5. APPLICATIONS OF THE INTEGRAL

Definite integrals are not used solely to study areas. They are used throughout mathematics and the sciences with a host of different interpretations. Indeed, such interpretations range from the concrete study of geometric quantities to the relatively abstract study of integral transforms.

In this chapter, we explore several applications of the integral, many of which are connected to the study of geometry. We begin by further exploring the concept of area, and then we move on to the concepts of volume, centroid, and arclength. In each of these applications, the definition of the integral plays a key role in deriving the formulas that are most important to our study.

However, our ultimate goal is one of the most important applications in modern science and mathematics—the use of calculus in the study of *probability.* To reach this goal, we have to introduce two additional concepts related to the integral—the concept of an *improper integral* and the technique of integration known as *integration by parts.*

By the end of the chapter, we feel that there will be little doubt as to the importance of both the integral and the definition of the integral. Indeed, the applications offered here are but a sampling from a vast arena the integral's uses. We only hope that this chapter affords a glimpse into how the integral can be and is used throughout science, engineering, mathematics, and technology.

5.1 Area Between Two Curves

If $f(x)$ is integrable over $[a,b]$, then its definite integral over $[a,b]$ is defined

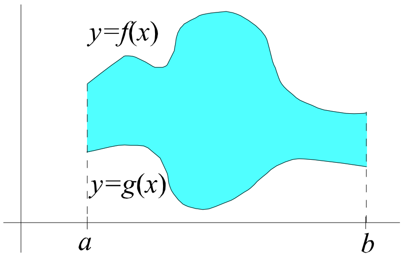
$$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{j=1}^{n} f(t_j)\,\Delta x_j \tag{5.1}$$

where the limit is over h-fine partitions of $[a,b]$. In this section, we use (5.1) to extend the integral to a tool for finding areas of regions bound between two curves.

Area of a Type I Region

To begin with, a *type I region* is a region with boundaries of the form $x=a$, $x=b$, $y=f(x)$ and $y=g(x)$, where $f(x)\geq g(x)$ for all x in $[a,b]$ and where $f(x)$ and

$g(x)$ are continuous on $[a, b]$.

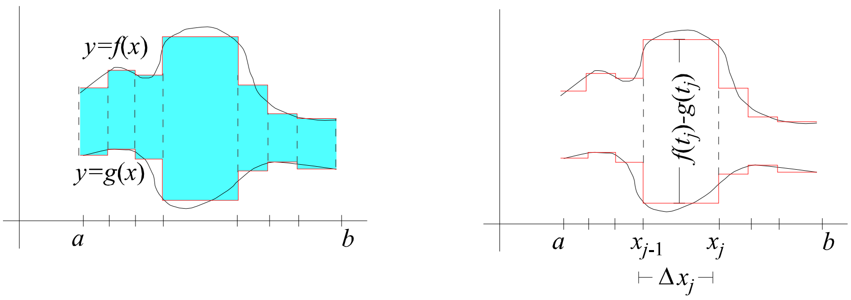


1-1: A Type I Region

To find the area of a type I region, let $h > 0$ and let $\{x_j, t_j\}_{j=1}^{n}$ be an h-fine partition of $[a, b]$. Then the simple functions

$$s_h(x) = \begin{cases} f(t_1) & if \quad x_0 \le x < x_1 \\ f(t_2) & if \quad x_1 \le x < x_2 \\ \vdots & \qquad \vdots \\ f(t_n) & if \quad x_{n-1} \le x < x_n \end{cases}, \qquad r_h(x) = \begin{cases} g(t_1) & if \quad x_0 \le x < x_1 \\ g(t_2) & if \quad x_1 \le x < x_2 \\ \vdots & \qquad \vdots \\ g(t_n) & if \quad x_{n-1} \le x < x_n \end{cases}$$

converge to f and g, respectively, as h approaches 0. Thus, the area of the region between the simple functions approximates the area of the region between $y = f(x)$ and $y = g(x)$ respectively.



1-2: Simple function approximation of f and g

Moreover, the region between $s_h(x)$ and $r_h(x)$ is a collection of rectangles with widths Δx_j and heights $f(t_j) - g(t_j)$. The total area of the rectangles is

$$area = [f(t_1) - g(t_1)]\,\Delta x_1 + \ldots + [f(t_n) - g(t_n)]\,\Delta x_n$$

and if we let h approach 0, the area of the rectangles converges to the area between the curves:

$$area = \lim_{h \to 0} \sum_{j=1}^{n} [f(t_j) - g(t_j)]\,\Delta x_j$$

By definition, this last limit converges to $\int_a^b (f(x) - g(x))\,dx$.

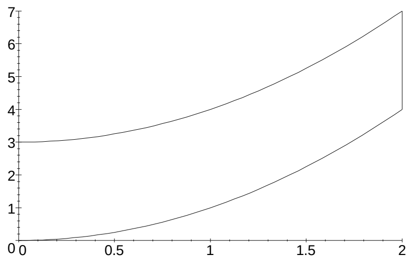
Theorem 1.1. If $f(x)$ and $g(x)$ are integrable on $[a, b]$ with $f(x) \ge g(x)$ for all x in $[a, b]$, then the region between $y = f(x)$ and $y = g(x)$ over $[a, b]$ has an area of

$$A = \int_a^b (f(x) - g(x))\,dx \tag{5.2}$$

That is, (5.2) is the formula for the area of a type I region.

EXAMPLE 1 Find the area of the region between $f(x) = x^2 + 3$ and $g(x) = x^2$ over $[0, 2]$.

Solution: Since $x^2 + 3 \geq x^2$ over $[0, 2]$, the region bound by $x = 0$, $x = 2$, $y = x^2$, and $y = x^2 + 3$ is a type I region:



1-3: Region between $y = x^2$ and $y = x^2 + 3$ over $[0, 2]$

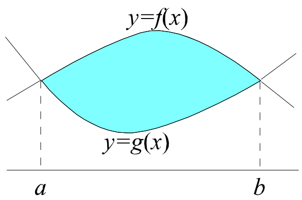
Thus, theorem 1.1 implies that the area of the region is

$$A = \int_0^2 \left(x^2 + 3 - x^2\right) dx = \int_0^2 3dx = 6$$

Check your Reading *What is the area of the region between $y = x + 2$ and $y = x$ over $[0, 2]$?*

Identifying Endpoints and Subdivisions of Regions

When an interval $[a, b]$ is not given, the graphs of the functions must intersect in at least two points in order to bound a type I region.

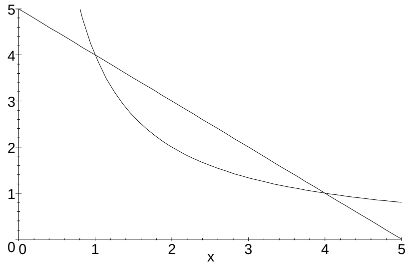


1-4: Limits of integration may be x-coordinates of intersections

To determine the interval $[a, b]$, we must set $f(x) = g(x)$ and solve for x.

EXAMPLE 2 Find the area of the region between the curves $y = 5 - x$ and $y = 4/x$.

Solution: We begin by graphing the two curves to determine if they intersect.



1-5: Region between $y = 5 - x$ and $y = 4/x$

The two graphs intersect when $5 - x = 4/x$. Multiplying both sides by x yields

$$x(5 - x) = x\left(\frac{4}{x}\right) \qquad \Longrightarrow \qquad 5x - x^2 = 4$$

The result can be transformed into $x^2 - 5x + 4 = 0$, which factors into

$$(x - 1)(x - 4) = 0$$

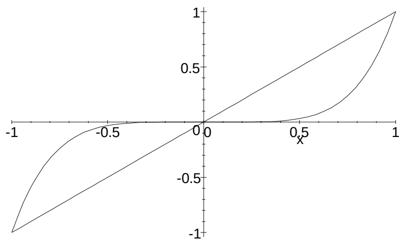
and which in turn has solutions of $x = 1$ and $x = 4$. The graph reveals that $5 - x \geq 4/x$ when x is in $[1, 4]$, so that the area of the region enclosed by $y = 5 - x$ and $y = 4/x$ is

$$area = \int_1^4 \left(5 - x - \frac{4}{x}\right) dx = 5x - \frac{x^2}{2} - 4\ln(x)\Big|_1^4 = \frac{15}{2} - 8\ln 2$$

If $y = f(x)$ and $y = g(x)$ intersect more than twice, then we find the area of the region between $y = f(x)$ and $y = g(x)$ over each interval implied by the points of intersection. That is, the area of the region between $y = f(x)$ and $y = g(x)$ is the sum of the areas of the individual subregions.

EXAMPLE 3 Find the area of the region between the curves $y = x^5$ and $y = x$.

Solution: Since $x^5 = x$ has solutions $x = -1, 0, 1$, the total region is the union of the regions over $[-1, 0]$ and over $[0, 1]$, respectively.



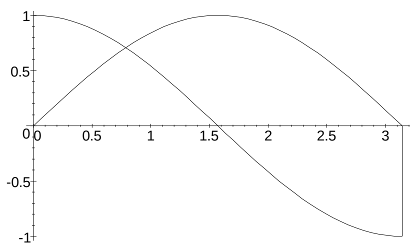
1-6: Region between $y = x$ and $y = x^5$

Notice that $x^5 \geq x$ when x is in $[-1,0]$, but that $x \geq x^5$ when x is in $[0,1]$. As a result, the area between the two curves is

$$\begin{aligned} area &= \int_{-1}^{0} \left(x^5 - x\right) dx + \int_{0}^{1} \left(x - x^5\right) dx \\ &= \frac{1}{3} + \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

EXAMPLE 4 Find the area of the region bound between $y = \sin(x)$ and $y = \cos(x)$ over $[0, \pi]$.

Solution: The graph below reveals that $\sin(x) = \cos(x)$ for some x in $[0, \pi]$.



1-7: Region between $y = \sin(x)$ and $y = \cos(x)$ over $[0, \pi]$

To determine the point of intersection, we solve the following:

$$\begin{aligned} \sin(x) &= \cos(x) \\ \tan(x) &= 1 \\ x &= \frac{\pi}{4} \end{aligned}$$

Since $\cos(x) \geq \sin(x)$ when x is in $\left[0, \frac{\pi}{4}\right]$ and $\sin(x) \geq \cos(x)$ when x is in $\left[\frac{\pi}{4}, \pi\right]$, the area of the region enclosed by the two functions is

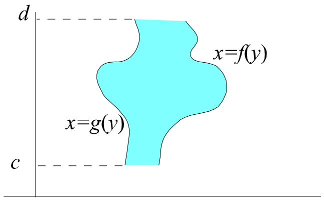
$$\begin{aligned} area &= \int_{0}^{\pi/4} (\cos(x) - \sin(x))\, dx + \int_{\pi/4}^{\pi} (\sin(x) - \cos(x))\, dx \\ &= \sqrt{2} - 1 + \sqrt{2} + 1 \\ &= 2\sqrt{2} \end{aligned}$$

Check your Reading *Why can't we compute the area between $y = x^5$ and $y = x$ with the integral*

$$\int_{-1}^{1} \left(x^5 - x\right) dx$$

Type II Regions

A *type II region* is a region with boundaries of the form $y = c$, $y = d$, $x = f(y)$ and $x = g(y)$, where $f(y) \geq g(y)$ for all x in $[c, d]$ and where f and g are continuous on $[c, d]$.



1-8: A type II region

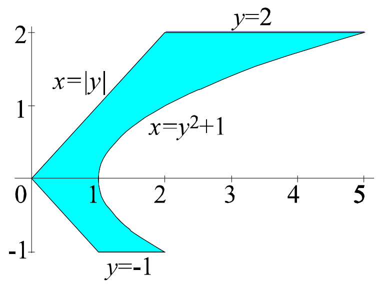
It follows that the area of a type II region is

$$A = \int_c^d [f(y) - g(y)]\, dy$$

That is, if x is considered a function of y, we simply interchange the roles of the two variables.

EXAMPLE 5 Find the area of the type II region between $x = y^2 + 1$ and $x = |y|$ over $[-1, 2]$.

Solution: The graphs of $f(y) = y^2 + 1$ and $g(y) = |y|$ reveal that $f(y) \geq g(y)$ over $[-1, 2]$.



1-9: Region between $x = y^2 + 1$ and $x = |y|$ over $[-1, 2]$.

The definition of the absolute value function says that

$$|y| = \begin{cases} y & if \quad y \geq 0 \\ -y & if \quad y < 0 \end{cases}$$

Thus, the area is computed using integrals over $[-1, 0]$ and $[0, 2]$, respectively.

$$\begin{aligned} area &= \int_{-1}^{0} \left(y^2 + 1 - |y|\right) dy + \int_0^2 \left(y^2 + 1 - |y|\right) dy \\ &= \int_{-1}^{0} \left(y^2 + 1 - -y\right) dy + \int_0^2 \left(y^2 + 1 - y\right) dy \\ &= \frac{5}{6} + \frac{8}{3} \\ &= \frac{7}{2} \end{aligned}$$

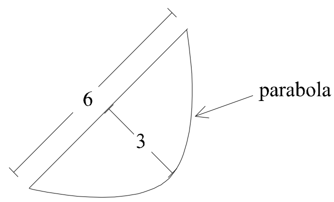
Check your Reading *Where did we use the definition of $|y|$ in the computation above?*

Creating a Coordinate System for a Given Region

When a region is given without benefit of a coordinate system, we must define our own coordinate system and then find representations for the curves in that coordinate system. In particular, let us consider regions bounded by parabolas using the fact that a parabola with nonzero roots r_1, r_2 and with a y-intercept of b has an equation of

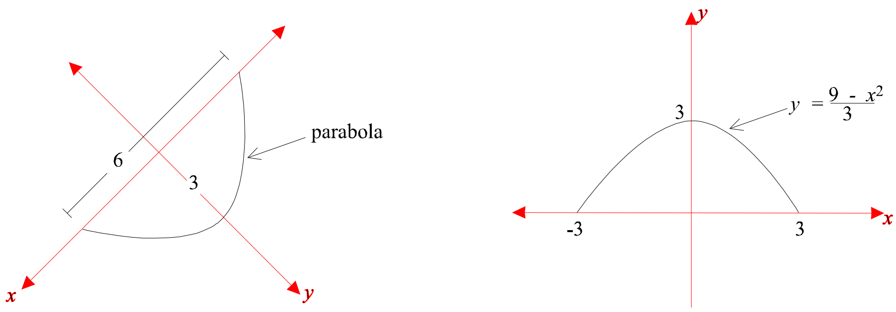
$$y = \frac{b}{r_1 r_2} (x - r_1)(x - r_2) \tag{5.3}$$

EXAMPLE 6 Find the area of the region shown below:



1-10: Parabola for example 6

Solution: We choose the x-axis to be concurrent with—i.e., on top of—the straight segment of length 6, and we choose the y-axis to run through the vertex perpendicular to the x-axis.



1-11: Parabola placed in xy-coordinate system

Orienting the x and y axes as is customary yields the diagram on the right above. The length of the "base" of the region is 6, which corresponds to 3 units to either side of the y-axis. Thus, we have a parabola with a y-intercept of 3 and two x-intercepts of 3 and -3. Its equation is

$$\begin{aligned} y &= \frac{3}{(-3)(3)} (x-3)(x--3) \\ &= \frac{-1}{3}(x-3)(x+3) \\ &= \frac{-1}{3}\left(x^2-9\right) \end{aligned}$$

which simplifies to $y = \frac{1}{3}\left(9 - x^2\right)$. As a result, the area of the region is

$$A = \frac{1}{3}\int_{-3}^{3}\left(9 - x^2\right)dx = 12$$

Exercises:

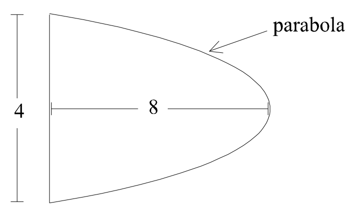
Find the area of the region between the two given curves. Some of these will be used again in section 6-2.

1. $y = x,\ y = x^2$
2. $y = 2x + 3,\ y = x^2$
3. $y = 4 - x^2,\ y = 3x$
4. $y = 2x^2 - x^3,\ y = x$
5. $y = 2 - x^2,\ y = x^2$
6. $y = 2x^2,\ y = x^2 + 1$
7. $y = x,\ \ y = x^4$
8. $y = x^4 + 1,\ y = 2x^2$
9. $y = x^3,\ \ y = x$
10. $y = -x^3,\ \ y = -x$
11. $y = x^3,\ \ y = x^2 + 2x$
12. $y = x^4,\ \ y = 5x^2 - 4$
13. $y = |x|\,,\ \ y = x^2$
14. $y = |x|\,,\ y = 2 - x^2$

15. $x = y,\ x = y^2$
16. $x = 2y + 3,\ x = y^2$
17. $x = 4 - y^2,\ x = 3y$
18. $x = 2y^2 - y^3,\ x = y$
19. $x = 2\,|y|\,,\ x = y + 3$
20. $x = y + |y|\,,\ x = y + 1$

21. $y = \sin(x),\ y = \sin(2x)$, over $[0, \pi]$
22. $y = \cos^2(x)\,,\ y = \sin^2(x)$, over $[0, \pi]$
23. $y = \cos(\pi x)\,,\ y = 0$ over $[0, 1]$
24. $y = \sin(\pi x)\,,\ y = 1$ over $[0, 1]$
25. $y = \sec^2(x),\ y = \tan^2(x)$, over $\left[0, \frac{\pi}{4}\right]$
26. $y = 2\cos^2(x)\,,\ y = 1$, over $[0, \pi]$
27. $y = x\sin\left(\pi x^2\right),\ y = x$, over $[0, 1]$
28. $y = 2\ln(x)$ and $y = \ln(x + 2)$
29. $y = e^{2x},\ y = 5e^x - 4$
30. $y = e^x + 4e^{-x},\ y = 5$
31. $y = e^{-x},\ y = e^{-2x}$, over $[0, \ln(2)]$
32. $y = x^{-3},\ y = x^{-2}$, over $[1, \infty)$

33. Let's find the area of the region shown in the figure below.



1-12: Exercise 33

(a) To begin with, define a coordinate system in which y can be written as a function of x.

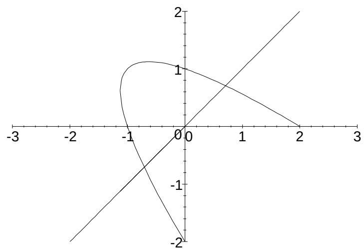
(b) Find the equation of the parabola in this coordinate system using (5.3).

(c) Find the area using a definite integral.

34. What is the area of the region between the line $y = x$ and the parabola

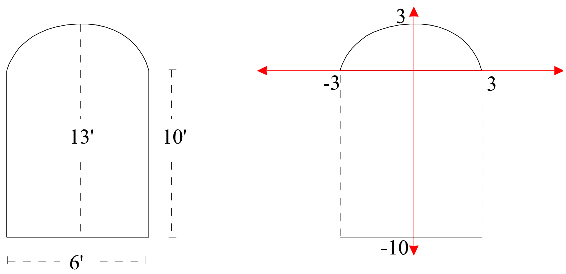
$$x^2 + 2xy + y^2 - x + y = 2$$

The graph of the region is shown below:



1-13: Exercise 34

35. A certain archway is to be filled with a steel-reinforced concrete slab which is to be $\frac{1}{2}$ foot thick.



1-14: Exercise 35

Let's determine how many cubic feet of concrete will be necessary for the job assuming the arc is a parabola

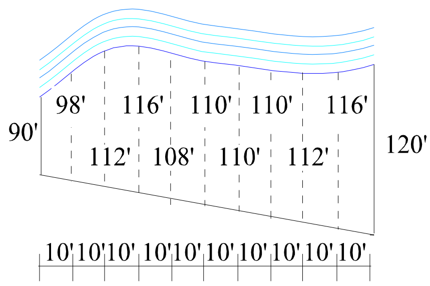
(a) Use the coordinate system shown on the right and (5.3) to define a function whose graph is the arc shown in the picture.

(b) Find the area between the two curves.

(c) Estimate the amount of concrete necessary for the job.

36. Repeat exercise 35 assuming the arc the arc of a circle. Which archway requires more concrete?

37. John buys an irregularly shaped lot bounded on one side by a creek.



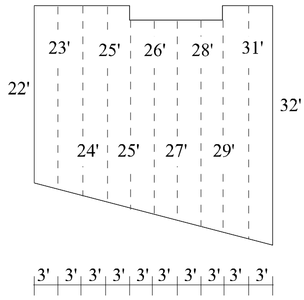
1-15: Exercise 37

He measures distances at 10 foot intervals across the lot.

(a) If the boundary near the stream is the graph of $f(x)$ and the diagonal boundary opposite the stream is the graph of $g(x)$, write an expression for the area of the lot assuming that $x = 0$ corresponds to the left vertical boundary in the plot above.

(b) Use the trapezoidal rule to estimate the area integral in (a). What is the approximate area of the lot in acres? (Note: 1 acre is equal to 43,560 square feet)

38. Below are the dimensions of an irregularly shaped room.

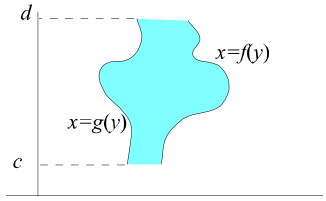


1-16: Exercise 38

(a) Describe the area of the room as a definite integral of the difference of two functions.

(b) Use the trapezoidal rule to compute the integral in (a). Explain why this yields the exact area of the room.

39. **Write to Learn:** Write a short essay which explains why the area of the region



1-17: Exercise 39

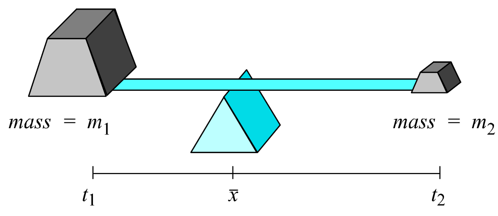
is given by $Area = \int_c^d [f(y) - g(y)]\, dy$. (Hint: mimic the simple function argument leading up to theorem 1.1).

5.2 Centroids

The x coordinate of the Centroid of a Region

In this section, we use definite integrals to define the center or *centroid* of a region in the xy-plane. We do so by extending a simple notion center of mass to a more general notion of *centroid* of a region.

To begin with, the principle of the lever says that a mass of m_2 can balance a larger mass $m_1$if the larger mass is closer to the fulcrum:



2-1: Smaller mass balancing the larger