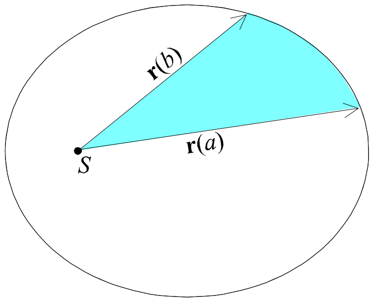
Kepler's Second Law

Since Keplerian motion is in a plane, we can assume that the plane containing the motion is the xy-plane. We also assume that the sun is at the origin of the xy-plane. Our immediate task then becomes that of determining the area of the "sector" shaded in blue below:



As was shown earlier in this semester, the area is

$$Area = \iint\limits_R dA$$

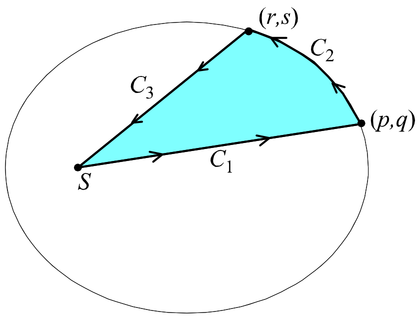
where R is the region shaded in blue above. However, by Green's theorem we have

$$\frac{1}{2}\oint\limits_{\partial R} ydx - xdy = \iint\limits_R dA \tag{0.1}$$

so that the area is given by

$$Area = \frac{1}{2}\int\limits_{C_1} ydx - xdy + \frac{1}{2}\int\limits_{C_2} ydx - xdy + \frac{1}{2}\int\limits_{C_3} ydx - xdy$$

where the curves C_1, C_2 and C_3 are as shown below:



Now, the curve C_1 is the line segment from $(0,0)$ to the point (p,q), so it is parametrized by

$$\begin{array}{ll} x = pt & \quad t \in [0,1] \\ y = qt & \end{array}$$

As a result, we have

$$\begin{aligned} \frac{1}{2}\int\limits_{C_1} ydx - xdy &= \frac{1}{2}\int_0^1 \left(y\frac{dx}{dt} - x\frac{dy}{dt} \right) dt \\ &= \frac{1}{2}\int_0^1 (qt \cdot p - pt \cdot q)\, dt \\ &= \frac{1}{2}\int_0^1 (pqt - pqt)\, dt \\ &= 0 \end{aligned}$$

Likewise, the integral over C_3 vanishes, and thus we have

$$Area = \frac{1}{2}\int_{C_2} xdy - ydx$$

Now, since $\mathbf{r}(t)$ is in the xy-plane, it must be of the form

$$\mathbf{r}(t) = [f(t), g(t), 0]$$

As a result, the area swept out by $r(t)$ over the interval $[a, b]$ is given by

$$\begin{aligned} Area &= \frac{1}{2}\int_a^b \left(x\frac{dy}{dt} - y\frac{dx}{dt} \right) dt \\ &= \frac{1}{2}\int_a^b \left(f(t) g'(t) - g(t) f'(t) \right) dt \end{aligned}$$

However, it is easy to see that

$$||\mathbf{r} \times \mathbf{v}|| = f(t) g'(t) - g(t) f'(t) \tag{0.2}$$

Thus, the area swept out by $\mathbf{r}(t)$ over the time interval $[a, b]$ is

$$Area = \frac{1}{2}\int_a^b ||\mathbf{r} \times \mathbf{v}|| \, dt$$

EXAMPLE 1 Find the area swept out by

$$\mathbf{r}(t) = [3\cos(t), 3\sin(t), 0]$$

over the time interval $[0, \pi]$.

Solution: First, we compute $\mathbf{r} \times \mathbf{v}$:

$$\mathbf{r} \times \mathbf{v} = [0, 0, 9]$$

and then we compute

$$Area = \frac{1}{2}\int_0^\pi ||\mathbf{r} \times \mathbf{v}|| \, dt = \frac{1}{2}\int_0^\pi 9dt = \frac{1}{2}9\pi$$

In fact, the region swept out by $\mathbf{r}(t)$ is the upper half circle of radius 3, which indeed has an area of $9\pi/2$.

If $\mathbf{r}(t)$ is a Keplerian motion, then we have already shown that $\mathbf{r} \times \mathbf{v}$ is constant. Thus, if

$$L = ||\mathbf{r} \times \mathbf{v}||$$

then the area swept out over a time interval $[a, b]$ is

$$Area = \frac{1}{2}\int_a^b Ldt = \frac{1}{2}L(b - a)$$

Suppose now that $[p, q]$ is a time interval with the same length as $[a, b]$—that is, that $p - q = b - a$. Then clearly, we must have

$$\frac{1}{2}L(b - a) = \frac{1}{2}L(p - q)$$

which implies that the area swept out over the time interval $[a, b]$ is the same as the area swept out over the time interval $[p, q]$. That is, equal areas are swept out in equal times.

Check Your Reading: What type of curve is given in example 1?

The Lenz Vector

The energy and angular momentum constants can be used to obtain a particularly nice form of the solutions, one that can subsequently be used to study orbits of satellites and other celestial phenomena.

To begin with, lets write the energy and angular momentum conservation laws in polar coordinates. In polar coordinates, $\mathbf{r} = r\mathbf{u}$, where $\mathbf{u} = [\cos(\theta), \sin(\theta), 0]$. It follows that

$$\frac{d\mathbf{u}}{dt} = [-\sin(\theta), \cos(\theta), 0]\frac{d\theta}{dt} = \mathbf{u}'\frac{d\theta}{dt} \tag{0.3}$$

It is then rather easy to show that

$$\mathbf{u}\times\frac{d\mathbf{u}}{dt} = \frac{d\theta}{dt}[0,0,1]$$

Let's now convert $\mathbf{L} = \mathbf{r} \times \mathbf{v}$ into polar coordinates. The velocity $\mathbf{v}$ is

$$\mathbf{v} = \frac{d}{dt}(r\,\mathbf{u}) = r\frac{d\mathbf{u}}{dt} + \frac{dr}{dt}\mathbf{u}$$

since $\mathbf{r} = r\,\mathbf{u}$. It follows that the angular velocity vector is

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{v} \\ &= r\,\mathbf{u}\times\left(r\frac{d\mathbf{u}}{dt} + \frac{dr}{dt}\mathbf{u}\right) \\ &= r^2\left(\mathbf{u}\times\frac{d\mathbf{u}}{dt}\right) + r\frac{dr}{dt}(\mathbf{u}\times\mathbf{u}) \end{aligned}$$

Since $\mathbf{u} \times \mathbf{u} = 0$, (0.3) implies that

$$\mathbf{L} = r^2\left(\mathbf{u}\times\frac{d\mathbf{u}}{dt}\right) = r^2\frac{d\theta}{dt}[0,0,1] \tag{0.4}$$

which further implies that the magnitude of $\mathbf{L}$ is

$$L = r^2\frac{d\theta}{dt} \tag{0.5}$$

Moreover, in polar coordinates, the acceleration vector is

$$\mathbf{r}'' = \frac{-k}{r^3}\mathbf{r} = \frac{-k}{r^3}[r\cos(\theta), r\sin(\theta), 0]$$

from which we get

$$\mathbf{r}'' = \frac{-k}{r^2}[\cos(\theta), \sin(\theta), 0]$$

As a result, we have

$$\begin{aligned} \mathbf{r}'' \times \mathbf{L} &= \frac{-k}{r^2}[\cos(\theta), \sin(\theta), 0] \times r^2\frac{d\theta}{dt}[0,0,1] \\ &= -k\frac{d\theta}{dt}[\cos(\theta), \sin(\theta), 0] \times [0,0,1] \\ &= k\,[-\sin\theta, \cos\theta, 0]\frac{d\theta}{dt} \end{aligned} \tag{0.6}$$

Thus, from (0.3) we know that

$$\mathbf{r}'' \times \mathbf{L} = k\frac{d\mathbf{u}}{dt}$$

But $\mathbf{L}$ is constant, so

$$\frac{d}{dt}(\mathbf{r}' \times \mathbf{L}) = \mathbf{r}'' \times \mathbf{L}$$

Combining these last two equations yields

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{L}) = k\frac{d\mathbf{u}}{dt}$$

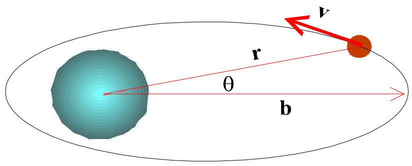
which implies that for some constant $\mathbf{b}$,

$$\mathbf{v} \times \mathbf{L} = k\mathbf{u} + \mathbf{b}$$

Finally, since $\mathbf{r} = r\mathbf{u}$, we have $\mathbf{u} = \mathbf{r}/r$ so that we get

$$\mathbf{v} \times \mathbf{L} = \frac{k}{r}\mathbf{r} - \mathbf{b} \tag{0.7}$$

The vector $\mathbf{b}$ in (0.7) is called the *Lenz vector*. From it we can derive the polar equation of motion. To do so, we assume that $\mathbf{p}$ lies along the x-axis, so that the angle between $\mathbf{r}$ and $\mathbf{b}$ is θ.



The dot product of $\mathbf{r}$ with the Lenz vector thus yields

$$\mathbf{r}\cdot(\mathbf{v} \times \mathbf{L}) = \frac{k}{r}(\mathbf{r}\cdot\mathbf{r}) - \mathbf{r}\cdot\mathbf{b}$$

However, a property of the cross and dot product implies that

$$\mathbf{r}\cdot(\mathbf{v} \times \mathbf{L}) = \mathbf{L}\cdot(\mathbf{r} \times \mathbf{v}) = \mathbf{L}\cdot\mathbf{L} = L^2$$

Moreover, since

$$\begin{aligned} L^2 &= \frac{k}{r}r^2 - rb\cos(\theta) \\ &= kr - rb\cos(\theta) \\ &= r(k - b\cos(\theta)) \end{aligned}$$

As a result, we have

$$r = \frac{L^2}{k - b\cos(\theta)}$$

If we now divide the numerator and denominator by k, then we have

$$r = \frac{L^2/k}{1 - \frac{b}{k}\cos(\theta)}$$

Thus, we let $\varepsilon = b/k$ and let $p = L^2/k$. As a result, the equation of the orbit is

$$r = \frac{p}{1 - \varepsilon\cos(\theta)} \tag{0.8}$$

Check Your Reading: What type of orbit does (0.8) reduce to when $\varepsilon = 0$?

Energy and Kepler's 3rd law

Let's derive two more very important concepts in the study of Kepler's problem. Equation (0.7) implies that the Lenz vector is given by

$$\mathbf{b} = \frac{k}{r}\mathbf{r} - \mathbf{v} \times \mathbf{L}$$

while the energy equation says that

$$H = \frac{1}{2}v^2 - \frac{k}{r}$$

Let's combine these two, along with the definition $b = k\varepsilon$ to find an equation relating the eccentricity to the total energy H.

To begin with, let's notice that

$$\begin{aligned}
\mathbf{b} \cdot \mathbf{b} &= \left(\frac{k}{r}\mathbf{r} - \mathbf{v} \times \mathbf{L}\right) \cdot \left(\frac{k}{r}\mathbf{r} - \mathbf{v} \times \mathbf{L}\right) \\
b^2 &= \frac{k^2}{r^2}\mathbf{r} \cdot \mathbf{r} - 2\frac{k}{r}\mathbf{r} \cdot (\mathbf{v} \times \mathbf{L}) + (\mathbf{v} \times \mathbf{L}) \cdot (\mathbf{v} \times \mathbf{L}) \\
k^2\varepsilon^2 &= k^2 - 2\frac{k}{r}(\mathbf{r} \times \mathbf{v}) \cdot \mathbf{L} + (\mathbf{v} \times \mathbf{L}) \cdot (\mathbf{v} \times \mathbf{L})
\end{aligned}$$

using the triple scalar product. By definition, $\mathbf{L} = \mathbf{r} \times \mathbf{v}$ and as a consequence, $\mathbf{v} \perp \mathbf{L}$. Thus, $\|\mathbf{v} \times \mathbf{L}\| = vL\sin\left(\frac{\pi}{2}\right) = vL$, so that

$$\begin{aligned}
k^2\varepsilon^2 &= k^2 - 2\frac{k}{r}\mathbf{L} \cdot \mathbf{L} + \|\mathbf{v} \times \mathbf{L}\|^2 \\
k^2\varepsilon^2 &= k^2 - 2\frac{k}{r}L^2 + v^2L^2 \\
&= k^2 + 2L^2\left(\frac{1}{2}v^2 - \frac{k}{r}\right) \\
k^2\varepsilon^2 &= k^2 + 2L^2H
\end{aligned}$$

As a result, we have

$$\varepsilon^2 = 1 + 2H\frac{L^2}{k^2} \quad or \quad \varepsilon = \sqrt{1 + 2H\frac{L^2}{k^2}}$$

The relationship between energy and eccentricity is fundamental to Kepler's problem, as is *Kepler's third law,* which follows from the fact that the equation of the orbit is

$$r = \frac{p}{1 - \varepsilon\cos(\theta)}$$

where $\varepsilon = b/k$ and $p = L^2/k$. The semi-major axis a satisfies the equation

$$r(0) + r(\pi) = 2a$$

so that it follows that

$$\frac{p}{1-\varepsilon} + \frac{p}{1+\varepsilon} = 2a$$

Solving for the parameter p leads to

$$p = a\left(1 - \varepsilon^2\right)$$

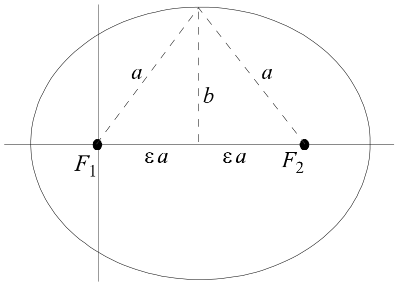
and since $p = L^2/k$, we have

$$L^2 = a\left(1 - \varepsilon^2\right)k \tag{0.9}$$

Now, we define T to be the period of the orbit—that is, the time required to sweep out the entire ellipse. Thus, the area of the entire ellipse is

$$Area = \frac{1}{2}\int_0^T ||\mathbf{r} \times \mathbf{v}||\, dt = \frac{1}{2}LT$$

However, the area of an ellipse is πab, where b is the *semi-minor axis,* which is the greatest distance from the y-axis to the ellipse:



By definition, the focus F_2 is a distance $2\varepsilon a$ from F_1, so that by symmetry and the Pythagorean theorem, we have

$$b = a\sqrt{1 - \varepsilon^2}$$

As a result, we have

$$\pi a^2\sqrt{1 - \varepsilon^2} = \frac{1}{2}LT$$

which upon solving for T and squaring yields

$$\frac{4\pi^2 a^4\left(1 - \varepsilon^2\right)}{T^2} = L^2 \tag{0.10}$$

Finally, we combine (0.9) with (0.10) to obtain

$$\frac{4\pi^2 a^4\left(1 - \varepsilon^2\right)}{T^2} = a\left(1 - \varepsilon^2\right)k$$

cancellation then yields

$$4\pi^2 a^3 = kT^2$$

which we rewrite as

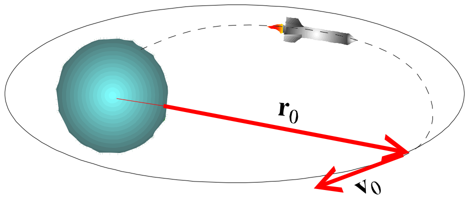
$$T^2 = \frac{k}{4\pi^2}a^3$$

That is, the *square of the period is proportional to the cube of the semi-major axis.*

Check Your Reading: What is T in days for the earth's orbit about the sun?

Orbits of Satellites

The basic idea in rocketry is that upon final engine cutoff, the rocket will have an position $\mathbf{r}_0$ and a velocity of $\mathbf{v}_0$. That is, at engine cutoff, a satellite will enter a Keplerian orbit with an initial position of $\mathbf{r}_0$ and an initial velocity of $\mathbf{v}_0$.



Thus, it is imperative that we know how to determine the equation of a Keplerian orbit given only k, the initial velocity $\mathbf{r}_0$ and the initial velocity $\mathbf{v}_0$. For convenience in this task, we will measure all distances in miles and all times in seconds. Thus, in terms of miles and seconds we have

$$k = GM = 95,194.14$$

and the radius of the earth as $R = 3963$ miles.

Given the initial data $\mathbf{r}_0$ and $\mathbf{v}_0$, the angular velocity is easily determined as

$$L = ||\mathbf{r}_0 \times \mathbf{v}_0||$$

However, in proving Kepler's first law,

$$r = \frac{L^2/k}{1 - \varepsilon \cos(\theta)}$$

the eccentricity ε arises as an arbitrary constant. Before Kepler's laws can be considered useful, we must have a means of determining the eccentricity from the initial data $\mathbf{r}_0$ and $\mathbf{v}_0$.

The eccentricity ε, on the other hand, is the result of two calculations. First, we use the energy equation

$$\frac{1}{2}v^2 - \frac{k}{r} = H \tag{0.11}$$

to determine the total energy H, after which the eccentricity follows from

$$\varepsilon = \sqrt{1 + 2H\frac{L^2}{k^2}}$$

which is verified in appendix 3. Notice that if H is positive, then $\varepsilon > 1$, if $H = 0$, then $\varepsilon = 1$ and if $H < 0$, then $\varepsilon < 1$. Thus, elliptical orbits correspond to a negative total energy, which is to say that the potential energy is always dominant.

EXAMPLE 2 Suppose that at time $t = 0$, a satellite is observed to have the initial data

$$\mathbf{r}_0 = [0, 4063, 0] \times [4, 0, 0] \qquad \text{and} \qquad \mathbf{v}_0 = [4, 0, 0]$$

—that is, 100 miles above the earth traveling at 4 miles per second. What is the equation of its orbit?

Solution: It is easy to show that

$$\mathbf{r}_0 \times \mathbf{v}_0 = [0, 0, -16252]$$

so that the parameter L is given by

$$L = ||\mathbf{r}_0 \times \mathbf{v}_0|| = 16,252$$

Likewise, it is obvious that $r_0 = 4063$ miles and $v_0 = 4$. Thus, the energy of the orbit is

$$H = \frac{1}{2}v_0^2 - \frac{k}{r_0} = \frac{1}{2}(4)^2 - \frac{95194.14}{4063} = -15.4295$$

and the eccentricity of the orbit is

$$\varepsilon = \sqrt{1 + 2(-15.4295)\frac{(16252)^2}{(95194.14)^2}} = 0.3171$$

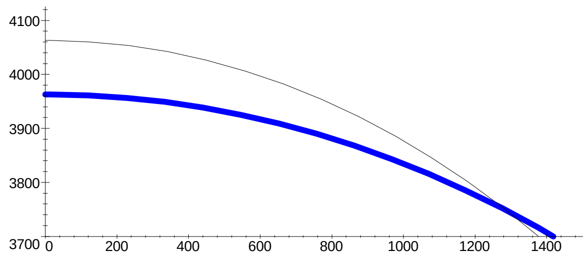
Because the orbit begins on the y-axis, we use $\sin(\theta)$ in place of $\cos(\theta)$ in the equation of the orbit.

$$r = \frac{L^2/k}{1 - 0.3171\sin(\theta)} = \frac{(16252)^2/(95194.14)}{1 - 0.3171\sin(\theta)}$$

which reduces to

$$r = \frac{2774.62}{1 - 0.3171\sin(\theta)}$$

Unfortunately, the orbit in example 2 is not a very good orbit in the sense that the rocket will definitely strike the earth. Indeed, as shown below, the rocket's orbit shown in black intercepts the "earth" shown in heavy blue.



Let's try again with slightly faster satellite.

EXAMPLE 3 Let $\mathbf{r}_0 = [4063, 0, 0]$ and suppose that

$$\mathbf{v}_0 = [0, 5, 0]$$

—that is, let's suppose it enters a Keplerian orbit at 5 miles per second. What is the equation of the orbit?

Solution: In this case, L is

$$L = ||\mathbf{r}_0 \times \mathbf{v}_0|| = ||[0, 0, 20315]|| = 20,315$$

Likewise, it is obvious that $r_0 = 4063$ miles and $v_0 = 5$. Thus, the energy of the orbit is

$$H = \frac{1}{2}v_0^2 - \frac{k}{r_0} = \frac{1}{2}(5)^2 - \frac{95194.14}{4063} = -10.9295$$

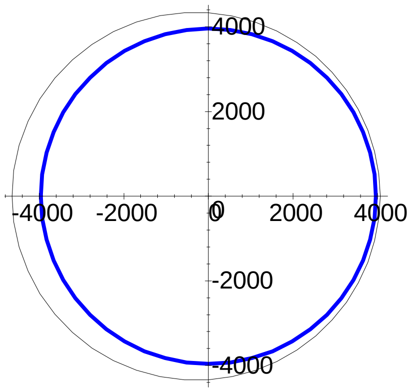
and the eccentricity of the orbit is

$$\varepsilon = \sqrt{1 + 2\left(-10.9295\right)\frac{\left(20315\right)^2}{\left(95194.14\right)^2}} = 0.067$$

This orbit begins on the x-axis, but let's use $-\cos(\theta)$ in place of $\cos(\theta)$:

$$r = \frac{L^2/k}{1 + 0.067\cos(\theta)} = \frac{\left(20315\right)^2/\left(95194.14\right)}{1 + 0.067\cos(\theta)} = \frac{4335.3427}{1 + 0.067\cos(\theta)}$$

which is shown below along with the "earth" shown in blue.



Moreover, we can use Kepler's third law to determine how long it takes for the satellite to orbit the earth. In particular, we will show in appendix 2 that

$$a = \frac{L^2/k}{1 - \varepsilon^2}$$

Consequently, for this orbit we have

$$a = \frac{\left(20315\right)^2/\left(95194.14\right)}{1 - \left(0.067\right)^2} = 4354.892$$

so that by Kepler's third law, we have

$$T = \sqrt{\frac{4\pi^2}{k}a^3} = \sqrt{\frac{4\pi^2}{95194.14}\left(4354.892\right)^3} = 5852.49 \text{ sec}$$

which is approximately an hour and 40 minutes.

Exercises

Construct the ellipse corresponding to the given initial conditions. Does the object ever hit the earth? If not, does it escape from the earth's gravitational field? If both questions have an answer of no, then determine the period of the satellite.

1. $\mathbf{r}_0 = [4063, 0, 0]$, $\mathbf{v}_0 = [0, 7, 0]$
2. $\mathbf{r}_0 = [5000, 0, 0]$, $\mathbf{v}_0 = [0, 1, 0]$
3. $\mathbf{r}_0 = [4163, 0, 0]$, $\mathbf{v}_0 = [0, 5, 0]$
4. $\mathbf{r}_0 = [4063, 0, 0]$, $\mathbf{v}_0 = [7, 0, 0]$
5. $\mathbf{r}_0 = [4163, 0, 0]$, $\mathbf{v}_0 = [0, 7, 0]$

6. The escape velocity of a particle at a distance r_0 from the earth's center is the speed v_0 necessary to make the total energy H equal to 0. Explain why this is a good definition of escape velocity and then determine the escape velocity of an object at the earth's surface $(r_0 = 3963)$.

7. When $\mathbf{r}_0 \perp \mathbf{v}_0$, then $L = r_0 v_0$. Suppose that $r_0 = 4063$ so that $L = 4063 v_0$. What initial velocity v_0 will result in a circular orbit (i.e., $\varepsilon = 0$). How fast is that in miles per hour, and what is the period of that orbit?

8. Verify (0.4).

9. Verify (0.6).