1. Evaluate the iterated integral

$$\int_0^{\pi} \int_0^{x} \sin(x)\, dydx$$

Solution: First we evaluate inner integral:

$$\int_0^{\pi} \int_0^{x} \sin(x)\, dydx = \int_0^{\pi} \sin(x)\, y|_0^x\, dx = \int_0^{\pi} x \sin(x)\, dx$$

and then integration by parts with $u = x$ and $dv = \sin(x)\, dx$ yields

$$\int_0^{\pi} \int_0^{x} \sin(x)\, dydx = -x\cos(x)|_0^{\pi} + \int_0^{\pi} \cos(x)\, dx = \pi$$

2. Find the volume of the solid bound between $z = 0$ and $z = x + 2y$ over

$$R: \begin{array}{ll} x = 0 & y = 0 \\ x = 2 & y = x^2 \end{array}$$

Solution: As a triple integral, we have

$$V = \iiint_S dV = \int_0^2 \int_0^{x^2} \int_0^{x+2y} dzdydx = \int_0^2 \int_0^{x^2} (x + 2y)\, dydx = \frac{52}{5}$$

3. Evaluate the following iterated integral by changing it from a Type I to a type II or vice versa:

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin(y)}{y} dydx$$

Solution: First, we convert to double integral over region R.

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin(y)}{y} dydx = \iint_R \frac{\sin(y)}{y} dA$$

The region R in type II is given by $y = 0$, $y = \pi$, $x = 0$, $x = y$, so that

$$\begin{aligned} \int_0^{\pi} \int_x^{\pi} \frac{\sin(y)}{y} dydx &= \iint_R \frac{\sin(y)}{y} dA \\ &= \int_0^{\pi} \int_0^{y} \frac{\sin(y)}{y} dxdy \\ &= \int_0^{\pi} \frac{\sin(y)}{y} y\, dy \\ &= \int_0^{\pi} \sin(y)\, dy \\ &= 2 \end{aligned}$$

4. Evaluate the following iterated integral by changing it from a Type I to a Type II or vice versa:

$$\int_0^1 \int_0^{1-x} \sec^2\left(2y - y^2\right) dydx$$

Solution: Convert to double integral over region R, which in type II is given by $y = 0$, $y = 1$, $x = 0$, $x = 1 - y$.

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \sec^2\left(2y - y^2\right) dydx &= \iint_R \sec^2\left(2y - y^2\right) dA \\
&= \int_0^1 \int_0^{1-y} \sec^2\left(2y - y^2\right) dxdy \\
&= \int_0^1 \sec^2\left(2y - y^2\right) \ (1 - y)\, dy
\end{aligned}$$

Now we let $u = 2y - y^2$ and $du = 2 - 2y$, so that $u(0) = 0$ and $u(1) = 1$ and

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \sec^2\left(2y - y^2\right) dydx &= \iint_R \sec^2\left(2y - y^2\right) dA \\
&= \int_0^1 \sec^2\left(2y - y^2\right) \ (1 - y)\, dy \\
&= \frac{1}{2} \int_0^1 \sec^2(u)\, du \\
&= \frac{1}{2} \tan(1)
\end{aligned}$$

5. Find the mass of the cylinder between $z = 0$ and $z = 1$ over the interior of the unit circle if its mass density is given by $\rho(x, y, z) = |y|$.

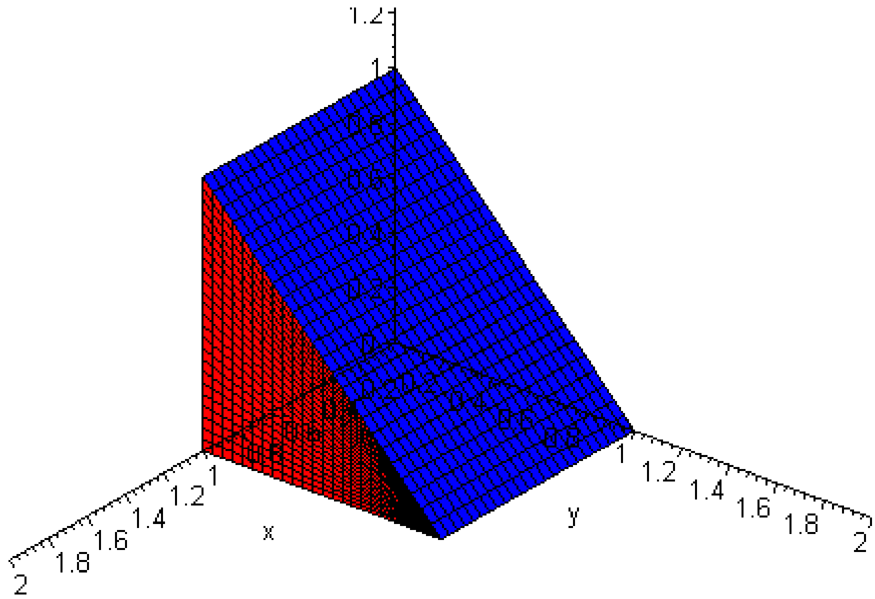
Solution: The mass is given by the triple integral

$$\begin{aligned}
M &= \iiint_S \rho(x, y, z)\, dV \\
&= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^1 |y|\, dzdydx \\
&= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} |y|\, dydx
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} y\,dy\,dx \\
&= 2\int_{-1}^{1} \left.\frac{y^2}{2}\right|_{0}^{\sqrt{1-x^2}} dx \\
&= \int_{-1}^{1}\left(1-x^2\right)dx \\
&= \frac{4}{3}
\end{aligned}
$$

6. What is the volume of the polyhedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,0,1)$, and $(1,0,1)$?

   Solution: The solid is a "wedge" bound between the xy-plane and the plane passing through the points $(0,1,0)$, $(1,1,0)$, $(0,0,1)$, and $(1,0,1)$. The equation of the latter plane is easily shown to be $z = 1 - y$.



   The region in the xy-plane over which the solid is defined is simply the unit square, so that

$$V = \int_0^1\int_0^1\int_0^{1-y} dz\,dx\,dy = \frac{1}{2}$$

7. Suppose that the probability density for the time required to complete the "A" component of an exam is given by

$$p_A(x) = \begin{cases} 0 & if \quad x < 0 \\ \frac{1}{30}e^{-x/30} & if \quad x \geq 0 \end{cases}$$

(time in minutes). Suppose the event of completing the "B" component of the exam has the same density. If the completion of the A and B sections are independent events, then what is the probability that a student will complete the entire exam (i.e., both sections) in less than an hour?

Solution: Independence of events "*completing A*" and "*completing B*" imply that the joint density of A and B is

$$p(x,y) = p_A(x)\, p_B(y)$$

Since the distributions are identical, the probability is

$$\Pr(x + y \leq 60) = \frac{1}{900} \iint_R e^{-x/30} e^{-y/30} dA$$

where R is the region defined by $x = 0$, $x = 60$, $y = 0$, and $y = 60 - x$. Thus,

$$\Pr(x + y \leq 60) = \frac{1}{900} \int_0^{60} \int_0^{60-x} e^{-x/30} e^{-y/30} dy dx = 1 - 3e^{-2}$$

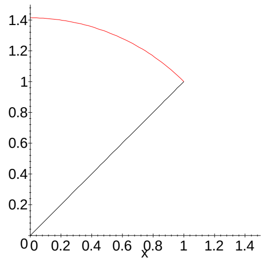
and since $1 - 3e^{-2} \approx 0.59399$, there is about a 59.4% chance of completing the entire exam in less than an hour.

8. Evaluate by converting to polar coordinates:

$$\int_0^1 \int_x^{\sqrt{2-x^2}} dy dx$$

Solution: The region $x = 0$, $x = 1$, $y = x$, $y = \sqrt{2 - x^2}$ is given in polar coordinates by

$$R: \quad \begin{array}{ll} \theta = \frac{\pi}{4} & r = 0 \\ \theta = \frac{\pi}{2} & r = \sqrt{2} \end{array}$$



so that changing to a double integral and using $dA = rdrd\theta$ yields

$$\int_0^1 \int_x^{\sqrt{2-x^2}} dydx = \iint_R dA = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} rdrd\theta = \frac{\pi}{4}$$

9. Evaluate by converting to polar coordinates

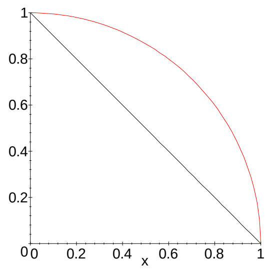
$$\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} \frac{dydx}{(x^2+y^2)^{3/2}}$$

Solution: The curve $y = 1 - x$ is the same as $x + y = 1$, and thus has a pullback of

$$r\cos(\theta) + r\sin(\theta) = 1, \qquad r = \frac{1}{\cos(\theta) + \sin(\theta)}$$

The curve $y = \sqrt{1-x^2}$ is the same as $x^2 + y^2 = 1$, or $r = 1$ in polar. Thus, the region $x = 0$, $x = 1$, $y = 1 - x$, $y = \sqrt{1-x^2}$ is given in polar coordinates by

$$R: \quad \begin{array}{ll} \theta = 0 & r = \frac{1}{\cos(\theta)+\sin(\theta)} \\ \theta = \frac{\pi}{2} & r = 1 \end{array}$$



so that changing to a double integral and using $dA = r dr d\theta$ yields

$$\begin{aligned}
\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} \frac{dydx}{\left(x^2+y^2\right)^{3/2}} &= \iint_R \frac{1}{\left(x^2+y^2\right)^{3/2}} dA \\
&= \int_0^{\pi/2} \int_{\frac{1}{\cos(\theta)+\sin(\theta)}}^{1} \frac{1}{\left(r^2\right)^{3/2}}\, r dr d\theta \\
&= \int_0^{\pi/2} \int_{\frac{1}{\cos(\theta)+\sin(\theta)}}^{1} \frac{1}{r^2} dr d\theta \\
&= \int_0^{\pi/2} \left. \frac{-1}{r} \right|_{\frac{1}{\cos(\theta)+\sin(\theta)}}^{1} d\theta \\
&= \int_0^{\pi/2} \left(\cos(\theta) + \sin(\theta) - 1\right) d\theta \\
&= 2 - \frac{\pi}{2}
\end{aligned}$$

10. Use the coordinate transformation $T(u,v) = \langle u, \sqrt{v} \rangle$ to evaluate

$$\int_0^{\sqrt{\pi}} \int_0^{\sqrt{\pi}} y \sin\left(y^2\right) dydx$$

Solution: If $x = 0$, then $u = 0$. If $x = \sqrt{\pi}$, then $u = \sqrt{\pi}$. If $y = 0$, then $v = 0$. If $y = \sqrt{\pi}$, then $v = \pi$. Thus, the pullback of the region of integration into uv-coordinates is

$$\begin{array}{lll} S: & u = 0 & v = 0 \\ & u = \sqrt{\pi} & v = \pi \end{array}$$

Since $T_u = \langle 1, 0 \rangle$ and since $T_v = \left\langle 0, \frac{1}{2} v^{-1/2} \right\rangle$, the Jacobian determinant is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2} v^{-1/2} \end{vmatrix} = \frac{1}{2} v^{-1/2}$$

Thus, we have

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \int_0^{\sqrt{\pi}} y \sin\left(y^2\right) dydx &= \iint_R y \sin\left(y^2\right) dA \\
&= \int_0^{\sqrt{\pi}} \int_0^{\pi} v^{1/2} \sin(v) \frac{1}{2} v^{-1/2} dvdu \\
&= \frac{1}{2} \int_0^{\sqrt{\pi}} \int_0^{\pi} \sin(v)\ dvdu \\
&= \sqrt{\pi}
\end{aligned}$$

11. Use the coordinate transformation $T(u,v) = \langle u, ve^{-u} \rangle$ to evaluate

$$\int_0^1 \int_0^1 ye^x dydx$$

Solution: The pullback of $x = 0$ is $u = 0$. The pullback of $x = 1$ is $u = 1$. The pullback of $y = 0$ is $ve^{-u} = 0$, or $v = 0$. The pullback of $y = 1$ is $ve^{-u} = 1$, or $v = e^u$. Thus, the pullback of the region of integration into uv-coordinates is

$$\begin{array}{lll} S: & u=0 & v=0 \\ & u=1 & v=e^u \end{array}$$

Since $T_u = \langle 1, -ve^{-u} \rangle$ and since $T_v = \langle 0, e^{-u} \rangle$, the Jacobian determinant is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -ve^{-u} \\ 0 & e^{-u} \end{vmatrix} = e^{-u}$$

Thus, $dA = e^{-u} dvdu$ and we have

$$\begin{aligned} \int_0^1 \int_0^1 ye^x dydx &= \iint_R ye^x dA \\ &= \int_0^1 \int_0^{e^u} ve^{-u} e^u \, e^{-u} dvdu \\ &= \int_0^1 \int_0^{e^u} ve^{-u} dvdu \\ &= \frac{1}{2}e - \frac{1}{2} \end{aligned}$$

12. Suppose $\rho(x,y,z) = xz(1-y)$ coulombs per cubic meter is the charge density of a "charge cloud" contained in the "box" given by $[0,1] \times [0,1] \times [0,1]$. What is the total charge inside the box?

Solution: If Ω denotes the box $[0,1] \times [0,1] \times [0,1]$, then

$$\begin{aligned} Q &= \int\int\int_\Omega xz(1-y)\,dV \\ &= \int_0^1 \int_0^1 \int_0^1 xz(1-y)\,dzdydx \\ &= \frac{1}{2} \int_0^1 \int_0^1 (x - xy)\,dydx \\ &= \frac{1}{8}\,C \end{aligned}$$

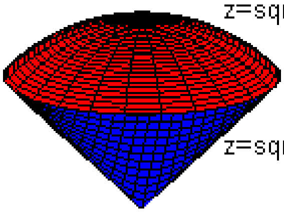
13. Evaluate by converting to spherical coordinates

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \frac{dzdydz}{z\sqrt{x^2+y^2+z^2}}$$

Solution: As a triple integral, we have

$$\iiint\limits_{S} \frac{dV}{z\sqrt{x^2+y^2+z^2}}$$

where S is the solid over the unit circle which is above the cone $z = \sqrt{x^2+y^2}$ and below the sphere with radius $\sqrt{2}$.



However, the cone in spherical coordinates is $\phi = \frac{\pi}{4}$, while the sphere is $\rho = \sqrt{2}$. Moreover, $z = \rho\cos(\phi)$, $x^2+y^2+z^2 = \rho^2$, and $dV = \rho^2 \sin(\phi)\, d\rho d\phi d\theta$ yields

$$\begin{aligned}\iiint_S \frac{dV}{z\sqrt{x^2+y^2+z^2}} &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \frac{\rho^2 \sin\phi d\rho d\phi d\theta}{\rho^2\cos(\phi)} \\ &= \sqrt{2}\int_0^{2\pi}\int_0^{\pi/4} \frac{\sin(\phi)}{\cos(\phi)} d\phi d\theta \\ &= \pi\sqrt{2}\,\ln(2)\end{aligned}$$

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14. What is the mass of the cone $x^2+y^2 = z^2$ between $z = 0$ and $z = 1$ if the mass density is constant at $\mu = 4$ kg per cubic meter?

Solution: In spherical coordinates, the cone $x^2+y^2 = z^2$ corresponds to the constant angle $\phi = \frac{\pi}{4}$. However, the plane $z = 1$ corresponds to

$$\rho\cos(\phi) = 1, \qquad or \qquad \rho = \sec(\phi)$$

Thus, the mass of the cone is

$$M = \iiint\limits_{cone} 4 \; dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec(\phi)} 4\rho^2 \sin(\phi) \, d\rho d\phi d\theta$$

Evaluating and simplifying the innter integral yields

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^{\pi/4} \frac{4\rho^3}{3} \sin(\phi) \Big|_0^{\sec(\phi)} \quad d\phi d\theta \\ &= \frac{4}{3} \int_0^{2\pi} \int_0^{\pi/4} \sec^3(\phi) \sin(\phi) \;\; d\phi d\theta \\ &= \frac{4}{3} \int_0^{2\pi} \int_0^{\pi/4} \sec(\phi) \quad \sec(\phi) \tan(\phi) \;\; d\phi d\theta \end{aligned}$$

since $\sin(\phi) \sec(\phi) = \tan(\phi)$. Letting $u = \sec(\phi)$, $du = \sec(\phi) \tan(\phi)$, $u(0) = 1$, and $u(\pi/4) = \sqrt{2}$ yields

$$\begin{aligned} M &= \frac{4}{3} \int_0^{2\pi} \int_1^{\sqrt{2}} u du d\theta \\ &= \frac{4}{3} \int_0^{2\pi} \frac{1}{2} d\theta \\ &= \frac{4\pi}{3} \end{aligned}$$