Uniform Circular Motion

Newton was fortunate in that the moon in orbit about the earth and the first five planets in orbit about the sun are almost in *uniform circular motion,* which is motion in a circle with a constant speed. In particular, let us assume that the radius of a uniform circular motion is r and that the period of the orbit—i.e., the time it takes to complete one revolution—is T. If the point (x, y) is on the orbit, then in polar coordinates we have

$$x = r\cos(\theta) \quad \textit{and} \quad y = r\sin(\theta)$$

As a result, the position vector is of the form

$$\mathbf{r} = \langle r\cos(\theta), r\sin(\theta)\rangle = r\ \langle \cos(\theta), \sin(\theta)\rangle$$

where θ is a function of the time parameter t. Often we let $\mathbf{u}(\theta) = \langle \cos(\theta), \sin(\theta)\rangle$ so that we have

$$\mathbf{r} = r\ \mathbf{u}(\theta)$$

Uniform circular is motion in a circle at a fixed rate of change, which is to say that

$$\frac{d\theta}{dt} = C$$

where C is a constant. As a result, we must have $\theta(t) = Ct + \theta_0$, where θ_0 is the angle at time $t = 0$. Since T is the period of the orbit, it follows that $\theta(T) = 2\pi + \theta_0$, so that

$$2\pi + \theta_0 = CT + \theta_0, \qquad C = \frac{2\pi}{T}$$

As a result, we have $\theta(t) = 2\pi t/p$, so that the position vector for uniform circular motion is

$$\mathbf{r} = r\ \mathbf{u}\left(\frac{2\pi}{T}t\right)$$

where r is the radius of the circular orbit and T is the period of the orbit.

> *EXAMPLE 1* Find the parameterization of the motion of a satellite which is 100 miles above the earth and has a period of $T = 5234.14$ seconds, or $T = 1$ hour, 27 minutes and 14 seconds.
>
> **Solution:** Since the radius of the earth is $R = 3963.21$ miles, the satellite's orbit is given by
>
> $$\mathbf{r}(t) = 4063.21\ \mathbf{u}\left(\frac{2\pi}{5234.14}\ t\right) = 4063.21\left\langle \cos\left(\frac{2\pi}{5234.14}\ t\right), \sin\left(\frac{2\pi}{5234.14}\ t\right)\right\rangle$$

Since $\mathbf{u}(\theta)$ is a unit vector, its derivative $\mathbf{u}'(\theta)$, which is given by

$$\mathbf{u}'(\theta) = \langle -\sin(\theta), \cos(\theta)\rangle$$

is orthogonal to $\mathbf{u}(\theta)$. It is also a unit vector. Moreover,

$$\frac{d}{dt}\mathbf{u}(\theta) = \frac{d}{dt}\langle \cos(\theta), \sin(\theta)\rangle = \left\langle -\sin(\theta)\frac{d\theta}{dt}, \cos(\theta)\frac{d\theta}{dt}\right\rangle$$

Thus, if we factor out $d\theta/dt$, then it is clear that

$$\frac{d}{dt}\mathbf{u}(\theta) = \mathbf{u}'(\theta)\frac{d\theta}{dt}$$

Likewise, we leave it to the reader to show that

$$\frac{d}{dt}\mathbf{u}'(\theta) = -\mathbf{u}(\theta)\frac{d\theta}{dt} \tag{0.1}$$

This allows us to compute the velocity and acceleration of an object in uniform circular motion.

EXAMPLE 2 Find the velocity and acceleration of the satellite whose orbit is

$$\mathbf{r}(t) = 4063.21\ \mathbf{u}\left(\frac{2\pi}{5234.\,14}\ t\right)$$

Solution: The chain rule implies that the velocity is

$$\mathbf{v}(t) = 4063.21\ \mathbf{u}'\left(\frac{2\pi}{5234.\,14}\ t\right)\frac{d}{dt}\left(\frac{2\pi}{5234.\,14}\ t\right) = 4.8776\ \mathbf{u}'\left(\frac{2\pi}{5234.\,14}\ t\right)$$

The acceleration vector is given by

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = 4.8766\frac{d}{dt}\ \mathbf{u}'\left(\frac{2\pi}{5234.\,14}\ t\right)$$

Since $u'' = -\mathbf{u}$, we have

$$\mathbf{a} = -4.8766\mathbf{u}\left(\frac{2\pi}{5234.\,14}\ t\right)\frac{d}{dt}\left(\frac{2\pi}{5234.\,14}\ t\right) = \frac{-4.8766\cdot 2\pi}{5234.14}\ \mathbf{u}\left(\frac{2\pi}{5234.\,14}\ t\right)$$

EXAMPLE 3 Consider that the moon is $R_m = 238,957$ miles from the center of the earth and that the period of the moon's orbit is

$$p = 27.321661\ days = 2,360,591.\,5104\ \sec$$

What is the parameterization of the Moon's orbit about the earth? What is the magnitude of the acceleration of the Moon in its orbit about the earth?

Solution: If we assume that the moon has a circular orbit about the earth, then

$$\mathbf{r}(t) = 238,957\ \mathbf{u}\left(\frac{2\pi}{2,360,591.5104}\ t\right)$$

As a result, the velocity and acceleration are, respectively,

$$\begin{aligned}\mathbf{v}(t) &= \frac{2\pi(238,957)}{2,360,591.5104}\ \mathbf{u}'\left(\frac{2\pi}{2,360,591.5104}\ t\right)\\ \mathbf{a}(t) &= \frac{-(2\pi)^2(238,957)}{(2,360,591.5104)^2}\ \mathbf{u}\left(\frac{2\pi}{2,360,591.5104}\ t\right)\end{aligned}$$

Since $\mathbf{u}$ is a unit vector, it follows that the magnitude of the acceleration of the Moon in its orbit about the earth is

$$a = \frac{(2\pi)^2(238,957)}{(2,360,591.5104)^2} = 8.\,93866\times 10^{-3}\frac{ft}{\sec^2}$$

Check Your Reading: The earth has an average radius of 3963.21 miles. How high above the earth is the satellite in example 1?

The Inverse Square Law

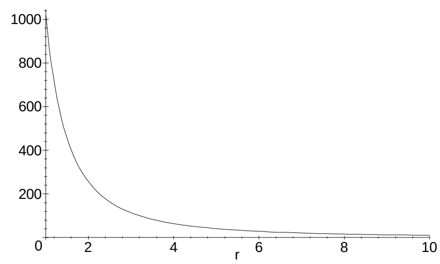
Once Galileo's mechanics had been generalized into the laws of motion, and once calculus had been developed as the setting for mathematical models of physical processes, Newton was able to deduce his law of universal gravitation with an approach similar to the one we use below.

To begin with, let us notice that the amount of gravity an object experiences depends only on its distance from the gravitational source (i.e., the weight of an object does not depend on whether it is in Paris or Pittsburgh). Thus, if r is the distance from an object to the center of the earth, then the magnitude of the gravitational force acting on the object is a function of r, which we denote by $F(r)$.

The question now becomes, "What does $F(r)$ look like mathematically?" First off, we notice that as an object moves further from the earth, the gravitational force acting on that object decreases. Indeed, since distant stars do not orbit the earth, we can assume that $F(r)$ tends to zero as r approaches ∞. That is, we must have

$$\lim_{r\to\infty} F(r) = 0$$

As a result, the graph of $F(r)$ is a decreasing function of the form



Because $F(r)$ has physical units, it must be a rational function of r. (that is, units like "the square root of a foot" or "cosine of a yard" do not make sense). Thus, we assume that the force is of the form

$$F(r) = \frac{mk}{r^n} \tag{0.2}$$

where k is a constant, m is the mass of the object and n is a positive integer. Our job now is to determine n and k.

We begin by noticing that an object at the earth's surface is $R = 3963.21$ miles from the earth's center, which translates into

$$R = 3963.21\, miles \times \frac{5280\ ft}{1\ mile} = 2.0926 \times 10^7\ feet$$

The acceleration of the object due to gravity is $g = 32\frac{ft}{\sec^2}$. Since force is the product of mass and acceleration, the force of gravity acting on the object is mg, which is the same as $32m$. Substitution into (0.2) thus yields

$$\frac{mk}{(2.09257 \times 10^7)^n} = 32m \qquad \text{or} \qquad k = 32\ \left(2.09257 \times 10^7\right)^n \tag{0.3}$$

Example 3 shows that the acceleration of the moon in its orbit about the earth is $8.93866 \times 10^{-3} \frac{ft}{\sec^2}$, so that the force of gravity acting on the moon is $8.93866 \times 10^{-3} m$. Since the moon is

$$R_m = 238,957 \, miles \times 5280 \frac{ft}{mile} = 1.26169 \times 10^9 \, ft$$

from the earth, substitution into (0.2) yields

$$\frac{mk}{\left(1.26169 \times 10^9\right)^n} = \left(8.93866 \times 10^{-3}\right) m \qquad \text{or} \qquad k = \left(8.93866 \times 10^{-3}\right) \left(1.26169 \times 10^9\right)^n \tag{0.4}$$

Combining (0.3) and (0.4) thus yields

$$32 \left(2.09257 \times 10^7\right)^n = \left(8.93866 \times 10^{-3}\right) \left(1.26169 \times 10^9\right)^n$$

We then rewrite this as a cross-ratio,

$$\frac{\left(2.09257 \times 10^7\right)^n}{\left(1.26169 \times 10^9\right)^n} = \frac{8.93866 \times 10^{-3}}{32}$$

which in turn simplifies to

$$\left(\frac{2.09257 \times 10^7}{1.26169 \times 10^9}\right)^n = 2.79333 \times 10^{-4}$$

As a result, we have

$$\left(1.65854\,526 \times 10^{-2}\right)^n = 2.79332\,998 \times 10^{-4}$$

which after applying the natural logarithm yields

$$n \ln \left(1.65854\,526 \times 10^{-2}\right) = \ln \left(2.79332\,998 \times 10^{-4}\right)$$

whose solution is $n = 1.996$. Since n must be an integer, we round up to obtain $n = 2$.

This leads to Newton's *Law of Universal gravitation*, which says that if two bodies of mass m and M are located a distance r apart, then the magnitude of the force of the gravitational attraction between them is

$$|\mathbf{F}| = G \frac{Mm}{r^2}$$

where $G = 6.67 \times 10^{-11} \, Nm^2/kg^2$ is the universal gravitational constant. Equivalently, Newton's law says that the potential between the two bodies is

$$U = -G \frac{Mm}{r} \tag{0.5}$$

We can also use geometry to show that any point source whose force spreads with spherical symmetry will obey the inverse square law. This is because as the force spreads, its intensity at a given distance r is spread uniformly across a sphere of radius r. If the total force on any sphere centered at the origin is the same—as would be expected with a constant source—then the force acting at a point is the ratio of the total force to the area of the sphere, so that force at a point is inversely proportional to r^2. It is because the inverse square law is geometric in nature that it applies to many different phenomena, including gravitational force, electrostatic force, and radiation.

Check Your Reading: The earth has an average radius of 3963.21 miles. How high above the earth is the satellite in example 1?

The Inverse Square Law

These calculations only show us that the earth's gravitational force satisfies an inverse square law. However, we need to show that the inverse square law of gravity is universal—i.e., every point with mass or *point-mass* produces an inverse square gravitational force vector field. Thus, we rederive the law of gravity for an arbitrary planet about the sun.

To begin with, let us again assume that the magnitude of the gravitational force is

$$F(r) = \frac{mk}{r^n}$$

where r is the distance from the sun to the planet, m is the mass of the planet, and k, n are constants which must be determined. Since $F = ma$, the acceleration of the planet due to the sun's gravity is

$$a = \frac{k}{r^n}$$

Applying the natural logarithm to both sides yields

$$\ln(a) = \ln(k) - n\ln(r)$$

Since k is constant, $C = \ln(k)$ is also constant, so that

$$\ln(a) = C - n\ln(r) \tag{0.6}$$

Moreover, if we let $Y = \ln(a)$ and $X = \ln(r)$, then we can rewrite (0.6) as

$$Y = C - nX$$

That is, Y and X are linearly related, which means that least squares can be used to predict n and C. To do so, we need only know the distances to the planets and their accelerations due to the sun's gravity.

Fortunately, the orbits of the first five planets are nearly circular, which means they can be modeled as uniform circular motions. Let us recall that uniform circular motion is parametrized by

$$\mathbf{r}(t) = \left\langle R_p \cos\left(\frac{2\pi}{p}t\right), R_p \sin\left(\frac{2\pi}{p}t\right)\right\rangle, \quad t \in [0, p]$$

Two time derivatives yield an acceleration of

$$\mathbf{a}(t) = \left\langle \frac{-4\pi^2 R_p}{p^2}\cos\left(\frac{2\pi}{p}t\right), \frac{-4\pi^2 R_p}{p^2}\sin\left(\frac{2\pi}{p}t\right)\right\rangle$$

As a result, the magnitude a of the acceleration vector is

$$a = \frac{4\pi^2 R_p}{p^2} \tag{0.7}$$

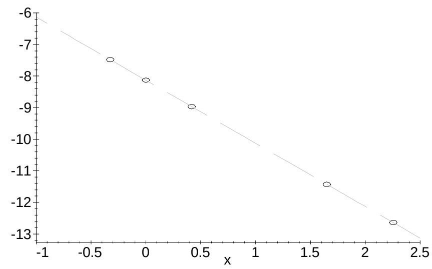
In the table below, the radii and periods of the 2^{nd} through 6^{th} planets are used to compute the accelerations due to the sun's gravity:

Planet	$R_p(AU)$	$p(days)$	$a = \frac{4\pi^2 R_p}{p^2}$
Venus	0.723	225	5.6381029×10^{-4}
Earth	1	365	$2.96328899 \times 10^{-4}$
Mars	1.52	687	$1.27142238 \times 10^{-4}$
Jupiter	2.5	4343	$1.08838719 \times 10^{-5}$
Saturn	9.54	10,767	3.2487679×10^{-6}

The data is then transformed using $X = \ln(R_p)$ and $Y = \ln(a)$:

Planet	$X = \ln(R_p)$	$Y = \ln(a)$
Venus	-0.324346	-7.480793
Earth	0	-8.124041
Mars	0.418710	-8.970204
Jupiter	1.648659	-11.428820
Saturn	2.255493	-12.637234

The (X, Y) data set does appear to lie on a straight line.



Application of the least squares algorithm to the (X, Y) predicts the linear model

$$y = -2.00001\,94805x - 8.12879\,62272$$

(which has a Pearson's r-value of $r = 0.99998$). That is, the model predicts that

$$n = 2.0000194805$$

and since n must be an integer, clearly we have $n = 2$. Moreover,

$$\ln(k) = -8.12879\,62272$$

which implies that $k = 2.94923\,0075 \times 10^{-4}$. Thus, the acceleration of the planets due to the sun's gravity is

$$a = \frac{2.94923\,0075 \times 10^{-4}}{r^2} \tag{0.8}$$

Exercises

Find the position vector, the speed and the magnitude of the acceleration of the uniform circular motion with radius r and period p. Convert all measurements to feet and seconds.

1. $r = 2$ feet, $p = 1$ second
2. $r = 2$ feet, $p = 2$ seconds
3. $r = 3$ feet, $p = 0.5$ second
4. $r = 2.5$ feet, $p = 0.2$ seconds
5. $r = 3963.21$ miles, $p = 24$ hours
6. $r = 238,957$ miles, $p = 27.322$ days
7. $r = 92,956,000$ miles, $p = 365.25$ days

8. Show that if an object is in uniform circular motion with period p and radius r, then

$$a = \frac{4\pi^2 r}{p^2}$$

9. In this exercise, we estimate the radius of the earth and then use uniform circular motion to determine the acceleration we feel due to the earth's spinning.

i. The sun rises in Knoxville at 7:14 a.m., but it rises in Nashville about 11 minutes later. If the distance from Knoxville to Nashville is 190 miles, then how fast in miles per minute is the earth spinning?

ii. There are $24 \cdot 60 = 1440$ minutes in a day. Multiply 1440 by the speed in the previous exercise to determine the circumference, and then divide by 2π to obtain the radius of the earth. (Note: We should be doing this at the equator.)

10. Kepler's third law says that the square of the period is proportional to the cube of the radius. That is,

$$p^2 = \lambda r^3$$

where λ is a constant determined by the planet being orbited.

i. Use the vector parametrization of a satellite 100 miles above the earth to determine λ.

ii. Use the vector parametrization of the moon to compute λ. Why is it the same as the estimate in (a)?

iii. How high above the earth must a satellite be in order to be in geosynchronous orbit? (Hint: what must its period be?)

11. Assume that the second through sixth planets have circular orbits, then we can use the result in exercise 8 to complete the following table, where the accelerations a are in units of Astronomical units per earth day per earth day.

Planet	$R_p(AU)$	$p(days)$	$a = \dfrac{4\pi^2 R_p}{p^2}$
Venus	0.723	225	
Earth	1	365	
Mars	1.52	687	
Jupiter	2.5	4343	
Saturn	9.54	10,767	

12. Plot the data points (R_p, a) in the first and third columns of the first table. Then graph the function in (0.8). Does the function appear to be a good fit of the original data?

13. In the derivation above, we omitted Mercury because we now know that a complete description of its motion requires general relativity, though Kepler's laws are a good approximation of Mercury's orbit. Below is the data for the first five planets. Transform using X and Y above, and then apply least squares to the transformed data. What is the prediction for n? Is it still close to 2?

Planet	$R_p(AU)$	$p(days)$	$a = \dfrac{4\pi^2 R_p}{p^2}$
Mercury	0.387	89	1.9288155×10^{-3}
Venus	0.723	225	5.6381029×10^{-4}
Earth	1	365	$2.96328899 \times 10^{-4}$
Mars	1.52	687	$1.27142238 \times 10^{-4}$
Jupiter	2.5	4343	$1.08838719 \times 10^{-5}$

14. Let $n = 2$ in (0.3) and solve for k.

15. Let $n = 2$ in (0.4) and solve for k.

16. The constant k is the product of the earth's mass M and the universal gravitational constant, G, which is equal to

$$G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$$

If k is translated into metric units, then we have

$$k = 3.98866 \times 10^{14} \frac{Nm^2}{kg}$$

Use this to determine the mass of the earth.