3. APPLICATIONS OF THE DERIVATIVE

Calculus was developed independently in the late seventeenth century by both Sir Isaac Newton of England and Gottfried Leibniz of Germany. However, Newton and Leibniz did not develop calculus in the context of functions. In fact, the function concept was not established until a two volume treatise by Euler published in 1748.[1] Instead, both Newton and Leibniz viewed calculus as a tool to be applied in the study of *analytic geometry.*

Unfortunately, the original formulations of calculus were based on intuition rather than rigor, and as a result, calculus seemed to produce contradictory results. In the early 1800's, the mathematician Augustin-Louis Cauchy sought to rectify this situation by establishing an axiomatic foundation of Calculus similar in form to the axiomatic foundations of geometry established by Euclid. His principal tool in this endeavor was the *Mean Value Theorem*, which even today is considered foundational to much of calculus.

In particular, the Mean Value theorem is the basis for the tools used in graphing functions and in finding their extreme values. In this chapter, we begin with the Mean Value theorem and its implications, which in turn sets the stage for the applications of the derivative in the second part of this chapter. The result will provide valuable insights into how concepts such as tangents and rates of change are applied to real-world problems.

3.1 The Mean Value Theorem

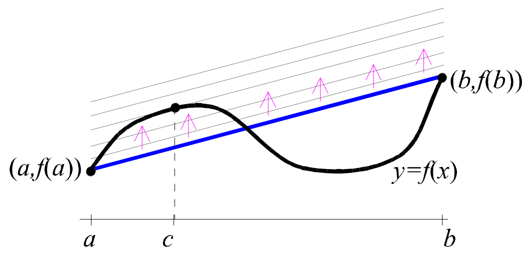
The Mean Value Theorem

While limits and derivatives are fundamental concepts in calculus, they alone are not sufficient to allow calculus to be as applicable and widely-used as it is today. Instead, as we will see in this section, the theoretical foundation of calculus is the *Mean Value Theorem*.

Suppose $f(x)$ is defined on $[a, b]$. If the secant line through $(a, f(a))$ and $(b, f(b))$ is translated vertically, then at least for the function in the figure below, the result is a family of secant lines that becomes closer and closer to a tangent

[1] However, Gottfried Liebniz did use the term "function" in a manner consistent with the function concept we use today.

line at some input c in (a, b).

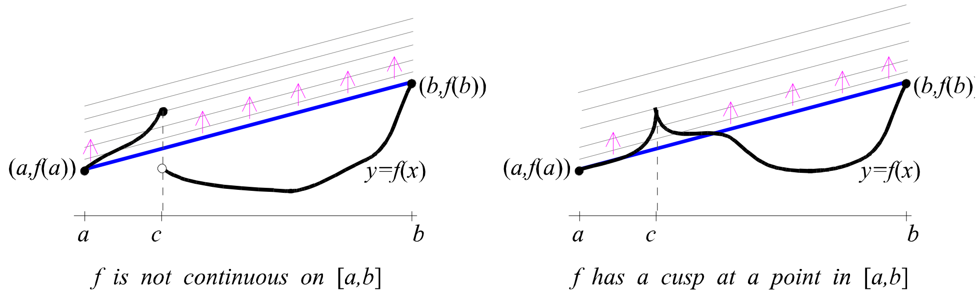


1-1: One of the parallels is a tangent line to the curve

Thus, the slope $f'(c)$ of the tangent line must be the same as the slope of the secant line, which means that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \tag{3.1}$$

However, this concept is not valid for all functions $f(x)$. In the plots below, the secant line through $(a, f(a))$ and $(b, f(b))$ is also translated vertically, but in these cases without implying the existence of a tangent line parallel to the original secant.

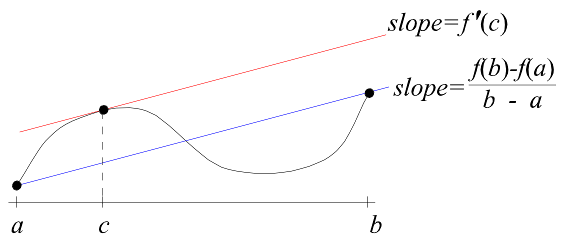


1-2: Figure 1-1 requires continuity and differentiability of f

Figure 1-2 illustrates why $f(x)$ must be continuous on $[a, b]$ and differentiable on (a, b) in order for (3.1) to hold. Given these conditions, however, the following theorem can be established (though the proof is beyond this text).

The Mean Value Theorem: If $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b), then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \tag{3.2}$$



1-3: The Mean Value Theorem

A more useful form of the Mean Value Theorem follows from 2 observations: First, if $f(x)$ is differentiable on an open interval containing $[a,b]$, then $f(x)$ must be continuous on $[a,b]$. Second, (3.2) is equivalent to $f(b) - f(a) = f'(c)(b-a)$.

The Mean Value Theorem (Restated): If $f(x)$ is differentiable on an open interval containing $[a,b]$, then there is a number c in $[a,b]$ such that

$$f(b) - f(a) = f'(c)(b-a) \tag{3.3}$$

This latter version of the Mean Value theorem (MVT) holds even when $a = b$ (i.e., even when $[a,b]$ is a single point). This makes it quite useful in proving the fundamental theorems in calculus, such as the one below:

Theorem 1.2: If $f'(x) = 0$ for all x in (p,q), then $f(x)$ is constant over (p,q).

Proof. If $[a,b]$ is contained in (p,q), then the MVT implies that

$$f(b) - f(a) = f'(c)(b-a)$$

for some c in $[a,b]$. However, $f'(c) = 0$ since c is in (p,q), which implies that

$$f(b) - f(a) = 0 \qquad or \qquad f(b) = f(a)$$

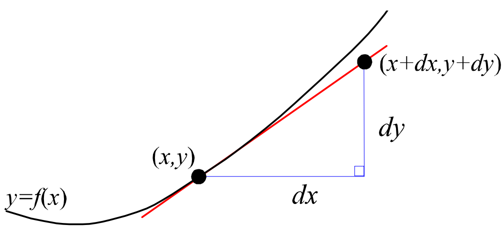
for all a, b in (p,q). Thus, all the outputs of $f(x)$ over (p,q) are the same, which is to say that $f(x)$ is constant. ■

We will not prove the Mean Value theorem here, as the proof requires some remarkable results from the study of the topology of the real line. However, a sketch of the proof is included in the appendix for completeness.

Check your Reading *How do we obtain (3.3) from (3.2)?*

Differentials Revisited

Let's look at another example in which the Mean Value theorem is used to establish an important result in calculus. Recall that the *differentials* of x and y, which are denoted by dx and dy, respectively, are defined to be small changes along the tangent line to the curve at a point (x,y).



1-4: The differentials dx and dy

Thus, if $y = f(x)$ passes through (x, y), then $f'(x)$ is the slope of the tangent line at (x, y) and consequently,

$$dy = f'(x)\, dx \tag{3.4}$$

Equivalently, we can write (3.4) in the form $dy = y'\, dx$.

EXAMPLE 1 Find the differential dy for the curve $y = x^3 + 2x$.

Solution: Since $y' = 3x^2 + 2$, the differential dy is $dy = \left(3x^2 + 2\right) dx$

EXAMPLE 2 Find dy for $y = x \sin(x)$.

Solution: Since $y' = \sin(x) + x\cos(x)$, the differential is

$$dy = [\sin(x) + x\cos(x)]\, dx$$

Suppose now that $f(x)$ has a continuous derivative on a neighborhood of a point p. If $a = p$ and $b = p + dx$, then the MVT implies that

$$f(p + dx) - f(p) = f'(c)\, dx$$

The quantity $\Delta y = f(p + dx) - f(p)$ is the *change* in $y = f(x)$ over $[p, p + dx]$. Moreover, the MVT implies that

$$\Delta y = f'(c)\, dx$$

for some c in $[a, b]$. Since f' is continuous, it follows that $f'(c) \approx f'(p)$ when dx is sufficiently close to 0. As a result, $\Delta y \approx dy$. That is, the MVT implies that the change Δy is closely approximated by the differential dy when dx is close to 0.

EXAMPLE 3 Find Δy and dy for $f(x) = x^3$ when $p = 1$ and $dx = 0.05$.

Solution: Since $\Delta y = f(p + dx) - f(p)$, we have

$$\Delta y = f(1.05) - f(1) = (1.05)^3 - 1^3 = 0.157625$$

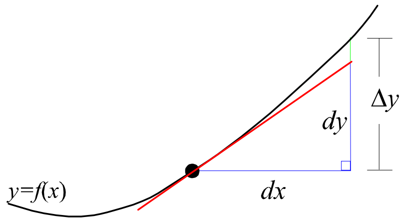
Since $f'(x) = 3x^2$, we have $f'(1) = 3$ and

$$dy = f'(1)\, dx = 3 \cdot 0.05 = 0.15$$

which closely approximates the actual value of $\Delta y = 0.157625$.

Check your Reading *By how much does Δy differ from dy in example 2?*

Graphically, dy is a change in y *along the tangent line,* while Δy is the resulting change in y *along the curve.*



1-5: dy is close to Δy for dx close to 0

The fact that $\Delta y \approx dy$ when dx is sufficiently close to 0 is thus a reflection of the fact that a tangent line is practically the same as a small section of the curve.

EXAMPLE 4 Find Δy and dy for $f(x) = x^3 + 2x$ when $p = 1$ and $dx = 0.1$, and then illustrate them graphically.

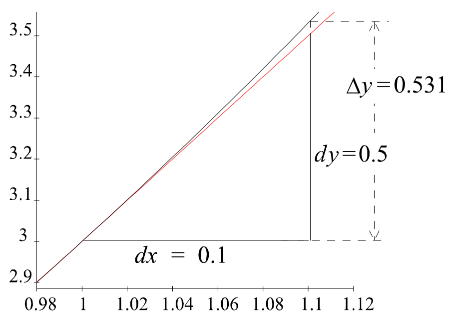
Solution: Since $p = 1$, and $p + dx = 1.1$, we have

$$\begin{aligned}\Delta y &= f(1.1) - f(1) \\ &= (1.1)^3 + 2(1.1) - \left(1^3 + 2 \cdot 1\right) \\ &= 0.531\end{aligned}$$

To find dy, we first notice that $f'(x) = 3x^2 + 2x$ so that $f'(1) = 5$. As a result, the differential dy is

$$dy = f'(x)\,dx = f'(1)\,(0.1) = 5 \cdot 0.1 = 0.5$$

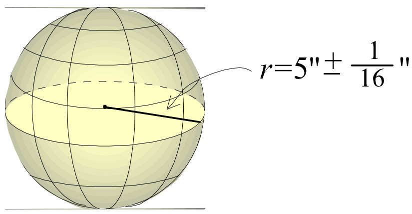
which closely approximates the actual value of $\Delta y = 0.531$.



1-6: dy approximates dy

As a result, dy can be used to approximate Δy in applications where dx is sufficiently close to 0. In particular, a *tolerance* is defined to be the maximum allowable error in the measurement of a quantity, so that if dx is the tolerance for a quantity x, then dy is often used as an approximation of the tolerance in y.

EXAMPLE 5 Find the volume V and approximate the tolerance dV of a sphere whose radius is measured to be 5 inches to within a tolerance of $\frac{1}{16}$ of an inch.



1-7: Sphere with radius of $5 \pm \frac{1}{16}$ inches.

Solution: The volume of a sphere with radius r is given by

$$V = \frac{4}{3}\pi r^3 \tag{3.5}$$

which implies that the volume of a sphere with radius 5 inches is

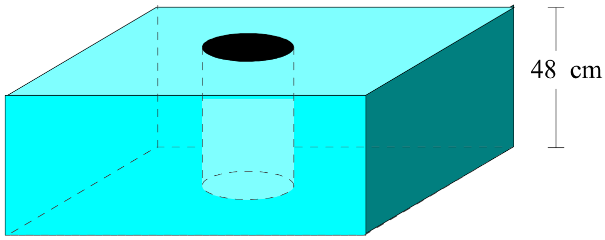
$$V = \frac{4}{3}\pi (5)^3 = \frac{500\pi}{3} \approx 523.6 \text{ in}^3$$

Since $V' = 4\pi r^2$, the differential is $dV = 4\pi r^2 dr$. Thus,

$$dV = 4\pi (5)^2 \frac{1}{16} = 19.6\ in^3$$

That is, the sphere's volume is 523.6 in^3, give or take about 19.6 in^3.

EXAMPLE 6 A right cylindrical hole is bored through a block of steel with a uniform width of 48 cm. If the radius of the hole is 5 cm to within a tolerance of 1 mm, then what is the tolerance in the volume of the resulting hole?



1-8: Hole drilled through a block

Solution: If r is the radius of the hole, then the volume of the hole is $V = \pi r^2 \cdot 48$ since the height of the hole is 48 cm. Moreover, $V'(r) = 96\pi r$ and thus, $V'(1) = 96 \cdot \pi \cdot 5 = 1507.96$. Since $dr = 1$ mm $= 0.1$ cm, we have

$$dV = V'(1)\, dr = 1507.96 \cdot 0.1 = 150.796 \text{ cm}^3$$

Check your Reading *What is the volume of the hole in example 6?*

Proofs Based on the Mean Value Theorem

The mathematician Augustin-Louis Cauchy tried to base all of calculus on the Mean Value theorem, much like Euclid based geomety on his 5 postulates. Although he was not completely successful, much of his program is still used today. That is, the Mean Value theorem continues to be used in graduate and undergraduate mathematics courses to prove many of the theorems of calculus.

EXAMPLE 7 Use the MVT to prove that if $b > 1$, then

$$\ln(b) \leq b - 1$$

Solution: To do so, we apply the MVT in the form (3.3) to the function $f(x) = \ln(x)$ on $[1, b]$. That is, since $f'(x) = \frac{1}{x}$, there is a number c in $[1, b]$ such that

$$\ln(b) - \ln(1) = \frac{1}{c}(b - 1)$$

However, c in $[1, b]$ implies that $c \geq 1$, which means that $\frac{1}{c} \leq 1$. Since $\ln(1) = 0$, this yields

$$\ln(b) = \frac{1}{c}(b - 1) \leq b - 1$$

As another example, let us use the Mean Value Theorem to develop a proof of the chain rule that uses the limit definition.

Suppose that $g(x)$ is differentiable at p and that $f(x)$ is differentiable on some neighborhood of $g(p)$. For each $h \neq 0$ sufficiently close to 0, the Mean Value theorem (3.3) implies that there is a number c_h between $g(p)$ and $g(p+h)$ such that

$$f[g(p+h)] - f[g(p)] = f'(c_h)[g(p+h) - g(p)] \tag{3.6}$$

Division by h and application of the limit as h approaches 0 then yields

$$\lim_{h \to 0} \frac{f[g(p+h)] - f[g(p)]}{h} = \left(\lim_{h \to 0} f'(c_h)\right)\left(\lim_{h \to 0} \frac{g(p+h) - g(p)}{h}\right)$$

Since c_h is between $g(p)$ and $g(p+h)$, the squeeze theorem says that

$$\lim_{h \to 0} g(p) \leq \lim_{h \to 0} c_h \leq \lim_{r \to 0} g(p+h) \qquad \Longrightarrow \qquad \lim_{h \to 0} c_h = g(p)$$

Thus, $f(c_h)$ approaches $f'(g(p))$ as $h \to 0$, and consequently

$$\lim_{h \to 0} \frac{f[g(p+h)] - f[g(p)]}{h} = f'[g(p)]\, g'(p)$$

Exercises:

Compute the differential dy.

1.	$y = x^2$	2.	$y = x^3$
3.	$y = \sin(x)$	4.	$y = \cos(x)$
5.	$y = \tan(x)$	6.	$y = \sec(x)$
7.	$y = e^x$	8.	$y = 2^x$
9.	$y = \sqrt{1 - x^2}$	10.	$y = \sin\left(x^2\right)$

Find dy and Δy for the given values of p and $\Delta x = dx$. Would you conclude that dy is a good approximation of Δy?

11.	$f(x) = x^2,\ p = 1,\ dx = 0.01$	12.	$f(x) = x^2 + 2x,\ p = 2,\ dx = 0.02$
13.	$f(x) = x^3 + x^2,\ p = 2,\ dx = 0.1$	14.	$f(x) = (x - 1)^2,\ p = 0,\ dx = 0.01$
15.	$f(x) = 2x + 3,\ p = 1,\ dx = 0.2$	16.	$f(x) = 3x - 1,\ p = -1.5,\ dx = 0.2$
17.	$y = -0.3x^2 + 0.7x + 0.5$ $p = 0.2,\ dx = 0.001$	18.	$y = 4x^2 + 35x - 15$ $p = 10,\ dx = 0.1$
19.	$xy = 1,\ x = 0.5,\ dx = 0.1$	20.	$y^2 = x,\ x = 1.44,\ dx = 0.1$

Exercises 21-28 involve tolerances and differentials as approximations.

21. A certain rectangle has a length which is twice its width. The length is 1 cm measured to within a tolerance of 0.1 cm. Estimate the maximum error in calculating the area of the rectangle using this length. (Hint: calculate dA.)

22. If a cube with has sides of length $x = 2$ cm to within a tolerance of 0.1 cm, then what is the tolerance in the volume of the cube?

23. A thin circular disk has a radius $r = 10$ cm measured to within a tolerance of 0.3 cm. Estimate the maximum error in calculating the area of the disk using this measurement. (Hint: calculate dA.)

24. The circumference of the top of a soup can is measured to be $9\frac{1}{4}''$ to within a 1/16 of an inch.

(a) Express the area of a circle as a function of the circumference of the circle (Hint: $r = C/2\pi$)

(b) What is the area of the top of the can?

(c) About how much variation in the area computation is possible given that the circumference is accurate only to a sixteenth of an inch?

25. The volume of a pyramid is one-third the height times the area of the base. If a pyramid has a square base with the length of one side being $100m$ measured to within a tolerance of $0.1m$, estimate the maximum error in the volume measurement as a function of the height, h.

26. A cylindrical hole is to be bored in a block of metal. The hole is to have a capacity of $50cc$ (cc is cubic centimeters), a height of $h = 6.351cm$ and a radius of $R = 1.583cm$. The volume must be accurate to within $0.2cc$, that is $|dV| < 0.2$. The volume of a cylinder is given by $V = \pi R^2 h$.

(a) If we assume that the height h is without error, to what tolerance dr must we keep the radius in order for $|dV| < 0.2$.

(b) If we assume that the radius R will be perfect, to what tolerance dh must we keep the height?

27. A copper wire 3 mm in diameter is to be covered with a plastic insulation 0.5 mm thick. Use differentials to estimate the volume of the plastic necessary to coat a roll of copper wire $100m$ ($100,000$ mm) in length.

28. A spherical shell has a radius of 100 cm and a thickness of 0.1 cm. Use differentials to approximate the volume of the shell.

Exercises 29-40 use the Mean Value Theorem to prove results in calculus.

29. Use the Mean Value Theorem to prove that if $f'(x) = 1$ for all x in (a, b), then

$$f(b) - f(a) = b - a$$

30. Use the Mean Value theorem to prove that if $b > a > 1$, then

$$b^{10} - a^{10} > 10(b - a)$$

(Hint: let $f(x) = x^{10}$ and notice that if $c > a$, then also $c > 1$).

31. Use the Mean value theorem to prove that if $b \geq a$, then

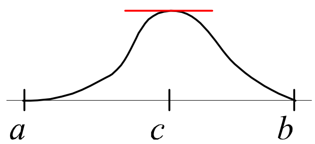
$$\sin(b) - \sin(a) \leq b - a$$

32. Use the Mean value theorem to prove that if $0 < a < b$, then

$$\sqrt{ab} < \frac{a + b}{2}$$

(Hint: let $f(x) = \sqrt{ax}$ and notice that $f(a) = a$, $f(b) = \sqrt{ab}$ and $\sqrt{\frac{a}{c}} < 1$.)

33. Rolle's Theorem: Show that if $f(a) = f(b) = 0$, if $f(x)$ is continuous on $[a, b]$, and if $f(x)$ is differentiable on (a, b), then there is a c in (a, b) such that $f'(c) = 0$.



1-9: Rolle's Theorem

34. Suppose that $f(x)$ is periodic with a minimum period $T > 0$. (i.e., T is the smallest number such that $f(x + T) = f(x)$ for all x). Show that if $f(x)$ is differentiable everywhere, then $f'(x)$ must have an infinite number of zeroes. (Hint: see exercise 33)

35. Use the Mean value theorem to prove the following: If there is a number $\delta < 1$ such that $|f'(x)| \leq \delta$ for all x in an interval (p, q), then

$$|f(b) - f(a)| \leq \delta |b - a| \tag{3.7}$$

for all intervals $[a, b]$ in (p, q). (A function that satisfies (3.7) is called a *contraction*).

36. Find the Error: If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$ and thus on the interval $[0, b]$, we have

$$\tan(b) - \tan(0) = \sec^2(c)(b - 0)$$

for some number c in $(0, b)$. However, $\sec^2(c) \geq 1$, so that

$$\tan(b) = \sec^2(c)(b - 0) \geq b - 0$$

That is, if $b \geq 0$, then $\tan(b) \geq b$ (Note: This conclusion can't be true since if $b = \pi$, then $\tan(\pi) = 0$).

37. Graph $f'(x)$ on the interval given. Estimate $\min[f'(x)]$ (i.e. the minimum value of $f'(x)$ on the given interval) and $\max[f'(x)]$ (the maximum value of $f'(x)$ on the given interval).[2] Compare each to the quantity $\frac{f(b)-f(a)}{b-a}$

(a)	$f(x) = 3x - 2x^3$ on $[-1, 1]$	(b)	$f(x) = 3x^{1/2} - x$ on $[1, 3]$
(c)	$f(x) = x^3 + 3x$ on $[-1, 1]$	(d)	$f(x) = e^x$ on $[0, \ln(2)]$

38. The results in the previous exercise lead to the following conjecture: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then

$$\min[f'(x)] \leq \frac{f(b) - f(a)}{b - a} \leq \max[f'(x)] \tag{3.8}$$

Prove this result with the Mean Value Theorem.

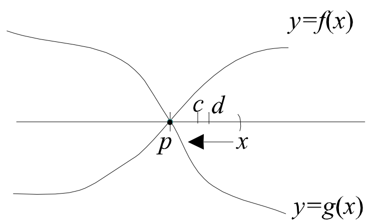
39. Use the theorem in exercise 38 to prove theorem 1.2. (i.e., use (3.8) to show that if $f'(x) = 0$ on (a, b), then $f(x)$ is a constant function on (a, b)).

40. **Write to Learn:** At 3:00 p.m., the odometer on your automobile reads 15,000 miles. At 4:00 p.m., the odometer on your automobile reads 15,060 miles. Let $r(t)$ denote your odometer reading in miles at time t in hours since noon, and then write a short essay in which you apply and interpret the Mean Value Theorem to $r(t)$ over $[3, 4]$. What instrument measures $r'(t)$? What does the Mean Value theorem imply about $r'(c)$ at some time c in $[3, 4]$?

3.2 L'Hôpital's Rule

L'Hôpital's Rule for $\frac{0}{0}$ Forms

Suppose that $f(x)$ and $g(x)$ are differentiable in a neighborhood of p and that both $f(p) = 0$ and $g(p) = 0$.



2-1: $f(p) = g(p) = 0$

[2] If available, the "trace" option can be used to estimate the maximum and the minimum of $f'(x)$.