1. LIMITS, TANGENTS, AND RATES OF CHANGE

We experience our world through our five senses, and yet there is more to our world than our five senses can reveal. We can extend our five senses with tools such as microscopes and telescopes, and yet there are many things that still remain hidden from us. We cannot see electricity. We cannot hear the wind blowing on Mars. We cannot taste, smell or feel the building blocks of matter. Even with the most advanced tools known to man, our five senses simply cannot reveal all there is to know about world around us.

We need a sixth sense, one which will allow us to see inside an atom or hear the pitch of a radio wave. Since the late seventeenth century, that sixth sense has been Calculus. No instrument in the world will allow us to see a planet outside of our solar system, and yet we know of several such planets. Their presence was inferred using Calculus. No extension of our five senses will allow us to explore the twists and turns in our world economy, but with our sixth sense—Calculus—we can make forecasts and predict recessions. We cannot see the wind. But with Calculus, we can describe it, model it and make predictions as to where it is going and where it has been.

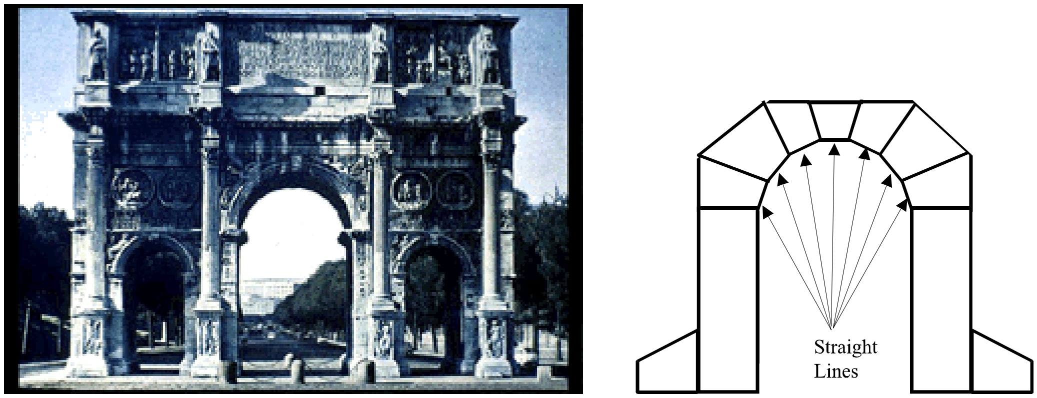
In this book, we learn to use the sixth sense of Calculus to investigate our world. We explore the role of Calculus in modern science and mathematics, and we revisit the discoveries of the past to see how Calculus has grown over the centuries. This chapter begins that exploration by laying the foundation upon which subsequent chapters will be built.

1.1 Tangent Lines

Introduction to Calculus

We have known calculus all of our lives. We use it daily to make sense of our world. When we measure short distances on the earth's surface with straight lines—yardsticks, tape measures, etc.—then we are using calculus. When we ignore the roundness of the earth with phrases like "a straight road" and "a flat field," then again we are using calculus. THE EARTH IS FLAT! ...or so we imagine when the distances being considered are small. This simple idea is the essence of Calculus.

In fact, the concepts underlying calculus have been used throughout history. Since antiquity, architects have used collections of short line segments to imply more complicated curves, such as when bricks and blocks with perfectly straight sides are used to construct semicircular arches.

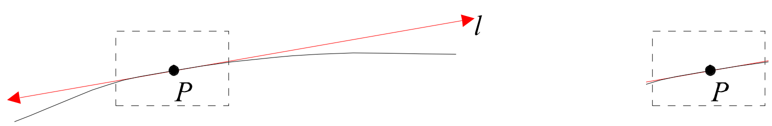


1-1: An arch is often formed by short line segments

That is, since antiquity, people have approximated small sections of curves with straight lines.

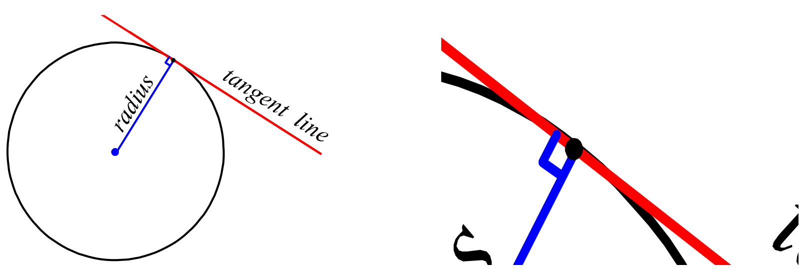
> **A Fundamental Concept in Calculus**
> Some curves can be divided into sections in which each section is nearly the same as a segment of a straight line

In calculus, curves are approximated by *tangent lines,* where a line l is tangent to a curve at the *point of tangency* P if the line and the curve are "practically the same" for small sections of the curve containing P.



1-2: The line l is tangent to the curve at point P.

Indeed, the Greek mathematician Euclid based much of his geometry on the fact that a tangent line to a circle is perpendicular to the radius—i.e., to the line through the origin and the point of tangency.



1-3: Tangent Line to a Circle

Notice again that the tangent line to a circle is "practically the same" as a small section of the circle which contains the point of tangency.

Check your Reading *What are some examples of a curve on the earth's surface being considered "practically the same" as a tangent line?*

Tangent Lines to Polynomials

Tangent lines are related to the fact that when h is close to 0, then higher powers like h^2, h^3, and so on are much, much closer to 0. For example, if $h = 0.001$, then

$$h^2 = 0.000001$$

which is 1000 times smaller than h. Likewise, if $h = 0.0001$, then

$$h^3 = 0.000000000001$$

which is much, much closer to 0 than h itself is

> **Neglible Powers of h**
> If h is sufficiently close to 0, then h^2, h^3, h^4, and so on are much, much closer to 0 than h is and thus can often be ignored.

If $f(x)$ is a polynomial, which is a function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

then this concept can be used to calculate a tangent line to the curve $y = f(x)$ at a given input $x = p$.

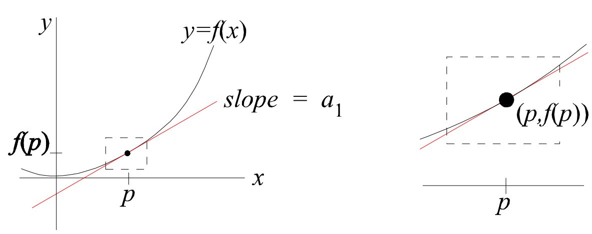
To do so, we let $x = p + h$, where h is assumed to be close to 0. Thus, $f(p+h)$ as a function of h is a polynomial of the form

$$f(p+h) = a_0 + a_1 h \quad + \quad \text{"higher powers of } h\text{"} \tag{1.1}$$

Since the higher powers are negligible, $y = f(p+h)$ is practically the same as $y = a_0 + a_1 h$. Finally, $x = p + h$ implies that $h = x - p$, so that $y = a_0 + a_1 h$ becomes

$$y = a_0 + a_1 (x - p)$$

Near the point of tangency $(p, f(p))$, the polynomial curve $y = f(x)$ is practically the same as the tangent line $y = a_0 + a_1 (x - p)$.



1-4: $y = f(x)$ is almost a straight line at $(p, f(p))$

Let's look at some examples.

EXAMPLE 1 Find the equation of the tangent line to $y = x^2$ when $p = 1$.

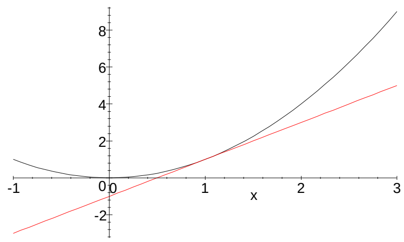
Solution: Since $p = 1$, we let $x = 1 + h$ to obtain $y = (1+h)^2$. Expanding the result leads to

$$y = (1+h)^2 = 1 + 2h + h^2$$

Since h^2 is negligible, the tangent line is $y = 1 + 2h$, which because $h = x - 1$ becomes

$$y = 1 + 2(x - 1) = 2x - 1$$

Thus, $y = 2x - 1$ is the tangent line to $y = x^2$ at $(1, 1)$, as is shown in figure 1-5:



1-5: $y = x^2$ versus $y = 2x - 1$

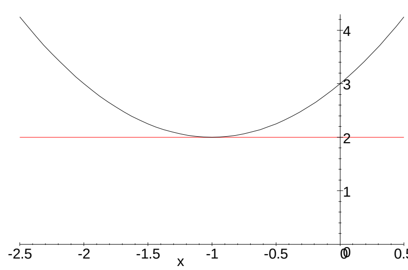
If $a_1 = 0$ in (1.1), then the tangent line is *horizontal*.

EXAMPLE 2 Find the equation of the tangent line to $y = x^2 + 2x + 3$ when $p = -1$.

Solution: Since $p = -1$, we let $x = -1 + h$ to obtain

$$\begin{aligned} y &= (-1 + h)^2 + 2(-1 + h) + 3 \\ &= 1 - 2h + h^2 - 2 + 2h + 3 \\ &= 2 + h^2 \end{aligned}$$

That is, $y = 2 + h^2$, but since h^2 is negligible, the tangent line is simply $y = 2$, which is a horizontal line (slope is 0). The parabola and its tangent line are shown in figure 1-6:



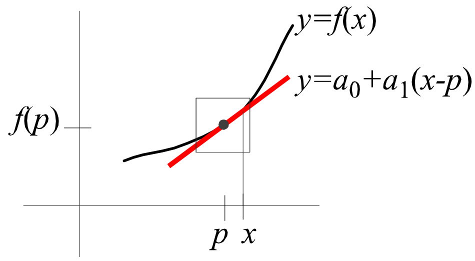
1-6: $y = x^2 + 2x + 3$ versus $y = 2$

Check your Reading *Why might horizontal tangent lines be important? (See figure 1-6 for help)*

In summary, to find the equation of the tangent line to the graph of an n^{th} degree polynomial when $x = p$, we use the following steps:

1. Let $x = p + h$ for h close to 0 and expand to obtain a polynomial in h of the form
$$y = a_0 + a_1 h + a_2 h^2 + \ldots + a_n h^n$$
2. Since h^2, h^3, h^4, and so on are negligible when h is close to 0, the polynomial is nearly the same as
$$y = a_0 + a_1 h$$
for x close to p.
3. Since $x = p + h$ implies that $h = x - p$, the *equation of the tangent line to* $y = f(x)$ *when* $x = p$ is
$$y = a_0 + a_1 (x - p)$$

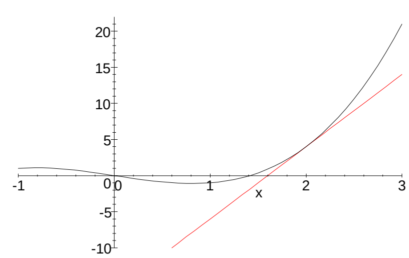
It follows that when x is close to p, which is when h is close to 0, then the tangent line is a good approximation of the curve itself.



1-7: Tangent is a good approximation to curve when x is close to p

EXAMPLE 3 Find the equation of the tangent line to $y = f(x)$ at $p = 2$ when $f(x) = x^3 - 2x$?

Solution: Substituting $x = 2 + h$ and expanding leads to
$$y = (2 + h)^3 - 2(2 + h) = 4 + 10h \quad + \quad 6h^2 + h^3$$
Since $6h^2 + h^3$ is negligible for h close to 0, the curves is practically the same as
$$y = 4 + 10h$$
Since $x = 2 + h$ implies $h = x - 2$, the equation of the tangent line is
$$y = 4 + 10(x - 2) = 10x - 16$$
That is, $y = 10x - 16$ is tangent to $y = x^3 - 2x$ at $(2, 4)$. Moreover, since the tangent line has a slope of 10, the derivative is $f'(2) = 10$.



1-8: $y = 10x - 16$ is tangent to $y = x^3 - 2x$ at $(2, 4)$

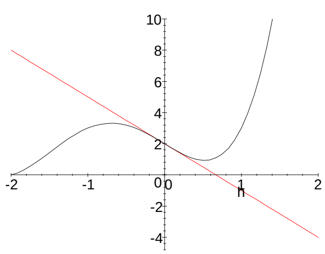
In particular, the tangent line is the line that becomes *ever more indistinguishable* from the curve as we choose shorter and shorter sections containing the point of tangency. A tangent line may even *cross the curve*, just as long as it does so by becoming arbitrarily close to the curve itself.

EXAMPLE 4 Find the equation of the tangent line to $y = x^4 + 3x^3 - 3x + 2$ when $p = 0$. Then graph both the curve and its tangent line over the intervals $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ and $[-0.2, 0.2]$.

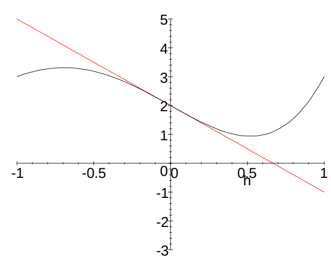
Solution: Since $x = 0 + h$, the curve is of the form

$$y = \overbrace{2 - 3h}^{\text{linear part}} + \overbrace{3h^3 + h^4}^{\text{higher powers of } h}$$

Since $x = h$, the tangent to the curve when $p = 0$ is $y = 2 - 3x$. Graphs of the curve and the line are shown over $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ and $[-0.2, 0.2]$ in figure 1-8a through 1-8d, respectively.



1-9a



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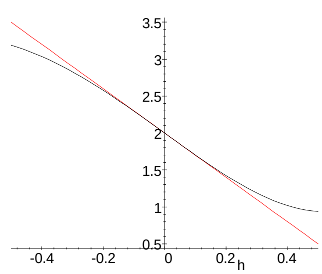

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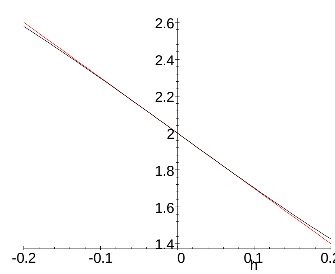


1-9d

As the intervals become shorter and shorter, the curve becomes more and more like the straight line. Thus, the line is tangent to the curve, even though it crosses over the curve itself.

Check your Reading *Can a line be tangent to a curve WITHOUT EVER INTERSECTING THE CURVE?*

In an xy-coordinate system, a "rise" is a change in the y coordinate and a "run" is a change in the x-coordinate. The slope of a line is a ratio of a "rise" to a "run", which means that the slope of the tangent line a_1 satisfies

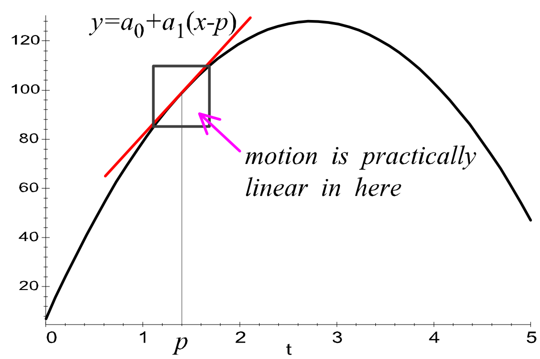
$$a_1 = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x}$$

That is, the slope a_1 is the *rate of change* of the tangent line. Since $y = f(x)$ is nearly the same as its tangent line at a point of tangency $(p, f(p))$, the rate of change of the function itself over a short interval $[p, p+h]$ is practically the same as a_1. That is, the *slope of the tangent line* at $x = p$ is *the rate of change of the function at* $x = p$.

For example, if a ball is thrown upward from an initial height of 7 feet with an initial velocity of 88 feet per second, then the height $r(t)$ of the ball at time t is given by

$$r(t) = 7 + 88t - 16t^2 \tag{1.2}$$

Over a short period of time, the motion of the ball is practically a straight line with slope a_1.



1-10: Over short time intervals, motion is practically linear with slope $r'(p)$.

Since the slope is the rate of change of the tangent line, a_1 is the *rate of change* of the ball at p seconds, which is also known as the *velocity* of the ball. That is, the slope of the tangent line tells us about how fast the ball is traveling at time $t = p$ seconds.

EXAMPLE 5 If $r(t) = 7 + 88t - 16t^2$ is the height in feet of a ball at time t in seconds, then how fast is the ball traveling at time $p = 0$ seconds? How fast at $p = 1$ seconds?

Solution: If $p = 0$, then we let $t = 0 + h$ and

$$r(0+h) = 7 + 88h - 16h^2$$

Since h^2 is negligible, we obtain $y = 7 + 88h$, so that the tangent line is

$$y = 7 + 88(t - 0)$$

which has a slope of 88. Thus, at $p = 0$, the ball has a velocity of 88 feet per second.

If $p = 1$, then we let $t = 1 + h$ and

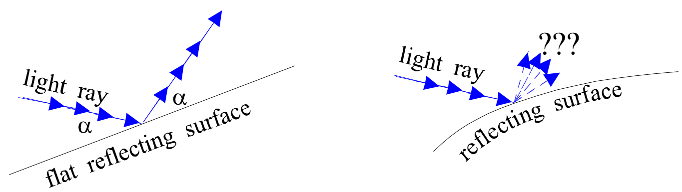
$$r(1+h) = 7 + 88(1+h) - 16(1+h)^2 = 79 + 56h - 16h^2$$

Since h^2 is negligible, we obtain $y = 79 + 56h$, so that the tangent line is

$$y = 79 + 56(t-1)$$

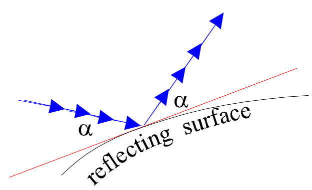
which has a slope of 56. Thus, at $p = 1$ seconds, the ball has a velocity of 56 feet per second.

Rates of change are not the only application of tangent lines. For example, the law of reflection says that if a ray of light reflects off of a flat surface, the angle of incidence is equal to the angle of reflection. But what is the direction of a reflected ray off of a *curved surface?*



1-11: Cross-sections of a flat and a curved surface

To answer that question, we zoom until the curve formed by the cross section of the reflecting surface can be replaced by its tangent line.



1-12: Ray of light reflected off of tangent line

We then compute the angle of reflection off of the flat tangent rather than the curved surface.

Exercises

Find the tangent line to $y = f(x)$ and identify $f'(p)$ for the given value of p. Graph both the curve and the line to verify tangency.

1. $f(x) = 3x^2, \quad p = 1$
2. $f(x) = -x^2, \quad p = 1$
3. $f(x) = x^2 - 1, \quad p = 1$
4. $f(x) = x^2 + 1, \quad p = 2$
5. $f(x) = x + 3x^2, \quad p = 2$
6. $f(x) = x^2 + 3x, \quad p = 1$
7. $f(x) = 3x + 2, \quad p = 1$
8. $f(x) = 3x + 2, \quad p = 2$
9. $f(x) = 1 + 3x - x^2, \quad p = 1$
10. $f(x) = 2 - 3x + x^2, \quad p = 2$
11. $f(x) = x^3 + 3$, $p = -2$
12. $f(x) = x^3 + 3x + 1$, $p = 2$
13. $f(x) = x^3 - 3x$, $p = 1$
14. $f(x) = x(x-1)^2$, $p = 1$

Grapher[1]**:** *Find the tangent line when $p = 0$, and then graph both the curve and the line over the intervals $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ and $[-0.1, 0.1]$. On which of*

[1] **Grapher** exercises require the use of a graphing calculator or computer to produce the graph of a function.

the intervals would you say that the curve and the line are indistinguishable?

15.	$y = 1 + 3x + x^2$	16.	$y = 2 - 3x + x^2$
17.	$y = x + 3x^2$	18.	$y = (2x+1)^2$
19.	$y = (3+x)(2-x)$	20.	$y = (3+x)(2-x)$
21.	$y = x\left(1+x^2\right)$	22.	$y = 1 + x + x^3$
23.	$y = 1 + x^3 + x^5$	24.	$y = 2x^2 + x + 1$

Each of the following functions represents the height $r(t)$ in feet of an object at time t in seconds. Find the velocity of the object at the given time p by finding the slope of the tangent line to the graph of the curve.

25.	$r(t) = 64 - 16t^2$ *at* $p = 1$ sec	26.	$r(t) = 64 - 16t^2$ *at* $p = 0$ sec
27.	$r(t) = 96t - 16t^2$ *at* $p = 0$ sec	28.	$r(t) = 96t - 16t^2$ *at* $p = 1$ sec
29.	$r(t) = 64 + 4t - 16t^2$ *at* $p = 1$ sec	30.	$r(t) = 32 + 12t - 16t^2$ *at* $p = 1$ sec

31. Grapher: Graph the following lines along with the curve

$$f(x) = \sqrt{4 + x + x^2}$$

on the intervals $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ *and* $[-0.1, 0.1]$. Which of the lines is tangent to the curve when $p = 0$?

(a) $y = \frac{x}{2} + 2$ (b) $y = \frac{x}{3} + 2$ (c) $y = \frac{x}{4} + 2$

32. Grapher: Graph the following lines along with the curve

$$f(x) = \sqrt{4 + x + x^2}$$

on the intervals $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ *and* $[-0.1, 0.1]$. Which of the lines is tangent to the curve when $p = 0$?

(a) $y = \frac{x}{2} + 2$ (b) $y = \frac{x}{3} + 2$ (c) $y = \frac{x}{4} + 2$

33. Grapher: In (a)-(c), only one of the lines is tangent to the given curve when $p = 0$. Graph both on the intervals $[-2, 2]$, $[-1, 1]$, $[-0.5, 0.5]$ *and* $[-0.1, 0.1]$. In which is the line tangent to the given curve when $p = 0$? (Be sure to use radians)

(a) $y = \cos(x)$, $y = x$
(b) $y = (1+x)^{10}$, $y = 1 + x$
(c) $y = \sqrt{1+x}$, $y = 1 + x/2$

34. Find the tangent lines to $y = x^2 - 2x + 3$ at $p = 0$, $p = 1$, $p = 2$, and $p = 3$. Then graph **only** the tangent lines on the interval $[-1, 4]$. What information might you infer about $y = x^2 - 2x + 3$ from these 4 tangent lines?

35. Find the tangent lines to the following curves when $p = 0$. The letters k and a are called *parameters* and should be treated as if they have a fixed numerical value. For example, if

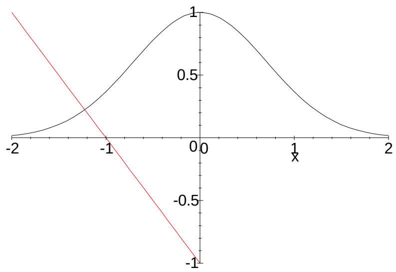
$$y = 1 + ax + x^2$$

then the tangent line when $p = 0$ is $y = 1 + ax$.

(a) $y = (x+a)^2$
(b) $y = (x+a)^3$
(c) $y = \left(1 + \frac{x}{k}\right)^2$
(d) $y = (1-kx)(1+kx)$

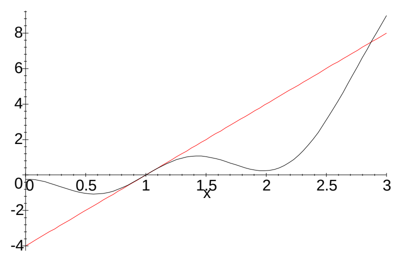
36. Show that in terms of the parameter p, the set of all tangent lines to $y = x^2$ at $x = p$ are given by $y = 2px - p^2$.

37. There are many myths and misunderstandings surrounding tangent lines. One of the most prevalent is that if a line intersects a curve at only one point, then it is a tangent line. The line in figure 1-13 intersects the curve at only one point. Why would we not consider this line to be a tangent line to the curve?



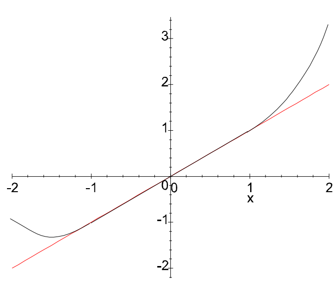
1-13: Is this a tangent line?

38. A related myth is that a tangent line cannot intersect a curve more than once. However, the line in figure 1-14 is tangent to the curve at $x = 1$,and yet it crosses the curve more than once. Explain why we would nonetheless consider it to be a tangent line to the curve at $x = 1$.



1-14: Is this a tangent line?

39. One last myth is that a line must be tangent to a curve at only one point. However, in figure 1-15, the curve in black is tangent to the red line for all x in $[-1, 1]$.



1-15: Is this a tangent line?

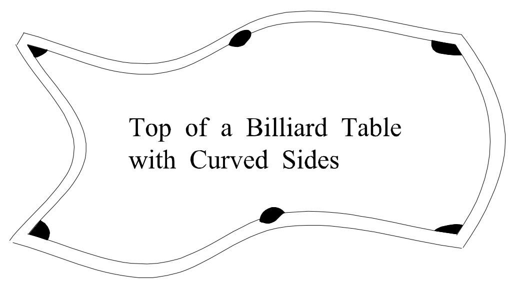
What, then, does it mean for a line to be tangent to a curve, and how do we see that concept illustrated in this example?

40. Computer Algebra System. If you have access to a computer algebra system, use it to find the tangent lines to the given curves at the given point. Then graph both the curve and the tangent line.

(a) $y = (1-x)(1-2x)(1-3x)$ when $p = 1$
(b) $y = x(x+1)(x+2)(x+3)$ when $p = 2$
(c) $y = x^2(x-1)^{10}$ when $p = 1$
(d) $y = 1 + x(1 + x(1+x))$ when $p = 3$

41. Write to Learn: The cross section of a satellite dish is a parabola with an equation of $y = x^2 + x + 4$. If a signal received from space travels down the positive y-axis, what will its angle of reflection off of the mirror be. Write a short essay explaining your results and how they were obtained.

42. Write to Learn: Suppose that a billiard table has curved sides that reflect billiard balls so that the angle of incidence is equal to the angle of reflection.



1-16: An Unsual Pool Table

Write a short essay in which you identify and explain *mathematically* which of the six pockets is the "easiest" to hit a ball into (in particular, be sure to use tangent lines in your explanation).

1.2 The Limit Concept

The Limit Process

Unfortunately, we cannot build a mathematical theory on the vague notion that "if h is close enough to 0, then h^2 and higher powers can be ignored." Instead, we base calculus on the more concrete and dynamic idea that "If h gets closer and closer to 0, then h^2 and higher powers become more and more negligible." The concept of h getting closer and closer to 0 is known as the *limit concept* in calculus and is the foundation on which the theory of calculus is built.

Let's be more specific. The symbol $\lim\limits_{x \to p}$, which is interpreted "the limit as x approaches p," denotes the act of letting x become closer and closer to p. Thus, the equation

$$\lim_{x \to p} f(x) = L \tag{1.3}$$