Total Domination Cover Rubbling

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Abstract

Let G be a connected simple graph with vertex set V and a distribution of pebbles on the vertices of V. The total domination cover rubbling number of G is the minimum number of pebbles, so that no matter how they are distributed, it is possible that after a sequence of pebbling and rubbling moves, the set of vertices with pebbles is a total dominating set of G. We investigate total domination cover rubbling in graphs and determine bounds on the total domination cover rubbling number.

Keywords: Pebbling; Rubbling; Domination cover rubbling; Total domination cover rubbling

1 Introduction

Graph pebbling first appeared in the literature in a 1989 paper of Chung [3]. A variation of graph pebbling called rubbling was introduced by Belford and Sieben in [2] and studied for example in [6, 7]. Let G be a connected simple graph with vertex set $V = V(G)$. A *configuration* of pebbles assigns a nonnegative integer number of pebbles to each vertex of G. In graph rubbling, two moves, namely a pebbling move and a rubbling move, are allowed and defined as follows. Let f be a pebble configuration on a graph G. Let u and v be vertices of G such that $f(u) \geq 2$ and v is adjacent to u. A *pebbling move*, denoted $p(u \to v)$, removes two pebbles from u and places one on v. This defines a new pebble configuration, f' for which $f'(u) = f(u) - 2$, $f'(v) = f(v) + 1$, and $f'(z) = f(z)$ for $z \in V \setminus \{u, v\}$. Let w be a vertex of G, and let v and x be distinct vertices adjacent to w such that $f(v) \geq 1$ and $f(x) \geq 1$. A *rubbling move*, denoted $r(v, x \to w)$, removes one pebble from each of v and x and places one pebble on w, giving a new pebble configuration f' for which $f'(v) = f(v) - 1$, $f'(x) = f(x) - 1$, $f'(w) = f(w) + 1$, and $f'(z) = f(z)$ for $z \in V \setminus \{v, w, x\}$. A vertex v is *reachable* from a configuration f if there is a way to place a pebble on v using a sequence, possibly empty, of pebbling and rubbling moves. We note that graph pebbling only allows pebbling moves, while graph rubbling allows both pebbling and rubbling moves.

In this paper, we consider a version of graph rubbling where the goal is to move pebbles to a total dominating set. A set $S \subseteq V$ is a dominating set of G if every vertex of $V \setminus S$ is adjacent to a vertex of S. And S is a *total dominating set* of G, abbreviated TD-set, if every vertex of V is adjacent to a vertex in S. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a TD-set of G, and a TD-set of cardinality $\gamma_t(G)$ is a *γ_t-set* of G.

In graph pebbling (respectively, graph rubbling), one is generally concerned with determining the minimum number of pebbles so that no matter how they are placed on the vertices of a graph G, there will always be a sequence of pebbling (respectively, pebbling and rubbling) moves that can move at least one pebble to any specified vertex of G. Crull et al. [4] introduced a stricter concept of graph pebbling called *cover pebbling*. The *cover pebbling number* of a graph G is the minimum number k of pebbles needed so that from any initial pebble configuration of k pebbles, after a series of pebbling moves, it is possible to have at least one pebble on every vertex of G. Gardner et al. [5] considered a version of cover pebbling, relaxing the restriction that every vertex in G receive a pebble to requiring that the vertices of a dominating set receive a pebble. Hence, instead of the outcome of at least one pebble on every vertex, the result is at least one pebble on every vertex of a dominating set, that is, a *domination cover*. Beeler, et al. [1] extended domination cover pebbling to domination cover rubbling by allowing both the pebbling and rubbling moves. The *domination cover rubbling number* $\psi_R(G)$ of a graph G is the minimum number of pebbles, so that no matter how they are distributed, it is possible to obtain a domination cover from the pebble configuration after a sequence of pebbling and rubbling moves. We study the analogous concept for total domination as follows.

The *total domination cover rubbling number* $\psi_R^t(G)$ of a graph G is the minimum number of pebbles, so that no matter how they are distributed, it is possible to obtain a total domination cover from the pebble configuration after a sequence of pebbling and rubbling moves.

We shall use the following terminology and notation. Let G be a graph with vertex set V and edge set E. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. A *universal vertex* has degree $|V| - 1$. A vertex of degree one is called a *leaf*, and its neighbor is a *support vertex*. The *distance* between two vertices u and v in a connected graph G, denoted by $d(u, v)$, is the length of a shortest (u, v)-path in G. The eccentricity of a vertex v is the maximum distance from v to any other vertex in G, and the maximum eccentricity of the vertices of G is the *diameter* of G, denoted $\mathrm{diam}(G)$. A *peripheral vertex* of G has eccentricity equal to $\mathrm{diam}(G)$.

2 Small Values

Since for any graph G without isolated vertices $\gamma_t(G) \geq 2$, no pebble configuration of one or two pebbles on a single vertex can reach a TD-set. Hence, we have the following observation.

Observation 1. *For any non-trivial connected graph G, $\psi_R^t(G) \geq 3$.*

We next characterize the graphs G having small total domination cover rubbling number, namely, $\psi_R^t(G) = 3$. If S is a set of vertices of G that does not total dominate G, then we call S a *non-total dominating set*, abbreviated NTD-set.

Theorem 2. *A connected graph G has $\psi_R^t(G) = 3$ if and only if for every NTD-set S, where $G[S]$ is connected and $|S| \leq 3$, there exists a vertex $x \in V \setminus S$ such that x is adjacent to every vertex in S and $\{x, v\}$ is a TD-set of G for some $v \in S$.*

Proof. Assume that for every NTD-set S of G with $G[S]$ is connected and $|S| \leq 3$, there exists a vertex $x \in V \setminus S$ such that x is adjacent to every vertex in S and $\{x, v\}$ is a TD-set for some $v \in S$. Since $\psi_R^t(G) \geq 3$, it suffices to show that any pebbling configuration of three pebbles can reach a TD-set of G. Suppose first that all three pebbles are placed on a single vertex v. Now $S = \{v\}$ is a NTD-set of G, so there exists a vertex $x \in N(v)$ such that $\{x, v\}$ is a TD-set of G. Hence, the pebbling move $p(v \to x)$ reaches the total domination cover $\{x, v\}$. Note that since v is an arbitrary vertex, every vertex in V is in a γ_t-set of G.

Next assume that two pebbles are placed on a vertex u and one is placed on a vertex v. If $S = \{u, v\}$ is a TD-set of G, then we are finished. Assume that S is a NTD-set. If $G[S]$ is connected, that is, u and v are adjacent, then by assumption, there exists a vertex x in $N(u) \cap N(v)$ such that $\{u, x\}$ is a TD-set or $\{v, x\}$ is a TD-set of G. If $\{u, x\}$ is a TD-set, then the rubbling move $r(u, v \to x)$ reaches the total domination cover $\{u, x\}$. On the other hand, if $\{v, x\}$ is a TD-set of G, then the pebbling move $p(u \to x)$ reaches it. If u and v are not adjacent, then since v is in a γ_t-set, say $\{v, x\}$, of G, it follows that x is adjacent to u. Thus, the pebbling move $p(u \to x)$ reaches the total domination cover $\{v, x\}$.

Finally assume that a pebble is placed on each of three vertices u, v, and w. If $S = \{u, v, w\}$ is a TD-set of G, then we have a total domination cover. Hence, assume that S is a NTD-set of G. Again, if $G[S]$ is connected, then there is a vertex $x \in V \setminus S$ such that $x \in N(u) \cap N(v) \cap N(w)$ and x is in a TD-set with one of u, v, and w. Regardless of which one of u, v, and w dominates with x, a rubbling move from the other two vertices in S to x reaches a total domination cover. If $G[S]$ is not connected, then there is an isolated vertex, say u, in $G[S]$. Since u is in a γ_t-set, say $\{u, x\}$ of G, it follows that x is adjacent to both v and w. Then the rubbling move $r(v, w \to x)$, reaches the total domination cover $\{u, x\}$. Thus, in all cases, $\psi_R^t(G) \leq 3$, and so $\psi_R^t(G) = 3$.

Conversely, assume that $\psi_R^t(G) = 3$. Consider a configuration of three pebbles on V, and let S be the set of vertices receiving pebbles. Clearly, $|S| \leq 3$.

Suppose that $S = \{v\}$, that is, three pebbles are placed on one vertex v. Since $\gamma_t(G) \geq 2$, $S = \{v\}$ does not totally dominate G. Thus, exactly one pebbling move must reach a total domination cover. That is, a pebbling move $p(v \to x)$ for some $x \in N(v)$ results in a total domination cover $\{v, x\}$. Since v is an arbitrary vertex, every vertex of G is in a γ_t-set with one of its neighbors, the condition holds for all subsets S with $|S| = 1$.

Next assume that $S = \{u, v\}$. Without loss of generality, we many assume that two pebbles are placed on u and one pebble is placed on v. If S is a TD-set of G, then we have a total domination cover. If S is a NTD-set, then we must be able to reach a total domination cover after exactly one move. Thus, either the pebbling move $p(u \to x)$ results in a total domination cover $\{x, v\}$ where $x \in N(u) \cap N(v)$, or the rubbling move $r(u, v \to x)$ results in a total domination cover $\{u, x\}$ where $x \in N(u) \cap N(v)$. Hence, the condition holds for any set S with $|S| = 2$.

Finally, let $S = \{u, v, w\}$. Then one pebble is placed on each of u, v, and w. As before, if S is a TD-set of G, then we are finished. If S is a NTD-set of G, then since $\gamma_t(G) \geq 2$, we must be able to reach a total domination cover of two vertices in a single rubbling move. Note that no subset of S is a TD-set. Thus, there exists a vertex x in $V \setminus \{u, v, w\}$ such that $\{x, y\}$ is a total domination cover of G for some $y \in \{u, v, w\}$. Hence, x is adjacent to y. Moreover, x is adjacent to both vertices of $\{u, v, w\} \setminus \{y\}$ as a rubbling move from these two vertices reaches x. This completes the proof of Theorem 2. $\square$

Our next corollaries follow directly from Theorem 2.

Corollary 3. *If G is a nontrivial graph with a universal vertex, then $\psi_R^t(G) = 3$.*

Corollary 4. *For $n \geq 2$, the star $K_{1,n-1}$ and the complete graph K_n,*

$$\psi_R^t(K_{1,n}) = \psi_R^t(K_n) = 3.$$

3 Examples: Paths and Blow-ups

In this section, we present a formula for the total domination cover rubbling number of a path. We also present a method to obtain infinite families of graphs with equal total domination cover rubbling numbers. To obtain the total domination cover rubbling number of a path, we recall the following lemma from [1].

Lemma 5 (Lemma 5, [1]). *Let G be either a path or a cycle. Suppose that a domination cover is reachable via pebbling and rubbling moves from some configuration of pebbles on G. Then, a domination cover is reachable from this same configuration using only pebbling moves.*

We note here that the proof of the above lemma follows in precisely the same way if domination cover is replaced by total domination cover. This is because the domination property was not needed in the proof, and in fact the lemma remains true if we replace the domination cover with any subset of the vertices.

Theorem 6. *For any non-trivial path P_n,*

$$\psi_R^t(P_n) = 3(2^{n+1})\left(\frac{1 - 16^{-\lfloor \frac{n}{4} \rfloor}}{15}\right) + \phi(n),$$

with

$$\phi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ 2 & \text{if } n \equiv 1 \pmod 4 \\ 3 & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$, where $n \equiv r \pmod 4$. Applying Lemma 5, we are able to eliminate the need for rubbling moves to reach a total domination cover of the path. We first note that the result holds for P_n, where $n \in \{2, 3\}$.

Now assume that $n \geq 4$. We begin by placing all pebbles on v_1. Using pebbling moves, we see that we need to initially place 2^{n-2} pebbles on v_1 in order to dominate v_n and an additional 2^{n-3} pebbles on v_1 to dominate v_{n-1}. Similarly, we need to initially place $2^{n-2}+2^{n-3}+2^{n-6}+2^{n-7}+2^{n-10}+2^{n-11}+\cdots+2^{r+2}+2^{r+1}$ to totally dominate $\{v_n, v_{n-1}, \ldots, v_{r+1}\}$. If $r=0$, then it is possible to reach a total domination cover of P_n from this initial pebble configuration. If $r=1$, then we need an additional two pebbles on v_1 in order to place a pebble on v_2 and obtain a total domination cover. If $r=2$ or $r=3$, then we need an additional three pebbles on v_1 in order to place a pebble on v_1 and v_2 in order to obtain a total domination cover.

With this in mind, we define the function ϕ by

$$\phi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ 2 & \text{if } n \equiv 1 \pmod 4 \\ 3 & \text{if } n \equiv 2,3 \pmod 4. \end{cases}$$

Combining, we obtain the lower bound

$$\begin{aligned} \psi_R^t(P_n) &\geq 2^{n-2}+2^{n-3}+2^{n-6}+2^{n-7}+2^{n-10}+2^{n-11}+\cdots+2^{r+2}+2^{r+1}+\phi(n) \\ &= (2^{n-2}+2^{n-3})(1+2^{-4}+2^{-8}+\cdots+2^{r-n+4})+\phi(n) \\ &= (2^{n-2}+2^{n-3})\sum_{k=0}^{\lfloor \frac{n}{4}\rfloor -1} 2^{-4k}+\phi(n) \\ &= (2^{n-2}+2^{n-3})\left(\frac{1-2^{-4\lfloor \frac{n}{4}\rfloor}}{1-2^{-4}}\right)+\phi(n) \\ &= (2^{n+2}+2^{n+1})\left(\frac{1-16^{-\lfloor \frac{n}{4}\rfloor}}{15}\right)+\phi(n) \\ &= 3(2)^{n+1}\left(\frac{1-16^{-\lfloor \frac{n}{4}\rfloor}}{15}\right)+\phi(n). \end{aligned}$$

Now, we proceed by induction on n to show that

$$\psi_R^t(P_n) \leq 3(2)^{n+1}\left(\frac{1-16^{-\lfloor \frac{n}{4}\rfloor}}{15}\right)+\phi(n).$$

We have established the base cases for $n \in \{2,3\}$. It is clear that $\psi_R^t(P_4)=6$ and the result holds for this base case.

Assume that the statement is true for all P_m with $2 \leq m \leq n-1$. We want to show the statement holds for P_n. First, note that we can obtain a TD-set of $\{v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$ with at most $2^{n-2}+2^{n-3}=3(2)^{n-3}$ pebbles. So, we still need to totally dominate P_{n-4} with the remaining

$$\begin{aligned} 3(2)^{n+1}\left(\frac{1-16^{-\lfloor \frac{n}{4}\rfloor}}{15}\right)+\phi(n)-3(2^{n-3}) &= 3(2)^{n-4+1}\left(\frac{16-16^{-\lfloor \frac{n}{4}\rfloor+1}}{15}-1\right)+\phi(n) \\ &= 3(2)^{n-4+1}\left(\frac{1-16^{-\lfloor \frac{n-4}{4}\rfloor}}{15}\right)+\phi(n-4) \end{aligned}$$

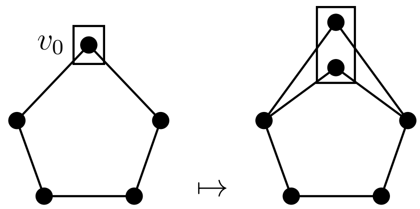

Figure 1: A blow-up $C_5(v_0; \overline{K_2})$

pebbles. Thus, by the inductive hypothesis, we have that this is enough pebbles to dominate P_{n-4}, and the result follows. $\square$

Let G be a connected graph and H be any arbitrary graph. For $v_0 \in V(G)$, we define the *blow-up* of G at v_0 by H, denoted $G(v_0; H)$, to be the graph with vertex set $V(G(v_0; H)) = (V(G)\backslash\{v_0\}) \cup V(H)$ and edge set

$$E(G(v_0; H)) = E(H) \cup \{vw : v \in N(v_0), w \in H\} \cup (E(G)\backslash\{vv_0 : v \in N(v_0)\}) .$$

An example of the blow-up $C_5(v_0, \overline{K}_2)$ is given in Figure 1. This construction gives us a simple way to produce infinite families of graphs having the same total domination cover rubbling number as the initial graph G.

Theorem 7. *For any graph H and connected graph G with $v_0 \in V(G)$,*

$$\psi_R^t(G(v_0; H)) = \psi_R^t(G).$$

Proof. Let $G' = G(v_0; H)$. To aid in the presentation, abusing notation slightly, we refer to the subgraph $G'[V(H)]$ in G' simply as H and to its vertices as $V(H)$. First, we show that $\psi_R^t(G') \leq \psi_R^t(G)$.

Consider any pebble configuration f of $\psi_R^t(G)$ pebbles on G'. From this configuration, we define a configuration of $\psi_R^t(G)$ pebbles on G given by the function

$$f_G(v) = \begin{cases} f(v) & \text{if } v \in V(G)\backslash\{v_0\} \\ \sum_{u \in H} f(u) & \text{if } v = v_0. \end{cases}$$

Since f_G defines a pebbling configuration of $\psi_R^t(G)$ pebbles on G, there is some sequence of pebbling/rubbling moves, denoted S, which reaches a total domination cover, say T, of G.

From T, we construct a TD-set $T' \subseteq V(G')$ of G' as follows. If $v_0 \notin T$, then let $T' = T$. If $v_0 \in T$, then we set $T' = (T\backslash\{v_0\}) \cup \{w_0\}$, for any vertex $w_0 \in V(H)$. Furthermore, we modify S on G to define a set of moves on G' as follows. For each move in S which involves

v_0, replace v_0 by w_0, otherwise, leave the move unchanged. Then, it follows that we can reach T' from this new set of moves. Thus, $\psi_R^t(G') \leq \psi_R^t(G)$.

Next, we show that $\psi_R^t(G') \geq \psi_R^t(G)$. Consider a configuration g of $\psi_R^t(G')$ pebbles on G. Fix a vertex of H, say w_0, in G'. Then, we define a configuration of $\psi_R^t(G')$ pebbles on G' given by the function

$$g_{w_0}(v) = \begin{cases} g(v) & \text{if } v \in V(G)\backslash\{v_0\} \\ g(v_0) & \text{if } v = w_0 \\ 0 & \text{otherwise.} \end{cases}$$

Since g_{w_0} is a pebbling configuration of $\psi_R^t(G')$ pebbles on G', there is some finite sequence of pebbling/rubbling moves S which reach a total domination cover of G'.

We modify the sequence of moves in S to eliminate moves involving vertices of $V(H) \setminus \{w_0\}$, while still reaching a total domination cover of G' as follows. First, we delete any moves from S which involve only vertices in $V(H)$. Second, for $u \notin V(H)$ and $v \in V(H)$, we replace the pebbling move $p(u \to v)$ (respectively, $p(v \to u)$) by $p(u \to w_0)$ (respectively, $p(w_0 \to u)$). Third, for $u, v \in V(H)$ and $w \notin V(H)$, we replace any rubbling move of the form $r(u, v \to w)$ by $p(w_0 \to w)$. Fourth, for $u, v \notin V(H)$ and $w \in V(H)$, we replace the rubbling move $r(u, v \to w)$ (respectively, $r(u, w \to v)$) by $r(u, v \to w_0)$ (respectively, $r(u, w_0 \to v)$). Finally, we consider the rubbling moves of the form $r(u, v \to w)$, where $u \in V(H) \setminus \{w_0\}$ and $v \notin V(H)$. If $w \in V(H) \setminus \{w_0\}$, then we delete this move. The only remaining possibility is that $w = w_0$. But we note that after all after these modifications have been made to S, no move places a pebble on a vertex in $V(H) \setminus \{w_0\}$. In particular, no pebble would be placed on u. Thus, we also delete any remaining rubbling moves in S that are in the form $r(u, v \to w_0\}$. We denote this new sequence of moves as S', and note that no move in S' involves a vertex of $V(H) \setminus \{w_0\}$.

Applying the new sequence of moves S' to the configuration g_{w_0}, we reach a total domination cover T' of G', which contains no elements of $V(H)\backslash\{w_0\}$. Using T', we define a TD-set T of G as follows. If $w_0 \notin T'$, then let $T = T'$. If $w_0 \in T'$, then let $T = (T'\backslash\{w_0\}) \cup \{v_0\}$.

Since no move in S' involves a vertex of $V(H) \setminus \{w_0\}$, we can adapt these moves to produce a sequence of pebbling/rubbling moves on G simply by replacing any appearance of w_0 by v_0. Then, the resulting sequence of moves reaches the total domination cover T. Hence, $\psi_R^t(G') \geq \psi_R^t(G)$, and so $\psi_R^t(G') = \psi_R^t(G)$, as desired. □

It is interesting to note that Theorem 7 does not hold for domination cover rubbling. To see this, consider the cycle $C_5 = (v_0, v_1, \ldots, v_4, v_0)$ for which $\psi_R(C_5) = 4$. But for the blowup $G = C_5(v_0; \overline{K}_2)$ (see Figure 1), placing four pebbles on v_1 makes it impossible to reach a domination cover, and hence $\psi_R(G) > 4$.

4 Bounds

In this section, we present bounds on $\psi_R^t(G)$.

Theorem 8. *If G is a graph with $\gamma_t(G) = 2$, then $\psi_R^t(G) \le 6$.*

Proof. Let $\{x, y\}$ be a γ_t-set of G. We now show that six pebbles are sufficient to achieve a total domination cover of G. Suppose that f is a configuration of six pebbles on the vertices of G. By the Pigeonhole Principle, any configuration of six pebbles to the vertices of G must place at least three pebbles in $N[x] \setminus \{y\}$ or at least three pebbles in $N[y] \setminus \{x\}$. Without loss of generality, assume that $N[x] \setminus \{y\}$ has at least three pebbles. If all six pebbles are placed on the vertices of $N[x] \setminus \{y\}$, then we can use a combination of pebbling moves to move at least three pebbles to x. Then the pebbling move $p(x \to y)$ completes a total domination cover. Suppose that five (respectively, four) pebbles are placed on the vertices of $N[x] \setminus \{y\}$ and the remaining one (respectively, two) pebble is placed on a vertex of $N[y] \setminus \{x\}$. In this case, we can move at least two pebbles to x using pebbling and rubbling moves. If necessary, we can then use a rubbling move to remove one pebble from x and one pebble from $N(y) \setminus \{x\}$ to place a pebble on y, completing a total domination cover. Finally, if three pebbles are placed on $N[x] \setminus \{y\}$ and another three pebbles are on $N[y] \setminus \{x\}$, then we can use a combination of pebbling and rubbling moves to place one pebble on each of x and y, completing a total domination cover. Hence, $\psi_R^t(G) \le 6$. $\square$

Of particular interest is to characterize those graphs G in which total domination number is $\gamma_t(G) = 2$ and $\psi_R^t(G) = 6$. Partial progress towards this result is given in the next couple of propositions.

Proposition 9. *Let G be a graph with $\gamma_t(G) = 2$. If there exists a vertex v of G such that no set consisting solely of v and at most two of its neighbors dominates G, then $\psi_R(G) = 6$.*

Proof. Let G be a graph with $\gamma_t(G) = 2$ and a vertex v satisfying the conditions of the hypothesis. By Theorem 8, $\psi_R^t(G) \le 6$. It suffices to show that there is an initial configuration of five pebbles that cannot reach a total domination cover.

Consider the configuration in which all five pebbles are placed on v. The only possible move is a pebbling move to place a pebble on a neighbor of v, say x. This move results in three pebbles on v and one pebble on x. Since $\{x, v\}$ is not a γ_t-set of G, another move must be made to reach a total domination cover. However, no matter which move is made, it must place a pebble on a vertex in $N(v)$. A rubbling move from v and x to another vertex in $N(v)$ does not make sense as it results in fewer pebbles on v, while leaving one pebble on exactly one neighbor of v. Thus, we consider the two other possibilites, which result in either one pebble on v and two pebbles on x, or one pebble on each of v, x, and y, where $x, y \in N(v)$. Since no set consisting of v and at most two of its neighbors dominates G, another move must be made. But then any move will result in exactly two vertices containing one pebble each. Furthermore, the two vertices containing pebbles are not adjacent or they are in $N[v]$. Since neither two non-adjacent vertices nor two vertices from $N[v]$ form a TD-set of G, we cannot reach a total domination cover. Thus, $\psi_R^t(G) \geq 6$, and so $\psi_R^t(G) = 6$. $\square$

Our next result follows directly from Proposition 9.

Corollary 10. *If G is a graph with $\gamma_t(G) = 2$ and $\operatorname{diam}(G) = 3$, then $\psi_R^t(G) = 6$.*

Proposition 11. *Let G be a graph with $\gamma_t(G) = 2$. If every vertex of G is in a γ_t-set or is adjacent to both vertices of a γ_t-set of G, then $\psi_R(G) \leq 5$.*

Proof. Let G be a graph satisfying the conditions of the lemma. We need to show that a total domination cover can be reached from any initial configuration f of five pebbles.

If at least four pebbles are placed on a single vertex v, then since v is in a γ_t-set or is adjacent to both vertices of a γ_t-set, either one or two pebbling moves from v can reach a total domination cover. Hence, we may assume that at most three vertices are on a single vertex. We consider all remaining possibilities.

Assume that f places three pebbles on a vertex v. If v is in a γ_t-set, then as before a pebbling move can reach a total domination cover. If v is not in a γ_t-set, then it dominates a γ_t-set, say $\{x, y\}$. Clearly, we are finished if either of the remaining two pebbles are placed on x or y as a pebbling move from v will result in pebbles on both x and y, and hence, a total domination cover. Thus, we may assume that the remaining two pebbles are placed in $V \setminus \{v, x, y\}$. Since $\{x, y\}$ is a γ_t-set of G, without loss of generality, x is adjacent to a vertex $u \in V \setminus \{v, x, y\}$ having at least one pebble. But then the moves $p(v \to y)$ and $r(u, v \to x)$ give a total domination cover.

Thus, we may assume that f assigns no single vertex more than two pebbles. Suppose that f places two pebbles on each of two vertices u and v. As before, either v is in a γ_t-set $\{v, x\}$ or v dominates a γ_t-set $\{x, y\}$. Clearly, we are finished after at most one move if there is a pebble on either x or y, so assume not. If $\{v, x\}$ is a γ_t-set, then u is adjacent to at least one of v and x. If u is adjacent to x, then the pebbling move $p(u \to x)$ gives a total domination cover. If u is adjacent to v, then the move $p(u \to v)$ followed by $p(v \to x)$ gives a total domination cover. We may assume that v is not in a γ_t, else we are finished. Thus, v dominates $\{x, y\}$. Since u is adjacent to at least one of x and y, say x, then the pebbling moves $p(v \to y)$ and $p(u \to x)$ give a total domination cover.

Suppose next that f places two pebbles on v and one pebble on each of three other vertices. If $\{v, x\}$ is a γ_t-set and a pebble is on x or on each of two vertices adjacent to x, then at most one rubbling move reaches a total domination cover. Hence, no pebble is on x and at least two of the vertices containing pebbles are adjacent to v. But then a rubbling move placing a third pebble on v following by a pebbling move from v to x yields a total domination cover. Thus, we may assume that v is not in a γ_t-set and that v dominates a γ_t-set $\{x, y\}$ of G. If either x or y has a pebble, then a pebbling move from v results in a total domination cover. Hence, the remaining three vertices with pebbles are in $V \setminus \{v, x, y\}$ and at least two of them are adjacent to the same vertex, say x, in $\{x, y\}$. Again, a rubbling move puts a pebble on x and a pebbling move from v puts one on y, reaching a total domination cover.

The only remaining possibility is that f assigns one pebble to each of five vertices. Let S be the set of vertices containing pebbles. If any subset S' of S a γ_t-set of G, then S is a total domination cover as every vertex of G is adjacent to a vertex in S'. Thus, we may assume that no two vertices of S form a γ_t-set. First assume that some vertex in S, say v, is in a γ_t-set $\{v, x\}$ of G. Then $x \notin S$, $x \in N(v)$, and every vertex in $S \setminus \{v\}$ is adjacent to at least one of x and v. If two or more vertices of $S \setminus \{v\}$ are adjacent to x, then a rubbling move to x reaches $\{v, x\}$, a total domination cover. Thus, at least three vertices of $S \setminus \{v\}$ are adjacent to v. That is, at most one vertex in $S \setminus \{v\}$, say u, is not adjacent to v. If u is adjacent to v, then two rubbling moves result in three pebbles on v. Then the pebbling move $p(v \to x)$ gives a total domination cover. If u is not adjacent to v, then u is adjacent to x and a rubbling move to v (from two of its three neighbors in S) followed by the rubbling move $r(v, u \to x)$ gives a total domination cover.

Thus, we may assume that no vertex of S is in a γ_t-set of G. Then for every $v \in S$, v dominates a γ_t-set $\{x, y\}$ and $\{x, y\} \subseteq V \setminus S$. The Pigeonhole Principle implies, without loss of generality, that at least two vertices of $S \setminus \{v\}$ are adjacent to x. If $S \subseteq N(x)$, then there exist rubbling moves to place at least two pebbles on x while leaving a pebble on v. But then the rubbling move $r(v, x \to y)$ reaches the total domination cover $\{x, y\}$. Thus, there is at least one vertex of S, say u, that is adjacent to y and not adjacent to x. But then a rubbling move from two neighbors of x in $S \setminus \{v\}$ followed by the move $r(v, u \to y)$ reaches a total domination cover. □

It is straightforward to see that if G is a graph with $\gamma_t(G) = 2$, then $\operatorname{diam}(G) \leq 3$. We note that the only graphs of diameter 1 are non-trivial complete graphs K_n, and by Corollary 4, $\psi_R^t(K_n) = 3$. By Corollary 10, graphs G with $\gamma_t(G) = 2$ and $\operatorname{diam}(G) = 3$ have $\psi_R^t(G) = 6$. Hence, all that remains to consider are graphs with $\gamma_t(G) = 2$ and $\operatorname{diam}(G) = 2$. Note that there are graphs G with $\gamma_t(G) = 2$, $\operatorname{diam}(G) = 2$, and $\psi_R^t(G) = 6$. One such example is the diameter-2 graph G given in Figure 2. We note that $\{u, v\}$ is a γ_t-set of G, so $\psi_R^t(G) \leq 6$ by Theorem 8. Consider a configuration of five pebbles in which all five pebbles are placed on u_1. It is easy to see that no series of moves from this starting position will yield a total domination cover of G, implying that $\psi_R^t(G) \geq 6$. Thus, $\psi_R^t(G) = 6$.

We conclude this section with the following bound concerning graphs having $\gamma_t(G) = 3$.

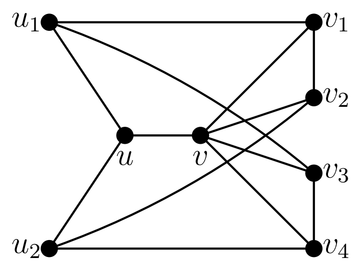

Figure 2: A graph G with $\gamma_t(G) = \mathrm{diam}(G) = 2$ and $\psi_R^t(G) = 6$

Theorem 12. *If G is a graph with $\gamma_t(G) = 3$, then $\psi_R^t(G) \leq 14$.*

Proof. Let $S = \{v_1, v_2, v_3\}$ be a γ_t-set of G. Then $G[S]$ is either a path P_3 or a triangle K_3. Assume, relabeling if necessary, that $v_1v_2, v_2v_3 \in E(G)$.

We consider a configuration f of fourteen pebbles on G. Since S is a γ_t-set of G, every pebble is in $N[v_i]$ for some $i \in [3]$. By the Pigeonhole Principle at least one closed neighborhood $N[v_i]$ receives at least five pebbles under f. Clearly, it requires at least as many pebbles to reach S from $N[v_1]$ (respectively, $N[v_3]$) as it does from $N[v_2]$. Thus, we assume, without loss of generality, that $N[v_1]$ has at least five pebbles under f.

If there are at least twelve pebbles on $N[v_1]$, then there is a sequence of pebbling and rubbling moves which places at least six pebbles on v_1. Repeating $p(v_1 \to v_2)$ twice results in at least two pebbles on each of v_1 and v_2. If three or more pebbles are on either v_1 or v_2, then either one or two pebbling moves can reach v_3 while leaving at least one pebble on each of v_1 and v_2. Also, if v_3 has a pebble, then we are finished. Hence, we may assume that there are exactly two pebbles on v_1, exactly two on v_2, and no pebbles on v_3, implying that there are at least two pebbles remaining on vertices in $V \setminus S$. If at this point, a vertex x in $N(v_3) \setminus S$ contains a pebble, then the rubbling move $r(v_2, x \to v_3)$ gives a total domination cover. Moreover, if a vertex $x \in N(v_2) \setminus S$ contains a pebble, then the rubbling move $r(x, v_1 \to v_2)$ gives three pebbles on v_1, and we can reach a total domination cover as before. Thus, we may assume that the remaining two pebbles are in $N(v_1) \setminus S$. Then either a rubbling or pebbling move can place at least three pebbles on v_1, and we can reach a total domination cover as before.

Henceforth, we may assume that $N[v_1]$ contains at most eleven pebbles, for otherwise, the result holds. By the Pigeonhole Principle, there are at least two pebbles on $N[v_2] \setminus N[v_1]$ or at least two pebbles on $N[v_3] \setminus N[v_1]$. If there are at least ten pebbles in $N[v_1]$, then there is a sequence of pebbling and rubbling moves using at most ten of these pebbles which places at least five pebbles on v_1. If there are at least two pebbles on $N[v_2] \setminus N[v_1]$, then by using two of these pebbles in a single pebbling or rubbling move, we can place at least one pebble on v_2. Thus, performing $p(v_1 \to v_2)$ twice and then $p(v_2 \to v_3)$, we have left at least one pebble on each of v_1 and v_2 and have placed a pebble on v_3, reaching a total domination cover. If there are at least two pebbles on $N[v_3] \setminus N[v_1]$, then at most one pebbling or rubbling move can place at least one pebble on v_3. Thus, after the pebbling move $p(v_1 \to v_2)$, we reach a total dominaton cover.

Henceforth, we may assume that $N[v_1]$ contains at most nine pebbles, for otherwise, the result holds. By the Pigeonhole Principle, there are at least three pebbles on $N[v_2] \setminus N[v_1]$ or at least three pebbles on $N[v_3] \setminus N[v_1]$. If there are eight or nine pebbles on $N[v_1]$, then, using at most eight of these pebbles, there is a sequence of pebbling and rubbling moves which places at least four pebbles on v_1. If there are at least three pebbles on $N[v_2] \setminus N[v_1]$, then using at most three of the pebbles on v_1 along with these pebbles, there is a sequence of pebbling and rubbling moves which places at least three pebbles on v_2. Thus, performing $p(v_2 \to v_3)$, we reach a total domination over. If there are at least three pebbles on $N[v_3] \setminus N[v_1]$, then at most one pebbling or rubbling move can result in at least one pebble on v_3. Thus, after the pebbling move $p(v_1 \to v_2)$, we reach a total dominatiion cover.

Henceforth, we may assume that $N[v_1]$ contains at most seven pebbles, for otherwise, the result holds. By the Pigeonhole Principle, there are at least four pebbles on $N[v_2] \setminus N[v_1]$ or at least four pebbles on $N[v_3] \setminus N[v_1]$. If there are six or seven pebbles on $N[v_1]$, then, using at most six of these pebbles on $N[v_1]$, there is a sequence of pebbling and rubbling moves which places at least three pebbles on v_1. If there are at least four pebbles on $N[v_2] \setminus N[v_1]$, then there is a sequence of pebbling and rubbling moves which places at least two pebbles on v_2. Thus, performing $p(v_1 \to v_2)$ and $p(v_2 \to v_3)$, we reach a total domination cover. If there are at least four pebbles on $N[v_3] \setminus N[v_1]$, then at most one pebbling or rubbling move using these pebbles gives at least one pebble on v_3. Thus, after the pebbling move $p(v_1 \to v_2)$, we reach a total domination cover.

Finally, we may assume that $N[v_1]$ contains exactly five pebbles, for otherwise, the result holds. Then, using at most four of the these pebbles, there is a sequence of pebbling and rubbling moves which places at least two pebbles on v_1. Again, by the Pigeonhole Principle, there are at least five pebbles on $N[v_2] \setminus N[v_1]$ or at least five pebbles on $N[v_3] \setminus N[v_1]$. If there are at least five pebbles on $N[v_2] \setminus N[v_1]$, then using a single pebble from v_1, there is a sequence of pebbling and rubbling moves which places at least three pebbles on v_2. Thus, performing $p(v_2 \to v_3)$, we reach a total domination cover. If there are at least five pebbles on $N[v_3] \setminus N[v_1]$, then after a sequence pebbling or rubbling moves we can place at least two pebbles on v_3. Thus, the rubbling move $r(v_1, v_3 \to v_2)$ reaches a total domination cover. This completes the proof. $\square$

5 Total Domination Cover Rubbling Versus Dominaton Cover Rubbling

We next investigate total domination cover rubbling versus domination cover rubbling. Since every total dominating set is a dominating set, it is immediate from the definition that $\psi_R(G) \le \psi_R^t(G)$ for every connected graph G with no isolated vertex. We show that the total domination cover rubbling number is at most three times the domination cover rubbling number.

Theorem 13. *If G is a connected graph with no isolated vertex, then $\psi_R^t(G) \le 3\psi_R(G)$.*

Proof. Let $k = \psi_R(G)$ and consider an arbitrary initial pebble configuration with $3k$ pebbles. We partition the set S of $3k$ pebbles into three subsets S_1, S_2, and S_3, each consisting of k pebbles. Since $\psi_R(G) = k$, there is a domination cover D_i from the pebble configuration f_i corresponding to S_i after a sequence of pebbling and rubbling moves for each $i \in [3]$. Let $D = D_1 \cup D_2 \cup D_3$ and let f be the pebbling configuration $f = \sum_{i=1}^{3} f_i$. Let I be the set of isolated vertices, if any, in the graph $G[D]$. Since each set D_i is a dominating set of G, we note that $I \subseteq D_1 \cap D_2 \cap D_3$. Each vertex in I is covered by at least one pebble resulting from the pebble configuration f_i corresponding to the set S_i for each $i \in [3]$, implying that each vertex in I is covered by at least three pebbles. For each vertex $v \in I$, let v' be an arbitrary neighbor of v, and so the vertex v' belongs to the set $V \setminus D$. Further, let $I' = \cup_{v \in I}\{v'\}$. We now apply the pebbling move $p(v \to v')$ which removes two pebbles from v and places one on v' for each vertex $v' \in I$. This defines a new pebble configuration, f', such that $f'(v) = f(v) - 2 \ge 1$ and $f'(v') \ge f(v) + 1 = 1$ if $v \in I$ and $f'(x) = f(x)$ for all vertices $x \in V \setminus (I \cup I')$. The pebble configuration f' produces a total domination cover, namely $D \cup I'$, from the initial pebble configuration of $3k$ pebbles, implying that $\psi_R^t(G) \le 3k = 3\psi_R(G)$. $\square$

We remark that the upper bound of Theorem 13 is best possible as may be seen by taking G to be a complete graph K_n where $n \ge 2$. In this case, $\psi_R(G) = 1$ and $\psi_R^t(G) = 3 = 3\psi_R(G)$.

6 Stacking in Trees

Recall that the periphery of a graph is the set of all vertices whose eccentricity equals its diameter. For trees, the peripheral vertices are leaves. We believe that the total domination rubbling number of a tree is equal to the number of pebbles necessary to reach a total domination cover from an initial configuration placing all the pebbles on a single peripheral leaf. Hence, we make the following conjecture.

Conjecture 14. (Stacking Theorem for Trees) *In order to determine the total domination cover rubbling number of a tree, it is sufficient to consider only pebble configurations which place all pebbles on a single peripheral vertex.*

Before proceeding, we should justify why the above conjecture is only made for trees. The prism $G \square P_2$ is the graph obtained from two copies of the graph G, say G_1 and G_2, with the same vertex labelings by adding edges such that each vertex of G_1 is adjacent to the vertex of G_2 which has the same label. Just as in [1], consider the graph $G = K_4 \square P_2$. Denote the vertices of one copy of K_4 in this graph by $\{u_1, u_2, u_3, u_4\}$ and the vertices of the other copy of K_4 by $\{v_1, v_2, v_3, v_4\}$. First, as every vertex of G is incident to a dominating edge, we can achieve a total domination cover by placing three pebbles on any vertex of G. Second, consider the NTD-set $S = \{u_1, u_2\}$. Note, the only vertices adjacent to both u_1 and u_2 are u_3 and u_4. Since neither $\{u_i, u_3\}$ nor $\{u_i, u_4\}$ is not a TD-set for $i \in [2]$, applying Theorem 2, we have that $\psi_R^t(G) > 3$. Thus, in order to obtain $\psi_R^t(G)$, it is necessary to consider configurations which begin with pebbles on more than one vertex.

As evidence for Conjecture 14, we present results on the total domination cover rubbling number for trees of diameter three (i.e., double stars) and trees of diameter four. In both of these cases, the worst case is obtained by stacking on a single peripheral vertex. The double star $S_{r,s}$ is the tree with exactly two non-leaf vertices x and y where x is adjacent to $r \geq 1$ leaves and y is adjacent to $s \geq 1$ leaves. Note that $\{x, y\}$ is a TD-set for the double star. Since $\gamma_t(S_{r,s}) = 2$ and its diameter is three, our next result follows immediately from Theorem 10.

Corollary 15. *For the double star $S_{r,s}$, $\psi_R^t(S_{r,s}) = 6$.*

Note that we have established the total domination cover rubbling number for trees of diameter two (namely stars) and trees of diameter three (namely double stars). Thus, a natural next step would be to determine the total domination cover rubbling number for trees of diameter four. Any tree of diameter four can be obtained by appending pendant vertices to the existing vertices of $K_{1,n}$ for $n \geq 2$. Label the center of the star as x and its leaves as $y_1, ..., y_n$. Suppose that we append $c \geq 0$ pendant vertices to x, namely $x_1, ..., x_c$, and $a_i \geq 1$ pendant vertices to y_i, namely $y_{i,1}, ..., y_{i,a_i}$ for $1 \leq i \leq n$. Note that for $i \neq j$ and for any ℓ and m, the vertices $y_{i,\ell}$, y_i, x, y_j, and $y_{j,m}$ induce a path of length four, and this construction gives all trees of diameter four. The resulting graph will be denoted $K_{1,n}(c; a_1, ..., a_n)$, and without loss of generality, we will assume that $a_1 \geq a_2 \geq ... \geq a_n$. An example is shown in Figure 3.

To aid in determining the total domination cover rubbling number for such trees, it is helpful to give the analogous result from [1].

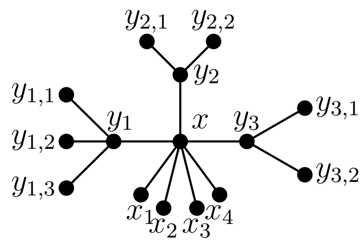

Figure 3: The graph $K_{1,3}(4; 3, 2, 2)$

Corollary 16. [1] *If $G = K_{1,n}(c; a_1, ..., a_n)$, where $a_1 \geq a_2 \geq \ldots \geq a_n$, then*

$$\psi_R(G) = \begin{cases} 8n-7 & \textit{if } c = 0 \textit{ and } a_1 = 1, \\ 8n-6 & \textit{if } c = 0 \textit{ and } a_1 \geq 2, \\ 8n-3 & \textit{if } c \geq 1 \textit{ and } a_1 = 1, \\ 8n-2 & \textit{if } c \geq 1 \textit{ and } a_1 \geq 2. \end{cases}$$

Note the set $\{x, y_1, ..., y_n\}$ is a γ_t-set of the tree $K_{1,n}(c; a_1, ..., a_n)$. This set is also a minimum dominating set and the domination cover reached for $K_{1,n}(c; a_1, ...a_n)$, when $c \geq 1$ and $a_1 \geq 2$. Hence, the result follows immediately from Corollary 16.

Corollary 17. *If $G = K_{1,n}(c; a_1, ..., a_n)$, then $\psi^t_R(G) = 8n - 2$.*

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