1. Evaluate the line integral

$$\int_C xdy - ydx$$

where C is the curve $\mathbf{r}(t) = \langle 2t, 3t \rangle$, t in $[0,1]$.

Solution: The pullback yields

$$\begin{aligned} \int_C xdy - ydx &= \int_0^1 \left(x\frac{dy}{dt} - y\frac{dx}{dt} \right) dt \\ &= \int_0^1 2t(3) - 3t(2)\, dt \\ &= \int_0^1 0dt \\ &= 0 \end{aligned}$$

2. Test for exactness. If exact, find its potential: $\mathbf{F}(x,y) = \langle x^2 + y^2, xy \rangle$

Solution: Since $M = x^2 + y^2$, $N = xy$, and $P = 0$, the curl of $\mathbf{F}$ is given by

$$curl\,(\mathbf{F}) = \langle 0 - 0, 0 - 0, 2x - x \rangle \neq \mathbf{0}$$

Thus, the field is not conservative and thus does not have a potential.

3. Test for exactness. If exact, find its potential: $\mathbf{F}(x,y) = \langle \sin(x+y), \sin(x+y) \rangle$

Solution: Since $M = N = \sin(x+y)$ and $P = 0$, the curl of $\mathbf{F}$ is given by

$$curl\,(\mathbf{F}) = \langle 0 - 0, 0 - 0, \cos(x+y) - \cos(x+y) \rangle = \mathbf{0}$$

Thus, $\mathbf{F}$ is conservative and its potential is given by

$$U(x,y) = \int \sin(x+y)\, dx = -\cos(x+y) + C(y)$$

Since $U_y = \sin(x+y) = N$, the function $C(y)$ must satisfy $C_y = 0$. Thus,

$$U(x,y) = -\cos(x+y) + G$$

for some constant G.

4. Test for exactness. If exact, find its potential: $\mathbf{F}(x,y,z) = \langle ye^x, e^x + 1, e^z \rangle$

Solution: The curl of $\mathbf{F}$ is given by

$$curl\,(\mathbf{F}) = \langle 0-0, 0-0, e^x - e^x \rangle = \mathbf{0}$$

Thus, the field is conservative and its potential is given by

$$U(x,y,z) = \int ye^x dx = ye^x + C(y,z)$$

However, $U_y = e^x + C_y$, so that

$$\begin{aligned} e^x + C_y &= e^x + 1 \\ C_y &= 1 \\ C &= y + k(z) \end{aligned}$$

Thus, the potential is given by $U(x,y,z) = ye^x + y + k(z)$. To determine k, we notice that

$$U_z = k'(z) = e^z \qquad \Longrightarrow \qquad k(x) = e^z + G$$

for some constant G. Thus, $U(x,y,z) = ye^x + y + e^z + G$.

5. Evaluate the integral below using the fundamental theorem for line integrals

$$\int_{(0,0,0)}^{(1,1,1)} (x+y+z)(dx+dy+dz)$$

Solution: The vector field is $\mathbf{F}(x,y,z) = \langle x+y+z, x+y+z, x+y+z \rangle$, which has a curl of

$$curl\,(\mathbf{F}) = \langle 1-1, 1-1, 1-1 \rangle = \mathbf{0}$$

Thus, the potential of $\mathbf{F}$ is

$$U(x,y,z) = \int (x+y+z)\,dx = \frac{x^2}{2} + xy + xz + C(y,z)$$

Since $U_x = x + C_y$, we have

$$\begin{aligned} x + C_y &= x + y + z \\ C_y &= y + z \\ C &= \frac{y^2}{2} + yz + k(z) \end{aligned}$$

Thus, the potential at this point is

$$U(x,y,z) = \frac{x^2}{2} + xy + xz + \frac{y^2}{2} + yz + k(z)$$

However, $U_z = x + y + k'(z)$, so that

$$\begin{aligned} x + y + k'(z) &= x + y + z \\ k'(z) &= z \\ k &= \frac{z^2}{2} + G \end{aligned}$$

Thus, the potential is

$$U(x,y,z) = \frac{x^2}{2} + xy + xz + \frac{y^2}{2} + yz + \frac{z^2}{2} + G$$

and the line integral is

$$\begin{aligned}\int_{(0,0,0)}^{(1,1,1)} (x+y+z)(dx+dy+dz) &= \left.\frac{x^2}{2} + xy + xz + \frac{y^2}{2} + yz + \frac{z^2}{2}\right|_0^1 \\ &= \frac{1}{2} + 1 + 1 + \frac{1}{2} + 1 + \frac{1}{2} \\ &= \frac{9}{2}\end{aligned}$$

6. Explain why the integral $\int_{(0,0,0)}^{(1,1,1)} xdy + ydx + zdz$ is independent of path. Then calculate the integral along two different paths from $(0,0,0)$ to $(1,1,1)$.

Solution: The vector field is $\mathbf{F}(x,y,z) = \langle y, x, z \rangle$. The curl of $\mathbf{F}(x,y,z)$ is

$$curl(\mathbf{F}) = \langle 0-0, 0-0, 1-1 \rangle = \mathbf{0}$$

so $\mathbf{F}$ is conservative. One path from $(0,0,0)$ to $(1,1,1)$ is given by

$$\mathbf{r}(t) = \langle t, t, t \rangle, \qquad t\ in\ [0,1]$$

The line integral over this curve is

$$\begin{aligned}\int_{(0,0,0)}^{(1,1,1)} xdy + ydx + zdz &= \int_0^1 \left(x\frac{dy}{dt} + y\frac{dx}{dt} + z\frac{dz}{dt} \right) dt \\ &= \int_0^1 3tdt \\ &= \left.\frac{3t^2}{2}\right|_0^1 \\ &= \frac{3}{2}\end{aligned}$$

Another curve that passes from $(0,0,0)$ to $(1,1,1)$ is given by

$$\boldsymbol{\rho}(t) = \left\langle t, t^2, t^3 \right\rangle, \qquad t\ in\ [0,1]$$

The line integral over this curve is

$$\begin{aligned}\int_{(0,0,0)}^{(1,1,1)} xdy + ydx + zdz &= \int_0^1 \left(x\frac{dy}{dt} + y\frac{dx}{dt} + z\frac{dz}{dt} \right) dt \\ &= \int_0^1 t(2t) + t^2(1) + t^3\left(3t^2\right) dt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 3t^2 + 3t^5 dt \\
&= t^3 + \frac{3t^6}{6}\Big|_0^1 \\
&= 1 + \frac{1}{2} \\
&= \frac{3}{2}
\end{aligned}
$$

7. Let R be the unit square. Use Green's theorem to evaluate the line integral

$$\oint_{\partial R} y^2 dx + x^2 dy$$

Solution: Green's theorem implies that

$$
\begin{aligned}
\oint_{\partial R} y^2 dx + x^2 dy &= \iint_R \frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}\left(y^2\right) dA \\
&= \iint_R (2x - 2y)\, dA \\
&= \int_0^1 \int_0^1 (2x - 2y)\, dy dx \\
&= 0
\end{aligned}
$$

8. Let R denote the upper half of the unit disk. Evaluate using Green's theorem:

$$\oint_{\partial R} (xy)\,(dx + dy)$$

Solution: Green's theorem implies that

$$
\begin{aligned}
\oint_{\partial R} xydx + xydy &= \iint_R \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(xy)\, dA \\
&= \iint_R (y - x)\, dA \\
&= \int_0^\pi \int_0^1 (r\sin(\theta) - r\cos(\theta))\, r dr d\theta \\
&= \int_0^\pi \int_0^1 \left(r^2 \sin(\theta) - r^2 \cos(\theta)\right) dr d\theta \\
&= \int_0^\pi \frac{r^3}{3}\sin(\theta) - \frac{r^3}{3}\cos(\theta)\Big|_0^1 d\theta \\
&= \frac{1}{3}\int_0^\pi (\sin(\theta) - \cos(\theta))\, d\theta \\
&= \frac{1}{3}\left(-\cos(\theta) - \sin(\theta)\right)|_0^\pi \\
&= \frac{2}{3}
\end{aligned}
$$

9. Evaluate by using Green's theorem to convert to a line integral over the boundary ($\mathbf{D}$ is the unit disk):

$$\int\int_{\mathbf{D}} \frac{-x}{(x^2+y^2+1)^{3/2}} dA$$

Solution: Let's let

$$N_x = \frac{-x}{(x^2+y^2+1)^{3/2}}, \qquad \textit{so that} \quad N = \int \frac{-x}{(x^2+y^2+1)^{3/2}} dx$$

Letting $u = x^2 + y^2 + 1$ implies that $du = -2xdx$, and

$$N = \frac{-1}{2} \int \frac{du}{u^{3/2}} = \frac{1}{u^{1/2}} = \frac{1}{(x^2+y^2+1)^{1/2}}$$

Thus, Green's theorem says that

$$\begin{aligned} \int\int_{\mathbf{D}} \frac{-x}{(x^2+y^2+1)^{3/2}} dA &= \oint_{\partial\mathbf{D}} \frac{1}{(x^2+y^2+1)^{1/2}} dy \\ &= \int_0^{2\pi} \frac{1}{(x^2+y^2+1)^{1/2}} \frac{dy}{dt} dt \end{aligned}$$

However, $x^2 + y^2 + 1 = 2$ on the unit circle, so that

$$\begin{aligned} \int\int_{\mathbf{D}} \frac{-x}{(x^2+y^2+1)^{3/2}} dA &= \frac{1}{\sqrt{2}} \int_0^{2\pi} \frac{dy}{dt} dt \\ &= \frac{1}{\sqrt{2}} \left. y(t) \right|_0^{2\pi} \\ &= \frac{1}{\sqrt{2}} (y(2\pi) - y(0)) \end{aligned}$$

Since the curve is closed—and thus, the endpoint and beginning point are the same—we must have $y(2\pi) = y(0)$. Thus,

$$\int\int_{\mathbf{D}} \frac{-x}{(x^2+y^2+1)^{3/2}} dA = 0$$

10. Find the area enclosed by the curve $\mathbf{r}(t) = \langle \cos^2(t), \cos(t)\sin(t) \rangle$, t in $[0, \pi]$, using Green's theorem.

Solution: The area is given by

$$\begin{aligned} Area &= \frac{1}{2} \oint_C x dy - y dx \\ &= \frac{1}{2} \int_0^{\pi} \left(x\frac{dy}{dt} - y\frac{dx}{dt} \right) dt \\ &= \frac{1}{2} \int_0^{\pi} \Big(\cos^2(t) (-\sin(t)\sin(t) + \cos(t)\cos(t)) - \cos(t)\sin(t) (-2\cos(t)\sin(t)) \Big) dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^\pi \left(-\sin^2(t)\cos^2(t) + \cos^4(t) + 2\cos^2(t)\sin^2(t)\right) dt \\
&= \frac{1}{2}\int_0^\pi \cos^2(t)\left(\cos^2(t) + \sin^2(t)\right) dt \\
&= \frac{1}{2}\int_0^\pi \cos^2(t)\, dt \\
&= \frac{1}{2}\int_0^\pi \frac{1}{2} + \frac{1}{2}\cos(2t)\, dt \\
&= \frac{1}{4}t + \frac{1}{8}\sin(2t)\Big|_0^\pi \\
&= \frac{\pi}{4}
\end{aligned}
$$

11. Calculate the surface area of the surface Σ parameterized by $\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), u^2\rangle$ for u in $[0,1]$ and v in $[0,2\pi]$.

Solution: To begin with, $\mathbf{r}_u = \langle\cos(v), \sin(v), 2u\rangle$ and $\mathbf{r}_v = \langle -u\sin(v), u\cos(v), 0\rangle$. Their cross-product is

$$\mathbf{r}_u \times \mathbf{r}_v = \left\langle -2u^2\cos(v), -2u^2\sin(v), u\right\rangle$$

Thus, $||\mathbf{r}_u \times \mathbf{r}_v||^2 = 4u^4\cos^2(v) + 4u^4\sin^2(v) + u^2 = 4u^4 + u^2$, so that

$$dS = \sqrt{4u^4 + u^2}dudv = u\sqrt{4u^2+1}dvdu$$

The surface area is thus given by

$$
\begin{aligned}
S &= \iint_\Sigma dS \\
&= \int_0^1\int_0^{2\pi} u\sqrt{4u^2+1}dvdu \\
&= 2\pi\int_0^1 u\sqrt{4u^2+1}du
\end{aligned}
$$

We let $w = 4u^2 + 1$, so that $dw = 8udu$ and

$$S = \frac{2\pi}{8}\int_1^5 w^{1/2}dw = \frac{5\pi\sqrt{5}}{6} - \frac{\pi}{6}$$

12. Compute the flux of the vector field $\mathbf{F}(x,y,z) = \langle y, x, z\rangle$ through the surface Σ parameterized by

$$\mathbf{r}(u,v) = \left\langle u\cos(v), u\sin(v), u^2\right\rangle, \qquad u\ in\ [0,1],\quad v\ in\ [0,2\pi]$$

Solution: To begin with, $\mathbf{r}_u = \langle\cos(v), \sin(v), 2u\rangle$ and $\mathbf{r}_v = \langle -u\sin(v), u\cos(v), 0\rangle$. Their cross-product is

$$\mathbf{r}_u \times \mathbf{r}_v = \left\langle -2u^2\cos(v), -2u^2\sin(v), u\right\rangle$$

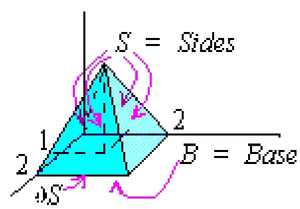
Thus, $d\mathbf{S} = \langle -2u^2\cos(v), -2u^2\sin(v), u\rangle\, dudv$ and

$$\begin{aligned} Flux &= \iint_\Sigma \mathbf{F}\cdot d\mathbf{S} \\ &= \int_0^{2\pi}\int_0^1 \langle y,z,x\rangle\cdot \mathbf{r}_u\times\mathbf{r}_v dudv \\ &= \int_0^{2\pi}\int_0^1 \left\langle u\sin(v), u^2, u\cos(v)\right\rangle\cdot\left\langle -2u^2\cos(v), -2u^2\sin(v), u\right\rangle dudv \\ &= \int_0^{2\pi}\int_0^1 \left(-2u^3\sin(v)\cos(v) - 2u^4\sin(v) + u^2\cos(v)\right) dudv \\ &= \int_0^{2\pi}\left(-\frac{1}{2}\sin(v)\cos(v) - \frac{2}{5}\sin(v) + \frac{1}{3}\cos(v)\right) dv \\ &= 0 \end{aligned}$$

13. Show that if $\mathbf{F}(x,y,z) = \langle xy+2z, yz+2x, xz+2y\rangle$, then $curl(\mathbf{F}) = \langle 2-y, 2-z, 2-x\rangle$. Then evaluate

$$\iint_S curl(\mathbf{F})\cdot d\mathbf{S}$$

when S is the surface of the pyramid with vertices $(2,0,0)$, $(2,2,0)$, $(0,2,0)$, $(0,0,0)$, and $(1,1,2)$ that is not contained in the xy-plane.



Solution: Stoke's theorem implies that

$$\iint_S curl(\mathbf{F})\cdot d\mathbf{S} = \oint_{\partial S}\mathbf{F}\cdot d\mathbf{r} = \iint_B curl(\mathbf{F})\cdot d\mathbf{S}$$

where ∂S is the square with vertices $(2,0,0)$, $(2,2,0)$, $(0,2,0)$, $(0,0,0)$ and B is base $[0,2]\times[0,2]$ in the xy-plane. In the base B, the unit normal is the unit vector $\mathbf{k}$, so that

$$\begin{aligned} \iint_S curl(\mathbf{F})\cdot d\mathbf{S} &= \iint_B \langle 2-y, 2-z, 2-x\rangle\cdot\mathbf{k}\, dydx \\ &= \int_0^2\int_0^2 (2-x)\, dydx \\ &= 4 \end{aligned}$$

14. Use Stoke's theorem for differential forms to calculate

$$\iint_{\partial S} xy\ dy\hat{}dz - z^2\ dy\hat{}dx$$

when S is the solid cube $[0,1]\times[0,1]\times[0,1]$.

Solution: Applying the d operator and using Stoke's theorem implies that

$$\begin{aligned}
\iint_{\partial S} xy\ dy \hat{} dz - z\ dy \hat{} dx &= \iiint_S d\left(xy\ dy \hat{} dz - z\ dy \hat{} dx\right) \\
&= \iiint_S d\left(xy\right) \hat{} dy \hat{} dz - d\left(z^2\right) \hat{} dy \hat{} dx \\
&= \iiint_S \left(ydx + xdy\right) \hat{} dy \hat{} dz - 2z\ dz \hat{} dy \hat{} dx \\
&= \iiint_S ydx \hat{} dy \hat{} dz + xdy \hat{} dy \hat{} dz + 2z\ dy \hat{} dz \hat{} dx \\
&= \iiint_S ydx \hat{} dy \hat{} dz - 2z\ dy \hat{} dx \hat{} dz \\
&= \iiint_S \left(y + 2z\right) dx \hat{} dy \hat{} dz \\
&= \int_0^1 \int_0^1 \int_0^1 \left(y + 2z\right) dxdydz \\
&= \frac{3}{2}
\end{aligned}$$

15. Compute the flux of the vector field $\mathbf{F}\left(x, y, z\right) = \langle x, y, z\rangle$ through the surface of a sphere Σ with radius R centered at the origin. Then show that the divergence theorem produces the same result.

Solution: The unit sphere is parameterized by

$$\mathbf{r}\left(\phi, \theta\right) = \langle R\sin\left(\phi\right)\cos\left(\theta\right), R\sin\left(\phi\right)\sin\left(\theta\right), R\cos\left(\phi\right)\rangle, \quad \phi\ in\ \left[0, \pi\right], \quad \theta\ in\ \left[0, 2\pi\right]$$

Thus, $\mathbf{r}_\phi = \langle R\cos\left(\phi\right)\cos\left(\theta\right), R\cos\left(\phi\right)\sin\left(\theta\right), -R\sin\left(\phi\right)\rangle$ and $\mathbf{r}_\theta = \langle -R\sin\left(\phi\right)\sin\left(\theta\right), R\sin\left(\phi\right)\cos$ and

$$\begin{aligned}
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \left\langle R^2\sin^2\left(\phi\right)\cos\left(\theta\right), R^2\sin^2\left(\phi\right)\sin\left(\phi\right), R^2\cos\left(\phi\right)\sin\left(\phi\right)\right\rangle \\
&= R\sin\left(\phi\right)\langle R\sin\left(\phi\right)\cos\left(\theta\right), R\sin\left(\phi\right)\sin\left(\phi\right), R\cos\left(\phi\right)\rangle \\
&= \langle x, y, z\rangle R\sin\left(\phi\right)
\end{aligned}$$

Thus, the flux is given by

$$\begin{aligned}
Flux &= \iint_{\partial\Sigma} \mathbf{F}\cdot d\mathbf{S} \\
&= \iint_{\partial\Sigma} \langle x, y, z\rangle \cdot \langle x, y, z\rangle\ \ R\sin\left(\phi\right) d\phi d\theta \\
&= \iint_{\partial\Sigma} \left(x^2 + y^2 + z^2\right)\ R\sin\left(\phi\right) d\phi d\theta
\end{aligned}$$

However, if (x, y, z) is on the unit sphere, then $x^2 + y^2 + z^2 = R^2$ and

$$\begin{aligned}
Flux &= \iint_{\partial\Sigma} R^3\sin\left(\phi\right) d\phi d\theta \\
&= R^3\int_0^{2\pi}\int_0^{\pi} \sin\left(\phi\right) d\phi d\theta \\
&= 4\pi R^3
\end{aligned}$$

The divergence theorem, on the other hand, implies that

$$\begin{aligned}
\iint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{S} &= \iiint_{\Sigma} div\,(\mathbf{F})\,dV \\
&= \iiint_{\Sigma} \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}zdV \\
&= \iiint_{\Sigma} 3dV \\
&= 3\,(Volume\ of\ sphere) \\
&= 3\frac{4\pi R^3}{3} \\
&= 4\pi R^3
\end{aligned}$$