\gamma(G) = 2$, we think the bound can be improved as follows for graphs having larger domination number.

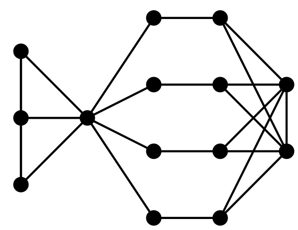

Figure 2: A diameter four graph G with $\gamma(G) = 2$ and $\psi_R(G) = 18$

Conjecture 19. *If G is a connected graph, then*

$$\psi_R(G) \leq 2\left(\frac{8^{\gamma(G)} - 1}{7}\right).$$

If Conjecture **??** is true, it is also sharp for paths P_n with $n \equiv 0 \pmod 3$.