ASTR-3415: Astrophysics
Course Lecture Notes
Section IV

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Abstract

These class notes are designed for use of the instructor and students of the course ASTR-3415: Astrophysics. This is the Version 1.2 edition of these notes.

IV. Stellar Interiors

A. The Laws of Stellar Structure

1. Hydrostatic Equilibrium (HSE).

a) Modeling the interiors of stars involves splitting the star up into thin layers (like an onion) $\Longrightarrow$ we have a series of concentric shells in such a model. In this section, we will print variables that are calculated for for a specific shell as *unsubscripted* (*e.g.*, ρ, T, P, N). For those variables whose values are set based on integrations from the center of the star to the specific shell, we will print with an "r" subscript (*e.g.*, M_r, L_r).

b) We can define the mass density, ρ ($[\rho] = \text{gm/cm}^3$) of gas in a given shell as the total mass in that shell, m, divided by the volume of the shell, $V = 4\pi r^2\, dr$.

i) From this definition, the mass in a given layer is found from

$$m = \int_{r_1}^{r_2} \rho\, 4\pi\, r^2\, dr \ , \qquad \text{(IV-1)}$$

where r_1 is the inner boundary radius of the shell (with respect to the center of the star) and r_2 is the outer boundary radius of the shell.

ii) The mass of this thin shell essentially represents the differential of the entire mass of the star interior to that shell: $m \rightarrow dM_r$.

c) From this integral equation, we can set up a differential equation that describes how the mass of the star changes with radial distance within the star:

$$\boxed{\frac{dM_r}{dr} = 4\pi\, r^2\, \rho\ ,} \qquad \text{(IV-2)}$$

which is referred to as the **mass equation** of stellar structure.

d) If we wish to find all of the mass interior to position r inside a star, we just need to integrate Eq. (IV-2) outward from the center, or

$$M_r = \int_0^r 4\pi\, r^2\, \rho\, dr\ . \qquad \text{(IV-3)}$$

e) From Newton's 2nd Law of Motion, we can derive an *equation of motion* of a gas element:

$$m\,\vec{a} = \vec{F} = \sum_{i=1}^{N} \vec{F}_i = \vec{F}_g + \vec{F}_{\rm P} + \vec{F}_{\rm fric}\ , \qquad \text{(IV-4)}$$

where N is the total number of individual force terms $\vec{F}_i$ giving rise to the total force $\vec{F}$ on the shell — note that here $\vec{F}$ is a force not a *flux* as was the case in Eq. (III-20, 21). There are three primary forces that play a role in the interiors of stars:

i) The gravitational force is given by Newton's Theory of Gravitation:

$$\vec{F}_g = -m\vec{\nabla}\Phi_r\ , \qquad \text{(IV-5)}$$

where m is the mass of an element of gas in a shell located at r, Φ_r is the **gravitational potential** given by

$$\Phi_r = -\frac{GM_r}{r}\ , \qquad \text{(IV-6)}$$

$G = 6.673 \times 10^{-8}$ dyne cm^2/gm^2 is the Universal Gravitational Constant, M_r is the amount of mass that lies interior to the shell, and the "del" or gradient operator is given by

$$\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{x} + \frac{\partial}{\partial y}\,\hat{y} + \frac{\partial}{\partial z}\,\hat{z} \qquad \text{(IV-7)}$$

in Cartesian coordinates and

$$\vec{\nabla} = \frac{\partial}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\,\hat{\theta} + \frac{1}{r\,\sin\theta}\frac{\partial}{\partial \phi}\,\hat{\phi} \qquad \text{(IV-8)}$$

in spherical coordinates. The gravitational force is negative since this vector is pointing inward opposite to the r coordinate.

- Since Φ_r is only a function of r,

$$\vec{\nabla}\Phi_r = \frac{\partial}{\partial r}\left(-\frac{GM_r}{r}\right)\,\hat{r} = \frac{d}{dr}\left(-\frac{GM_r}{r}\right)\,\hat{r} = \frac{GM_r}{r^2}\,\hat{r}\;. \qquad \text{(IV-9)}$$

- Note that the acceleration due to gravity is calculated with

$$\vec{g} = \vec{\nabla}\Phi = \frac{GM_r}{r^2}\,\hat{r}\;. \qquad \text{(IV-10)}$$

- Finally, the **potential energy** is related to the potential by the relation

$$\text{PE} \;= m\,\Phi_r = -\frac{GM_r m}{r}\;. \qquad \text{(IV-11)}$$

ii) $\vec{F}_{\rm P}$ represents the force due to the internal pressure inside the shell (see below). This pressure and force results from particles (both matter and energy) colliding with the gas particles.

iii) $\vec{F}_{\rm fric}$ is the force due to various friction forces that exist in the gas. This friction is called *viscosity stress*, where viscosity is the name given to internal friction in a fluid. This viscosity is typically separated into 2 components:

- *Kinematic viscosity* which is friction on a microscopic level $\Longrightarrow$ individual gas particles rubbing against each other.

- *Bulk viscosity* which is friction on a macroscopic level $\Longrightarrow$ "blobs" gas rubbing against each other.

f) The force due to the gas pressure can be deduced as follows. Assume we slice up the shell of gas into segments show in the figure below.

i) The internal pressure ($P = F_{\rm P}/A$, where A = surface area) will try to expand this segment of the shell. Since the lateral forces, F_ℓ are all in opposition to each other, they completely cancel out when integrated over the entire shell.

ii) The force due to this pressure at the bottom of the shell segment, $F_{\rm P,b}$, will be pointing in the opposite direction as that at the top of the shell segment, $F_{\rm P,t}$, and both of these forces are in the radial direction.

iii) Hence the total force due to this pressure will just be

$$F_{\rm P} = F_{\rm P,b} - F_{\rm P,t} \ ,$$

where $F_{\rm P,b} > F_{\rm P,t}$.

iv) Since both $\vec{F}_{\rm P,b}$ and $\vec{F}_{\rm P,t}$ are in the radial direction, they will only have an $\hat{r}$ unit vector associated with them.

v) If we let the thickness of the shell approach zero, this total force due to this pressure becomes a differential in force, $F_{\rm P} \rightarrow -dF_{\rm P}$, or in terms of pressure, $-d\vec{F}_{\rm P} = -A\ dP\ \hat{r}$. Here we have introduced a negative sign since radial derivatives are typically integrated from the inner radius (*i.e.*, the initial position) to the outer radius (*i.e.*, the final position), opposite of our definition of $F_{\rm P}$.

g) If we set $\vec{F}_{\rm fric} = m\,\vec{a}_f = m\,d\vec{v}_f/dt$ and the total force equal to $\vec{F} = m\,d\vec{v}/dt$, we can rewrite the equation of motion (Eq. IV-4) of gas in the shell as

$$m\,\frac{d\vec{v}}{dt} = -\frac{GM_r m}{r^2}\,\hat{r} - A\,dP\,\hat{r} + m\,d\vec{v}_f/dt \ . \qquad \text{(IV-12)}$$

i) If we divide each term by the volume of the shell V, we can then write an equation of force per unit volume:

$$\begin{aligned}
\frac{m}{V}\,\frac{d\vec{v}}{dt} &= -\frac{GM_r m}{r^2\,V}\,\hat{r} - \frac{A}{V}\,dP\,\hat{r} + \frac{m}{V}\,\frac{d\vec{v}_f}{dt} \\
\rho\,\frac{d\vec{v}}{dt} &= -\frac{GM_r\rho}{r^2}\,\hat{r} - \frac{4\pi r^2}{4\pi r^2\,dr}\,dP\,\hat{r} + \rho\,\frac{d\vec{v}_f}{dt} \\
\rho\,\frac{d\vec{v}}{dt} &= -\frac{GM_r\rho}{r^2}\,\hat{r} - \frac{dP}{dr}\,\hat{r} + \rho\,\frac{d\vec{v}_f}{dt} \ ,
\end{aligned}$$

which results in a force per unit volume equation of motion. Such an equation is nothing more that the **conservation of linear momentum**.

ii) Shortly, we will be assuming that the gas is ideal $\Longrightarrow$ such gas has no internal friction, hence $d\vec{v_f}/dt = 0$.

iii) Since all of the remaining terms are in the radial direction, we can drop the vector notation giving

$$\rho \frac{dv}{dt} = -\frac{GM_r\rho}{r^2} - \frac{dP}{dr} \; . \qquad \text{(IV-13)}$$

h) If there are no accelerations in the gas ($a = dv/dt = 0$), Eq. (IV-13) reduces to

$$\boxed{\frac{dP}{dr} = -\frac{G\,M_r\,\rho}{r^2} \; ,} \qquad \text{(IV-14)}$$

which is called the **hydrostatic equilibrium** (HSE) equation. Here P is the total pressure of the gas. When HSE is valid for stellar interiors, $P = P_g$ (*i.e.*, the gas pressure) except for very hot stars when $P = P_g + P_{\rm rad}$, where the radiation pressure is given by Eq. (III-17):

$$P_{\rm rad} = \frac{1}{3}\,a\,T^4 \; , \qquad \text{(IV-15)}$$

since the radiation field will be in local thermal equilibrium with the gas due to the high density of the gas. In this equation, T is the temperature of the gas and $a = 7.56591 \times 10^{-15}$ erg/s^3/K^4 is the radiation constant.

i) With this HSE equation, we can immediately get an estimate of the central pressure in the Sun. Letting $\rho = \overline{\rho_\odot}$ (*i.e.*, the average mass density of the Sun =

$M_\odot/\frac{4}{3}\pi R_\odot^3) = 1.4$ gm/cm^3, $P_{\rm photo} \ll P_c(\odot)$ (*i.e.*, the photospheric pressure is negligible with respect to the central pressure), and using $M_r = M_\odot = 1.99 \times 10^{33}$ gm and $R_\odot = 6.96 \times 10^{10}$ cm, we get

$$\begin{aligned}
\frac{dP}{dr} &= -\frac{G\,M_r\,\rho}{r^2} \\
\frac{\Delta P}{\Delta r} &\approx -\frac{G\,M_\odot \overline{\rho_\odot}}{R_\odot^2} \\
\frac{P_{\rm photo} - P_c(\odot)}{R_\odot - 0} &\approx -\frac{G\,M_\odot \overline{\rho_\odot}}{R_\odot^2} \\
-\frac{P_c(\odot)}{R_\odot} &\approx -\frac{G\,M_\odot \overline{\rho_\odot}}{R_\odot^2} \\
P_c(\odot) &\approx \frac{G\,M_\odot \overline{\rho_\odot}}{R_\odot} \\
P_c(\odot) &\approx 2.7 \times 10^{15}\ \text{dyne/cm}^2 \\
&\approx 2.7 \times 10^{9}\ \text{atm}\ ,
\end{aligned}$$

where 1 atm is the Earth's atmospheric pressure which is equal to $P_\circ = 1.013 \times 10^5$ Pa (Pascals = N/m^2) = 1.013×10^6 dyne/cm^2. Using more accurate modeling techniques gives the actual value of 50 times this amount.

2. **Equation of State (EOS).**

   a) To solve the HSE equation, we need an accurate density profile $\rho(r)$. For exotic matter that exists in white dwarfs and neutron stars, this profile can be fairly complicated and will require quantum mechanics to describe it. However, for normal gas, we can assume the gas is *perfectly ideal* and use the **ideal gas law** to describe the equation of state:

$$P_g = N\,k_B T\ , \tag{IV-16}$$

where P_g (dyne/cm^2) is the gas pressure at depth r, N is the particle number density (particles/cm^3) at this depth, T the temperature, and $k_B = 1.3806 \times 10^{-16}$ erg/K is Boltzmann's constant.

b) As shown in the stellar atmospheres section of the notes (§III), the *free particle* number density can be related to the gas density by

$$N = \frac{\rho}{\mu\, m_{\rm H}} \ , \qquad \text{(IV-17)}$$

where $m_{\rm H} = 1.673 \times 10^{-24}$ gm is the mass of the hydrogen nucleus and μ is the *mean molecular weight* of the gas.

c) There are a variety of techniques for determining the value of the particle (*i.e.*, number) density. Before describing these, we need to define a few terms.

i) Let's start by introducing mass fractions:

- $X = m_{\rm H}/m_{\rm tot}$ is the mass fraction of hydrogen atomic nuclei $\Longrightarrow$ the fraction of matter, by weight, that consists of hydrogen.

- $Y = m_{\rm He}/m_{\rm tot}$ is the mass fraction of helium atomic nuclei $\Longrightarrow$ the fraction of matter, by weight, that consists of helium.

- $Z = m_{\rm metals}/m_{\rm tot}$ is the remaining fraction consisting of everything heavier than helium, the so-called *metals*.

- Note that since $m_{\rm tot} = m_{\rm H} + m_{\rm He} + m_{\rm metals}$, the following relation occurs:

$$X + Y + Z = 1 \ . \qquad \text{(IV-18)}$$

ii) The **atomic weight** of an atom is the mass of the atomic nucleus measured in *atomic mass units* (amu). Unfortunately, there are two definitions for the amu:

- All elements have various *isotopes* associated with them $\Longrightarrow$ they have the same number of protons but different numbers of neutrons.

- Some define the amu such that the mass of ^{16}O (the most abundant isotope of oxygen, 8 protons + 8 neutrons = 16 nucleons) is exactly 16.000000 amu.

- More often it is defined such that the mass of ^{12}C (the most abundant isotope of carbon, 6 protons + 6 neutrons = 12 nucleons) is exactly 12.000000 amu.

- Look carefully when viewing a data table of elements or a periodic table for their definition of atomic weight.

- In this class, we will use the ^{12}C definition of the amu such that

$$\boxed{\begin{aligned} 1 \text{ amu} &= 1.660540 \times 10^{-24} \text{ gm} \\ &= 931.49432/c^2 \text{ MeV.} \end{aligned}}$$

  Note that when energy units (such MeV = million electron volts) are used (via $E = mc^2$), the c^2 is usually not mentioned (but it is there none the less).

- Atomic weights are non-integer numbers because various isotopes of the same element can naturally occur. When viewing these weights on a periodic table, these weights (*e.g.*, H is 1.0079, He is 4.0026) include the cosmic abundance fraction of all isotopes that naturally occur in the total weight.

iii) Another value that is used to describe nuclear mass is the *nucleon number*, A, which is equal to the total number of nucleons (*i.e.*, protons plus neutrons) in an atomic nucleus. Note that some authors call this the atomic weight by mistake. Whenever you see integers listed for the atomic weight (except for ^{12}C or perhaps ^{16}O), it is the nucleon number that is being presented.

iv) Another number of interest is the *atomic number*, Z, which is equal to the number of protons in the nucleus, which is just the charge of the nucleus. Z defines the element (*e.g.*, $Z = 1$ for hydrogen, 2 for helium, etc.). Since metalicity also uses Z, we will use Z_a from this point forward to indicate the atomic number to avoid confusion.

d) The mean molecular weight is related to the *mean mass* of the particles that make up the gas by

$$\overline{m} = \mu \, m_{\rm H} = \sum_{\beta} m_{\beta} \, N_{\beta} / \sum_{\beta} N_{\beta} \; , \qquad \text{(IV-19)}$$

where β is an counter of all of the atomic and molecular species that make up the gas, m_{β} is the mass of that species, and N_{β} is the number density of that species.

i) Note that the total number density is $N = \sum_\beta N_\beta$, so we can calculate the mean molecular weight with

$$\mu = \sum_\beta m_\beta \, N_\beta / \left(m_{\rm H} \, N\right) \ . \qquad \text{(IV-20)}$$

ii) We can simplify the calculation of μ by making use of *mass fractions* as described earlier. Recall that in terms of mass fractions (X = hydrogen, Y = helium, and Z = *metals*). In terms of mass fractions, the number densities can be expressed as (using hydrogen for example)

$$N_{\rm H} = \frac{1}{V_{\rm H}} = \frac{1}{m_{\rm H}} \cdot \frac{m_{\rm H}}{m_{\rm tot}} \cdot \frac{m_{\rm tot}}{V} = \frac{1}{m_{\rm H}} \cdot X \cdot \rho = \frac{X\rho}{m_{\rm H}} \ .$$

Hence the number densities for hydrogen, helium, and the metals are

$$\begin{aligned} N_{\rm H} &= \frac{X\rho}{m_{\rm H}} \\ N_{\rm He} &= \frac{Y\rho}{m_{\rm He}} = \frac{Y\rho}{4m_{\rm H}} \\ N_{\rm metals} &= \frac{Z\rho}{\overline{m}_{\rm metals}} = \frac{Z\rho}{\overline{A}\ m_{\rm H}} \ , \end{aligned}$$

where $\overline{m}_{\rm metals}$ is the average mass of all of the metals and $\overline{A}$ is the average nucleon number of the metals.

iii) Since the interior of stars have high enough temperatures to completely ionize H and He, and completely ionize <u>most</u> of the more common metals, hydrogen will supply an additional electron to the total number density for each atom ionized and helium will supply two. Since the most stable isotopes of the metals have $A = 2Z_a$, the metals will supply $\frac{1}{2}A$ electrons. Hence, we can tabulate the total number of particles supplied to the gas as

Element:	**Hydrogen**	**Helium**	**Metals**
No. of atoms:	$\dfrac{X\rho}{m_{\rm H}}$	$\dfrac{Y\rho}{4m_{\rm H}}$	$\left[\dfrac{Z\rho}{\overline{A}\, m_{\rm H}}\right]$
No. of electrons:	$\dfrac{X\rho}{m_{\rm H}}$	$2\,\dfrac{Y\rho}{4m_{\rm H}}$	$\dfrac{1}{2}\overline{A}\,\dfrac{Z\rho}{\overline{A}\, m_{\rm H}}$

Note that for the last column, that the average nucleon number for the most common metals seen in stars is at least 16 (hence, $\overline{A} \geq 16$). Since $\overline{A}$ cancels in the total electron number from metals, we see that the total number of electrons from metals will far exceed the total number of metal nuclei (*i.e.*, atoms). As such, the total number of particles from each species type is

Element:	**Hydrogen**	**Helium**	**Metals**
No. of particles:	$2\,\dfrac{X\rho}{m_{\rm H}}$	$3\,\dfrac{Y\rho}{4m_{\rm H}}$	$\dfrac{Z\rho}{2m_{\rm H}}$,

or in equation format,

$$N = \left(2\,X + \frac{3}{4}Y + \frac{1}{2}Z\right)\frac{\rho}{m_{\rm H}}\,. \qquad \text{(IV-21)}$$

iv) By making use of Eq. (IV-17), we can now express the mean molecular weight in terms of mass fractions as

$$\mu = (2\,X + \frac{3}{4}\,Y + \frac{1}{2}\,Z)^{-1} \;. \qquad \text{(IV-22)}$$

One should keep in mind that these equations for N and μ (Eqs. IV-21, 22) are only valid when temperatures are high enough to completely ionized hydrogen and helium and most of the metals, hence deeper in the star. Near the surface (*i.e.*, photosphere), the Saha equation must be used to get an accurate electron number density from the ionization of the atomic species.

v) For younger, high-metalicity stars, the so-called Population I stars, $X = 0.70$, $Y = 0.28$, and $Z = 0.02$. Plugging these values into Eq. (IV-22) gives $\mu = 0.62$.

e) Using the definitions above, we use the standard form of the ideal gas law for our EOS:

$$\boxed{P = \frac{\rho\, k_B\, T}{\mu\, m_{\rm H}}\;,} \qquad \text{(IV-23)}$$

where here μ is the mean molecular weight in the shell at distance r from the center of the star.

f) From the EOS, we can get an estimate of the Sun's central temperature. Using $P_c(\odot)$ and $\overline{\rho_\odot}$ from above and the fact that $\mu \approx 0.6$ for the Sun (see above), we get

$$T_c(\odot) = \frac{P_c(\odot)\,\mu\, m_{\rm H}}{k_B\,\overline{\rho_\odot}} = 1.2 \times 10^7 \text{ K} \;. \qquad \text{(IV-24)}$$

3. Modes of Energy Transport.

a) To determine T, we must discover how energy is transported from the hot central regions to the cooler photosphere (5800 K) as it must according to the 2nd law of thermodynamics. There are 3 modes of heat energy transport that occur in nature.

b) **Conduction** occurs when energetic (*i.e.*, hot) atoms communicate their agitation to nearby cooler atoms via *collisions*.

i) The conductive flux was given by Eq. (III-22), which we will express here as a scalar equation in the radial direction as

$$F_{\rm cond} = -\kappa_\circ \, T^{5/2} \, \frac{dT}{dr} \ , \qquad \text{(IV-25)}$$

where $\kappa_\circ \approx 8 \times 10^{-7}$ erg/cm/s/K$^{7/2}$ is the conductivity constant.

ii) Note that the conductive flux $F_{\rm cond}$ will be a positive number inside of a star since the temperature gradient dT/dr will be negative since temperature drops as one moves away from the center of the star.

iii) This mode is only efficient in solids, degenerate matter, and stellar coronae. As such, it is typically ignored in stellar interior modeling.

c) **Convection** is the transport of heat by mass motions in fluids.

i) If dT/dr is great enough, fluid can become unstable and *boil*. Hot fluid masses (*i.e.*, bubbles) rise up from hotter regions to cooler regions and give up their energy to the surrounding environment.

ii) Unfortunately, no completely adequate theory of stellar convection yet exists (although a few physicists are working on it), and we must take approximate measures to determine the convective flux.

iii) Pages 356 through 365 of your textbook by Carroll and Ostlie describe the details of the **mixing-length theory** which is the most widely used approximate theory for convection. The resulting equation from the mixing-length theory for the convective flux is

$$F_{\rm conv} = \rho\, C_P \left(\frac{k_B}{\mu\, m_{\rm H}}\right)^2 \left(\frac{T}{g}\right)^{3/2} \beta^{1/2} \left[\delta\left(\frac{dT}{dr}\right)\right]^{3/2} \alpha^2 \,, \tag{IV-26}$$

where

$$\delta\left(\frac{dT}{dr}\right) dr = \delta T \equiv \left(\left.\frac{dT}{dr}\right|_{\rm ad} - \left.\frac{dT}{dr}\right|_{\rm act}\right) dr \;, \tag{IV-27}$$

and $\left.\frac{dT}{dr}\right|_{\rm ad}$ is the adiabatic temperature gradient (essentially the gradient in the "bubble" of gas) and $\left.\frac{dT}{dr}\right|_{\rm act}$ is the actual temperature gradient of the surrounding gas that the bubble is rising through.

iv) The rest of the parameters in Eq. (IV-26) are the standard parameters we have been using in this section of the notes. In addition to these, C_P is the specific heat of the gas at constant pressure given by Eq. (III-28), and g is the acceleration due to gravity.

v) Two *free* (*i.e.*, guessed) *parameters* are also in Eq. (IV-26). $\alpha = \ell/H_P$ is the ratio of the mixing length ℓ (which is the "guessed" parameter) to the pressure scale height. From comparisons of numerical stellar models with observations, values of $0.5 < \alpha < 3$ are typical. The mixing length is that distance that the bubble travels before giving its heat energy up to the surrounding gas. The second free parameter is β, which is a kinetic energy scale factor that has a range of $0 < \beta < 1$. It essentially gives the amount of the kinetic energy of the bubble that gets converted to heat.

vi) The **pressure scale height** H_P used in the calculation of α is defined as the distance that the pressure drops by a value of e^{-1} with height. It is defined by

$$\frac{1}{H_P} \equiv -\frac{1}{P}\frac{dP}{dr} \; . \qquad \text{(IV-28)}$$

If H_P doesn't vary much in a shell of gas, we can integrate the equation above to show that

$$P = P_\circ \, e^{-r/H_P} \; . \qquad \text{(IV-29)}$$

Finally, if we make use of the HSE equation (Eq. IV-14) in Eq. (IV-28), one can show that

$$H_P = \frac{P}{\rho g} \; . \qquad \text{(IV-30)}$$

vii) After all of this, we still haven't described an equation for the temperature gradient when convection is important in moving the energy from the center of the star to the outer regions. Once again I refer you to pages 350 through 361 of your textbook (Carroll and Ostlie). Here we will give the equation for the description of the temperature gradient when convection is important as the adiabatic temperature gradient:

$$\boxed{\frac{dT}{dr} = \left.\frac{dT}{dr}\right|_{\rm ad} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P}\frac{dP}{dr} \ ,} \qquad \text{(IV-31)}$$

where γ is the ratio of specific heats given by Eq. (III-26).

d) **Radiation Transport** is the 3rd mode and is usually the most important means of energy flow in a star (see §I of the notes on radiative transfer). Energy is transported from *photon* flow. The ease at which the photons flow depends upon the opacity of the gas.

i) We start by differentiating Eq. (IV-15) with respect to r which gives

$$\frac{dP_{\rm rad}}{dr} = \frac{4}{3}\, a\, T^3 \, \frac{dT}{dr} \ . \qquad \text{(IV-32)}$$

ii) Though we didn't show this in the radiative transfer portion of the notes, the equation of radiative transfer can be manipulated to show that

$$\frac{dP_{\rm rad}}{dr} = -\frac{k\rho}{c}\, F_{\rm rad} \ . \qquad \text{(IV-33)}$$

iii) Equating these two equations for the radiation pressure derivative gives

$$\frac{dT}{dr} = \frac{-3k\rho}{4acT^3}\, F_{\rm rad} \ .$$

iv) Now since

$$F_{\rm rad} = \frac{L_r}{4\pi r^2} \ , \qquad \text{(IV-34)}$$

the temperature gradient for radiation transport becomes

$$\boxed{\frac{dT}{dr} = \frac{-3k\,\rho}{16\pi\,a\,c\,r^2\,T^3}\,L_r\ .} \qquad \text{(IV-35)}$$

v) As was the case for stellar atmospheres, getting the opacity correct is one of the most fundamental needs in interior modeling.

4. Energy Generation.

a) We next need to describe how the luminosity changes with depth. Actually for layers of the interior under radiative and convective equilibrium, the luminosity remains constant with depth. However, if an energy source is present, RE will not be valid and we need to calculate the change in luminosity with depth.

b) If the rate of energy production per unit mass of stellar material (erg/s/gm) is denoted by ε, the additional luminosity supplied to a shell from this production is

$$\boxed{\frac{dL_r}{dr} = 4\pi\,r^2\,\rho\,\varepsilon\ .} \qquad \text{(IV-36)}$$

c) This equation expresses a balance between the net energy loss from the shell, dL_r, and that generate within the shell $\Longrightarrow$ the **conservation of energy** equation (also called the **thermal equilibrium** equation).

d) But what is this energy source? In 1861, Helmholtz and Kelvin suggested that gravitational contraction of the Sun might be the source of its luminosity.

i) The kinetic energy of a mass m in orbit around a larger mass M is

$$\text{KE} = m\,v^2/2. \qquad \text{(IV-37)}$$

ii) Its gravitational potential energy is

$$\text{PE} = -G\,M\,m/r. \qquad \text{(IV-38)}$$

iii) Any object in a curved trajectory experiences centripetal acceleration $F_c = m\,v^2/r$. In a gravitational field, $F_c = F_g = G\,M\,m/r^2$, the force due to gravity. From this equality, we immediately see that

$$\text{KE} = m\,v^2/2 = G\,M\,m\;/2r = \frac{1}{2}\,\text{PE}, \qquad \text{(IV-39)}$$

this equation is called the **virial theorem**.

iv) If that body moves inward to a smaller, stable orbit at $r - dr$, the increase in KE is equal to only $\frac{1}{2}$ PE. To conserve total energy (KE + PE), the other half of PE must be transmitted to the agent which alters the satellite orbit — in the case of a star, this is the energy that is radiated away.

v) Kelvin and Helmholtz applied this theorem to the Sun, where

$$\mathcal{E}_{\rm rad} = G\,M_\odot/2R_\odot = 9.54 \times 10^{14}\ \text{erg/gm} \qquad \text{(IV-40)}$$

is the amount of PE available to be radiated away for each gram of solar material.

vi) The amount of time that this contraction will take place to give a constant luminosity is called the **Kelvin-Helmholtz time**:

$$t_{\rm KH} = \mathcal{E}_{\rm rad}\, M_\star / L_\star. \qquad \text{(IV-41)}$$

In the case of the Sun, $t_{\rm KH} = 1.57 \times 10^7$ yrs $\Longrightarrow$ a much shorter time than the age of the Earth. Some other energy source must be at work to produce the Sun's energy.

B. Particle Physics

1. Before investigating the thermonuclear reactions that produce the energy in the Sun and stars, we need to describe the physics of subatomic particles.

2. An important parameter of particles from quantum mechanics that will describe how particles interact with each is **spin**. In *classical* mechanics, a rigid body admits two kinds of angular momentum:

   a) **Orbital**: $\vec{L} = \vec{r} \times \vec{p}$ (*i.e.*, distance from an axis times the linear momentum), associated with the motion *of* the center of mass. Such motion is referred to as a **revolution** about the center of mass.

   b) **Spin**: $\vec{S} = I\vec{\omega}$ (*i.e.*, moment of inertia times an angular velocity), associated with the motion *about* the center of mass. Such motion is referred to as a **rotation** about an axis.

   c) One then talks about the **total angular momentum**: orbital ($\vec{L}$) + spin ($\vec{S}$) angular momenta.

   d) Here, we will define "classical" fundamental particles (*i.e.*, the atoms) as those whose movement is described by classical themodynamics $\Longrightarrow$ their velocities follow a

Maxwell-Boltzmann distribution (see Eq. III-5) whose characteristics have already been described in §III.F.3.c of the course notes.

3. By analogy, we have the same description on the microscopic level for *quantum* mechanics:

a) **Orbital**: The motion of an electron about the nucleus of an atom as described by spherical harmonics (take PHYS-4617 *Quantum Physics* to learn about these) with the orbital angular momentum quantum number ℓ and the magnetic (or azimuthal) quantum number m. This is sometimes referred to as the *extrinsic* angular momentum ($\vec{L}$).

b) **Spin**: Unlike the classical case, this isn't the spin of the electron about an axis, since the electron is a point particle (even though we describe electron spin about an axis in elementary physics). Here "spin" is nothing more than the *intrinsic* angular momentum ($\vec{S}$) of the electron. Since this is intrinsic, spin angular momentum is independent of spatial coordinates (r, θ, ϕ).

4. It is the "spin" of a particle that primarily dictates how particles interact.

a) It so happens that every elementary particle has a *specific and immutable* value of s which we call the **spin** of that particular species.

b) Particles with *half-integer* spins are called **fermions** since they follow **Fermi-Dirac** statistics instead of the classical Maxwell-Boltzmann statistics. Fermions are said to have **antisymmetrical** wave functions $\Longrightarrow$ a wave function helps to describe the probability distribution functions for the position and energy of subatomic particles. Wave functions are determined from a solution of the Schrödinger equation (as presented in PHYS-4617).

c) Particles with *integer* spins are called **bosons** since they follow **Bose-Einstein** statistics (see page 327 of your textbook for details). Bosons are said to have **symmetrical** wave functions.

d) Antisymmetrical wave functions go to zero as two identical particles approach each other. As a result, two fermions in the same quantum state exhibit mutual repulsion to avoid their combined wave functions going to zero. This effect is known as the **Pauli Exclusion Principle**. Bosons do not suffer from this problem.

5. Particle physics is directly related to the 4 known forces in nature (*i.e.*, gravity, weak (nuclear), electromagnetic (E/M), and strong (nuclear)) — particles are classified depending on whether they participate in any of the above forces.

a) **Strong interactions**: Force that binds nucleons together with a characteristic range of $\sim 10^{-13}$ cm. Particles that participate in the strong force are called **hadrons**. The smallest component particle of a hadron is called a **quark**.

b) **E/M interactions**: Force between charged particles which has an infinite range that falls off as $1/r^2$. This force is 100 times weaker than the strong force, however it is what holds atoms and molecules together.

c) **Weak interactions**: These are responsible for β-decay of nuclei (*i.e.*, radioactivity). They are about 10^{-13} times as strong as the strong interactions with a range $\ll 10^{-13}$ cm.

d) **Gravitational interactions**: These are by far the weakest of the interactions on the microscopic scale, typically about 10^{-40} times as strong as the strong interactions on nuclear scales. Gravity is another infinite, $1/r^2$ force, except it is charge independent — as such, this force dominates all others on a cosmic scale.

6. The particles that make up matter and energy can be classified into 2 broad categories:

a) **Field particles**: These particles mediate the 4 natural forces and are sometimes referred to as the *energy* particles — all are bosons. Here they are listed in order from strongest to weakest:

i) **Gluons**: Mediate the strong [nuclear] force. Strength of this force is described by six different **color charges**.

ii) **Photons**: Mediate the electromagnetic force. Strength of this force is described by the two **electric charges** (*i.e.*, '+' and '–').

iii) The **weakons** mediate the weak [nuclear] force. Strength of this force is described by the **weak charges** carried by the W and Z intermediate vector bosons.

iv) Gravitons: Mediate the force of gravity. Strength of this force is described by a <u>single</u> charge called the **mass**.

b) Elementary particles: These are particles that make up matter. They are subdivided into 3 groups:

i) Leptons (which means *little things*) include the *electron* (e^-, $m_e = 511$ keV), *muon* (μ, $m_\mu = 107$ MeV), and *tau particle* (τ, $m_\tau = 1784$ MeV), each with a negative charge; their respective neutrinos: *electron neutrino* (ν_e, $m_{\nu-e} < 30$ eV), *muon neutrino* (ν_μ, $m_{\nu-\mu} < 0.5$ MeV), and *tau neutrino* (ν_τ, $m_{\nu-\tau} < 250$ MeV), each with no charge; and the antiparticles of each: e^+ (called a *positron*), $\overline{\mu}, \overline{\tau}, \overline{\nu_e}, \overline{\nu_\mu}$, and $\overline{\nu_\tau}$. These particles do <u>**not**</u> participate in the strong interactions. All leptons have spin of 1/2 hence are fermions.

ii) Mesons (*middle things*) are particles of intermediate mass that are made of quark-antiquark pairs and include *pions*, *kaons*, and *η-particles.* All are unstable and decay via weak or E/M interactions. All mesons have either 0 or integer spin hence are bosons.

iii) Baryons (*big things*) include the nucleons n (*neutrons* — neutral particles) and p (*protons* — positive charged) and the more massive *hyperons* (*i.e.*, Λ, Σ, Ξ, and Ω). Baryons are composed of a triplet of quarks and has a spin of either 1/2 or 3/2, and as such, are fermions. Each baryon has an antibaryon associated with it.

7. From the above list of elementary particles, there seems to be only 2 types of basic particles: *leptons* which do not obey the strong force and *quarks* which do obey the strong force. There are 6 *flavors* of leptons (as describe above). As such, it was theorized and later observed, that 6 flavors or *colors* of quarks (and an additional 6 antiquarks) must exist:

a) **Up** (u) quark has a rest energy of 360 MeV and a charge of $+\frac{2}{3}\,e$.

b) **Down** (d) quark has a rest energy of 360 MeV and a charge of $-\frac{1}{3}\,e$.

c) **Charmed** (c) quark has a rest energy of 1500 MeV and a charge of $+\frac{2}{3}\,e$.

d) **Strange** (s) quark has a rest energy of 540 MeV and a charge of $-\frac{1}{3}\,e$.

e) **Top** (t) quark has a rest energy of 170 GeV and a charge of $+\frac{2}{3}\,e$.

f) **Bottom** (b) quark has a rest energy of 5 GeV and a charge of $-\frac{1}{3}\,e$.

8. Note that a proton is composed of 2 u and a d quark and a neutron composed of an u and 2 d quarks.

9. The theory on how quarks interact with each other is called **quantum chromodynamics**. One interesting result of this theory is that quarks cannot exist in isolation, they must always travel in groups of 2 to 3 quarks.

10. As described above, the four forces in relativistic quantum mechanics are mediated by the exchange of integer-spin particles (bosons).

a) Of the four forces, only gravity gives rise to attractive forces between *like* particles (same type of color charge, electric charge, weak charge, or mass).

b) This difference arises because the graviton is spin 2, whereas the gluon, photon, and weakon are spin 1 as shown in the following table.

Spin quantum numbers for a sample of elementary and field particles.

Common Name	**Symbol†**	**Particle Type**	**Spin (s)**	**Spin Family**
Pion	π^+	meson	0	boson
	π^0	meson	0	boson
Electron	e^-	lepton	$\frac{1}{2}$	fermion
Muon	μ^-	lepton	$\frac{1}{2}$	fermion
Neutrino	ν_e	lepton	$\frac{1}{2}$	fermion
Proton	p	baryon	$\frac{1}{2}$	fermion
Neutron	n	baryon	$\frac{1}{2}$	fermion
Gluon	G	field	1	boson
Photon	γ	field	1	boson
Weakon	W	field	1	boson
Delta	Δ^+	baryon	$\frac{3}{2}$	fermion
Graviton	g	field	2	boson

† – The superscipt in the symbol corresponds to the charge of the particle: '+' = positive, '–' = negative, '0' = neutral. Symbols with no superscript are neutral, except for the proton which is positively charged, and the weakons which can have a +, –, or no electric charge.

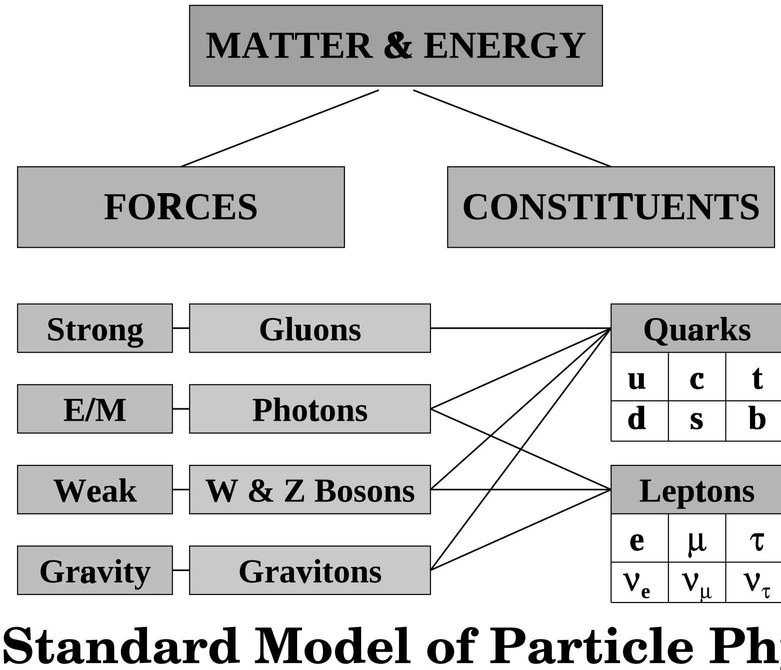

Figure IV–1: The Standard Model is the current best description of the subatomic world.

11. Figure IV-2 summarizes the interconnections between matter represented by the fundamental particles of *quarks* and *leptons* and energy represented by the force-field particles of *gluons, photons, weakons,* and *gravitons.*

12. Conservation Laws.

a) If we assign a **baryon number** B of +1 to each baryon (nucleon or hyperon) and –1 to an antinucleon or anti-hyperon, then in a closed system

$$\sum B = \text{ constant.} \qquad \text{(IV-42)}$$

b) Similarly, if we assign a **lepton number** L of +1 to each lepton ($i.e.$, e^-, μ, ν, etc.) and of –1 to antileptons ($i.e.$, $e^+, \overline{\mu}, \overline{\nu}$, etc.), then in a closed system

$$\sum L = \text{ constant.} \qquad \text{(IV-43)}$$

c) A similar conservation law does not exist for bosons — the mesons and field particles (*i.e.*, photons).

d) Charge must be conserved in a nuclear reaction.

e) Mass-energy, via $E = m\,c^2$, must be conserved in a nuclear reaction.

f) Momentum must be conserved. Hence a matter-antimatter reaction must create two photons to conserve momentum ($e.g.$, $e^- + e^+ \rightarrow 2\gamma$).

C. Thermonuclear Reactions.

1. In 1938, it became clear that the long-term energy source for stars must be *thermonuclear fusion reactions*. In these reactions, lighter elements *burn* to form heavier elements $\Longrightarrow$ **nucleosynthesis**.

2. Two nuclei will fuse to form one nuclei if they come within 10^{-13} cm of each other — but they must be moving fast enough to overcome the *Coulomb repulsion* that exists between like charged particles.

   a) Particles must be at a high temperature to be moving fast.

   b) This high temp completely ionizes all of the nuclei.

   c) Temps must build even more to get the kinetic energy to overcome the **Coulomb barrier**.

3. In main sequence stars, H is fused into He. Since H is composed of 1 baryon and He, 4 baryons (2p + 2n), 4 H nuclei must be used to construct one He nuclei.

$$
\begin{array}{rcrcl}
4 \times m_H & = & 4 \times 1.0078 \text{ amu} & = & 4.0312 \text{ amu} \\
-m_{He} & & & = & 4.0026 \text{ amu} \\
\hline
\Delta m & & & = & 0.0286 \text{ amu}
\end{array}
$$

a) This mass deficit, Δm, is converted into energy:

$$
\begin{aligned}
E = \Delta m\, c^2 &= (0.0286) \times (1.66 \times 10^{-24} \text{ gm}) \\
&\quad \times (9.00 \times 10^{20} \text{ cm}^2/\text{s}^2) \\
&= 4.3 \times 10^{-5} \text{ erg}
\end{aligned}
$$

b) From this calculation, we see that the energy release efficiency, η, of this H$\rightarrow$He reaction is $0.0286/4.0312 = 0.0071 \Longrightarrow$ only 0.71% of the original mass of H is converted to energy!

c) With this in mind, we can rewrite Einstein's famous equation as

$$
\boxed{E = \eta\, m\, c^2,} \tag{IV-44}
$$

where η is the efficiency of the reaction and m is the initial mass in the reaction.

d) We can calculate the total energy the Sun will release during its main sequence lifetime. Since the reactions are only being carried out in the core of the Sun and this core contains about 10% of the Sun's mass, the total energy release will be

$$
\begin{aligned}
E_{\rm tot} &= 0.1 M_\odot\, \eta\, c^2 \\
&= 1.28 \times 10^{51} \text{ ergs.}
\end{aligned}
$$

e) The present luminosity of the Sun is 3.90×10^{33} erg/s. If the Sun's luminosity remains somewhat constant while on the main sequence, we can determine its **main sequence lifetime**:

$$
t_{\rm MS}(\odot) = E_{\rm tot}/L_\odot = 3.28 \times 10^{17} \text{ sec } = 1.04 \times 10^{10} \text{ yrs,}
$$

the Sun's main sequence lifetime is about 10 billion years. Since the Sun is currently 5 billion years old, it is at mid-life.

4. As previously mentioned, high temperatures are needed to overcome the Coulomb repulsion of the charged nuclei. But how high a temperature?

a) From classical physics, we can set the average kinetic energy of the particles involved equal to the thermal energy of the particles and solve for the temperature (see page 334 of Carroll and Ostlie). This gives

$$T_{\rm classical} = \frac{2 Z_1 Z_2 e^2}{3 k_B r} \; , \qquad \text{(IV-45)}$$

where $Z_1 e$ is the charge on particle 1, $Z_2 e$ is the charge on particle 2, and r is the distance where a nuclear reaction will occur (about the size of a nucleus, 10^{-13} cm).

b) For 2 protons coming together ($Z_1 = Z_2 = 1$), this gives a temperature of 10^{10} K, whereas the Sun's central temperature is only 1.58×10^7 K.

c) We could also investigate this in terms of energy. For the Sun's central temperature, each proton will have a thermal energy of 1 keV, whereas the Coulomb potential barrier is 1000 keV (1 MeV)! Not all the particles have this energy, some are moving at much quicker velocities, hence have higher thermal energies and temperatures following the Maxwell-Boltzmann distribution of velocities. Unfortunately, the number of particles in the tail of this Maxwellian is insufficient to account for the Sun's luminosity.

d) We can now turn to quantum mechanics to solve the problem. As discussed in the subsection on particle spin, in reality elementary particles are not little billiard balls colliding with each other as a result of following trajectories. Instead, they follow probability distributions described by their wave functions. In quantum mechanics, there is a small probability that wave functions can penetrate energy barriers that are higher than the energy of the wave function. This effect is known as quantum **tunneling**.

e) Using quantum mechanics, we can describe a temperature needed to produce a sufficient number of tunneling events to sustain a nuclear reaction (see page 335 of the textbook) as

$$T_{\rm quantum} = \frac{4\mu_m Z_1^2 Z_2^2 e^4}{3k_B h^2} \,, \tag{IV-46}$$

where μ_m is the reduced mass of the colliding "particles" and h is Planck's constant.

f) In this equation, two protons can come together (*i.e.*, fuse) at a temperature of 10^7 K, which is consistent with the central temperature deduced for the Sun.

g) A more detailed calculation from statistical mechanics shows that the bulk of the energy is being liberated by reactions involving particles in the high energy tail of the Maxwellian distribution as shown in Figure 10.6 on page 339 in your textbook.

i) What is shown in this figure is that particles with energies at the **Gamow Peak** will be the ones that supply most of the energy through thermonuclear reactions.

ii) The Gamow Peak corresponds to a local maximum in the two probability functions: the $e^{-E/k_B T}$ Maxwell-Boltzmann distribution term and the $e^{-bE^{-1/2}}$ quantum tunneling penetration term, where,

$$b \equiv \frac{2^{3/2}\pi^2 \mu_m^{1/2} Z_1 Z_2 e^2}{h} \ .$$

iii) The Gamow Peak for a given temperature will occur at the energy of

$$E_\circ = \left(\frac{bk_B T}{2}\right)^{2/3} , \qquad \text{(IV-47)}$$

for the Sun, the Gamow Peak is at 6 keV.

5. By making use of statistical mechanics in conjunction with quantum mechanics, stellar interior modelers set up power laws that describe the energy production rate per unit mass of the form

$$\varepsilon = \varepsilon_\circ X_i X_x \rho^\alpha T^\beta , \qquad \text{(IV-48)}$$

where the X's are the mass fractions of the fusing particles, and $\varepsilon_\circ, \alpha$. and β are constants that depend upon the reactions involved (more to come on this).

D. Various Reaction Chains

1. Two different fusion processes convert H into He, the first is important for stars with $T_c \lesssim 1.8 \times 10^7$ K ($M \lesssim 1.3 M_\odot$, ∼F5 V star) and is called the **proton-proton chain**.

a) The first of this reaction chain is called the **PP I** chain:

Reaction			Energy Released (MeV)	Reaction Time
$^1H + {}^1H$	$\longrightarrow$	$^2H + e^+ + \nu_e$	1.442	1.4×10^9 yr
$^1H + {}^2H$	$\longrightarrow$	$^3He + \gamma$	5.493	6 sec
$^3He + {}^3He$	$\longrightarrow$	$^4He + {}^1H + {}^1H$	12.859	10^6 yr

i) ^{1}H = hydrogen atom (1 proton).

ii) ^{2}H = heavy hydrogen (1 proton + 1 neutron) = deuterium.

iii) ^{3}He = light helium (2 protons + 1 neutron).

iv) ^{4}He = helium (2 protons + 2 neutron) = alpha particle.

v) γ = Gamma ray photon.

vi) e$^+$ = positron (positive charge) = anti-electron (antimatter). This positron interacts with the free electrons in the core virtually immediately which produces 2 additional gamma ray photons.

vii) ν_e = electron neutrino (neutral particle). The neutrino's absorption cross section is negligible and leaves the stellar core (and star) immediately without further interaction. The energy loss from the neutrino is 0.263 MeV which has not been included in the *Energy Released* column.

viii) In this PP I change, please note that the following reaction can take place 1.4% of the time that the first reaction takes place

$$^1\mathrm{H} + \mathrm{e}^- + {}^1\mathrm{H} \longrightarrow {}^2\mathrm{H} + \nu_e$$

the so-called "pep" (proton-electron-proton) reaction which releases 1.4 MeV and loses an additional 1.4 MeV in energy loss from the neutrino.

ix) Note that the *Energy Released* column is a combination of the energy of any gamma rays created, the kinetic energy = thermal energy picked up by the resulting nuclei as a result of this reaction, and the energy gained by the positron annihilation.

x) The times listed for the *Reaction Time* column is that for the Sun's central temperature.

xi) Note that the first two reaction steps must occur twice before the last one can take place.

xii) The last step of this reaction chain is occurs 69% of the time in comparison to the other two PP chains in the production of ^{4}He in the Sun.

xiii) This PP I chain dominates the other PP chains in stars with central temperatures of $T \lesssim 1.6 \times 10^7$ K.

xiv) Note the long *average time* it takes for the first and third reaction steps to take place for a single particle. However when normalized by the total number of particles in the Sun's (or star's) core, about 9.0×10^{37} of these reactions take place per second!

b) A second chain, called **PP II**, also can occur (31% of the time in the Sun) in the production of ^{4}He once the first two steps of the PP I chain occur.

Reaction			Energy Released (MeV)	Reaction Time
^{3}He + ^{4}He	$\longrightarrow$	^{7}Be + γ	1.586	1.0×10^6 yr
^{7}Be + e$^-$	$\longrightarrow$	^{7}Li + ν_e	0.861	0.4 yr
^{7}Li + ^{1}H	$\longrightarrow$	^{4}He + ^{4}He	17.347	10^4 yr

i) ^{7}Be = beryllium-7 atom (4 protons + 3 neutrons).

ii) ^{7}Li = lithium-7 atom (3 protons + 4 neutrons).

iii) The neutrino energy loss in this PP chain is 0.80 MeV.

iv) This PP II chain dominates the other PP chains in stars with central temperatures of $1.6 \times 10^7 \lesssim T \lesssim 2.5 \times 10^7$ K.

v) The *Energy Released* and the *Reaction Time* have the same meaning as they did for the PP I chain.

c) A third and final PP chain, called **PP III**, occurs only 0.3% of the time in the Sun in the production of ^{4}He once the first two steps of the PP I chain occurs and the first step of the PP II chain occurs.

Reaction			Energy Released (MeV)	Reaction Time
^{7}Be + ^{1}H	$\longrightarrow$	^{8}B + γ	0.135	70 yr
^{8}B	$\longrightarrow$	^{8}Be + e$^+$ + ν_e	17.98	1 sec
^{8}Be	$\longrightarrow$	^{4}He + ^{4}He	0.095	1 sec

i) ^{8}B = boron-8 atom (5 protons + 3 neutrons).

ii) ^{8}Be = beryllium atom (4 protons + 4 neutrons).

iii) The neutrino energy loss in this PP chain is 7.2 MeV. The Davis *solar neutrino experiment*, which detected only 1/3-rd of the predicted solar neutrinos was most sensitive to these ^{8}B beta decay neutrinos. Recently, neutrinos have been found to oscillate between the 3 known neutrino states which accounts for the low detection rate of the Davis experiment.

iv) This PP III chain dominates the other PP chains in stars with central temperatures of $T \gtrsim 2.5 \times 10^7$ K in the production of helium.

v) The *Energy Released* and the *Reaction Time* have the same meaning as they did for the PP I chain.

2. For more massive stars ($T_c \gtrsim 1.8 \times 10^7$ K, $M \gtrsim 1.3 M_\odot$, ~F5 V star), the **CNO cycle** is the dominant reaction chain.

a) This reaction chain uses carbon as a catalyst:

Reaction			Energy Released (MeV)	Reaction Time
^{12}C + ^{1}H	$\longrightarrow$	^{13}N + γ	1.95	1.3×10^7 yr
^{13}N	$\longrightarrow$	^{13}C + e$^+$ + ν_e	2.22	7 min
^{13}C + ^{1}H	$\longrightarrow$	^{14}N + γ	7.54	2.7×10^6 yr
^{14}N + ^{1}H	$\longrightarrow$	^{15}O + γ	7.35	3.2×10^8 yr
^{15}O	$\longrightarrow$	^{15}N + e$^+$ + ν_e	2.71	82 sec
^{15}N + ^{1}H	$\longrightarrow$	^{12}C + ^{4}He	4.96	1.1×10^5 yr

i) ^{12}C = carbon-12 (6 protons + 6 neutrons).

ii) ^{13}C = carbon-13 (6 protons + 7 neutrons).

iii) ^{13}N = nitrogen-13 (7 protons + 6 neutrons) [radioactive].

iv) ^{14}N = nitrogen-14 (7 protons + 7 neutrons).

v) ^{15}N = nitrogen-15 (7 protons + 8 neutrons).

vi) ^{15}O = oxygen-15 (8 protons + 7 neutrons) [radioactive].

vii) ^{16}O = oxygen-16 (8 protons + 8 neutrons).

viii) The neutrino energy loss in the ^{13}N beta decay is 0.710 MeV and the neutrino energy loss is 1.000 MeV for the second ^{15}O beta decay.

ix) The *Energy Released* and the *Reaction Time* have the same meaning as they did for the PP I chain.

b) Note that this reaction sequence does not make any new elements other than He!

c) For the last step in the CNO cycle, an additional set of reactions can take place:

Reaction			Energy Released (MeV)	Reaction Time
$^{15}N + {}^{1}H$	$\longrightarrow$	$^{16}O + \gamma$	12.126	1.0×10^{7} yr
$^{16}O + {}^{1}H$	$\longrightarrow$	$^{17}F + \gamma$	0.601	3.0×10^{10} yr
^{17}F	$\longrightarrow$	$^{17}O + e^{+} + \nu$	2.762	3 min
$^{17}O + {}^{1}H$	$\longrightarrow$	$^{14}N + {}^{4}He$	1.193	2.0×10^{11} yr

i) ^{17}F = fluorine-17 (9 protons + 8 neutrons).

ii) ^{18}O = oxygen-16 (8 protons + 10 neutrons).

iii) The neutrino energy loss in the ^{17}F beta decay is 0.94 MeV.

iv) The resulting ^{14}N isotope can then be used back in the

$$^{14}N + {}^{1}H \longrightarrow {}^{15}O + \gamma$$

reaction in the primary CNO cycle.

3. The thermonuclear reaction rate, ε (in erg/gm/s), is very sensitive to temperature. For the two hydrogen to helium reaction chains (*i.e.*, the proton-proton chain and CNO cycle), we can write Eq. (IV-48) as two separate equations:

$$\varepsilon_{pp} = \varepsilon_{\circ}\, \rho X^2 \left(\frac{T}{10^6}\right)^{\beta} \quad \text{(IV-49)}$$

$$\varepsilon_{cc} = \varepsilon_{\circ}\, \rho X X_{\rm CN} \left(\frac{T}{10^6}\right)^{\beta} , \quad \text{(IV-50)}$$

where X is the mass fraction of hydrogen (as defined on page IV-8 of the course notes), $X_{\rm CN}$ is the weighted average of the combined mass fraction of carbon and nitrogen (since these species are the two lead-off species of the two CNO cycles), note that typically

$$X_{\rm CN} = \frac{1}{3} Z \; , \qquad \text{(IV-51)}$$

with the Z being the metalicity mass fraction. Finally, $\varepsilon_\circ$ and α are temperature dependent constants given in the following table (from Bosmas-Crespin, Fowler, Humblet 1954, *Bull. Soc. Royale Sciences Liege*, No. 9-10, 327).

ε_{pp}			ε_{cc}		
$T/10^6$	$\log \varepsilon_\circ$	α	$T/10^6$	$\log \varepsilon_\circ$	α
4 – 6	–6.84	6	12 – 16	–22.2	20
6 – 10	–6.04	5	16 – 24	–19.8	18
9 – 13	–5.56	4.5	21 – 31	–17.1	16
11 – 17	–5.02	4	24 – 36	–15.6	15
16 – 24	–4.40	3.5	36 – 50	–12.5	13

In this table, for overlapping temperatures, a weighted average is used to get the final rate.

4. Figure IV-3 shows the rate of nuclear reactions as a function of temperature for the 3 processes described above. Note that the Sun is right on the borderline of having the CNO cycle being an important component to its energy output. Also note the much steeper temperature dependence that the CNO cycle has with respect to the PP chain.

5. We will see shortly, that when a main sequence uses up all of its H fuel in its core, the now He-rich core will contract and heat up. When temperatures exceed 10^8 K, helium fusion can begin. Helium fuses via the **triple-α process**. The ash of this reaction is carbon.

Figure IV–2: Temperature dependence of the rate of nuclear energy release for three reaction processes. From Martin Schwarzschild, *Structure and Evolution of the Stars* (Princeton, NJ; Princeton Univ. Press) 1958.

a) The reaction is as follows:

Reaction			Energy Released (MeV)
^{4}He + ^{4}He	$\longleftrightarrow$	^{8}Be + γ	-0.0921
^{8}Be + ^{4}He	$\longrightarrow$	^{12}C + γ	7.37

i) Since this reaction chain is not occurring in the Sun, we have not reported on any reaction times here since they are very temperature dependent.

ii) As can be seen, three α-particles (*i.e.*, He nuclei) fuse to become one carbon nuclei.

iii) ^{8}Be is unstable and quickly decays, so there is not much berylium around for the 2nd chain to take place. For every 1 berylium nuclei, there are

10^{10} α-particles, however this ratio is more than enough to release enough energy to power a red giant star. Note that this reaction actually *drains* energy away from the gas $\Longrightarrow$ it is an **endothermic** reaction. All reactions we have mentioned up to now have been **exothermic** $\Longrightarrow$ they release energy.

b) The 3α process has an even greater sensitivity to temperature than the CNO cycle:

$$\varepsilon_{3\alpha} \approx 10^{-8} \rho Y^3 \left(\frac{T}{10^6}\right)^{30} \text{ erg/gm/s}, \qquad \text{(IV-52)}$$

where Y is the fractional abundance of helium per unit mass.

c) This is the way the Universe makes carbon. As such, the C atoms that make up our DNA were created in an ancient red giant star that no longer exists. To quote Carl Sagan, *we are star stuff!*

d) Should the core of a red giant obtain temperatures that exceed a few hundred million Kelvins, another reaction can take place via an **alpha** (α) **capture**:

$$^{12}\text{C} + {}^{4}\text{He} \longrightarrow {}^{16}\text{O} + \gamma$$

which releases 7.161 MeV of energy.

e) Most of the ^{16}O in the Universe is made in this fashion.

6. Finally, if somewhat higher temperatures are ever encountered inside a star, which happens during stellar evolution of massive stars, even heavier elements can be created from the fusion of additional α-particles and α-particle by-products:

Reaction			Energy Released (MeV)	Minimum Temperature Required (10^6 K)
^{14}O + ^{4}He	$\longrightarrow$	^{20}Ne + γ	4.730	700
^{20}Ne + ^{4}He	$\longrightarrow$	^{24}Mg + γ	9.317	1500
^{24}Mg + ^{4}He	$\longrightarrow$	^{28}Si + γ	9.981	1800
^{28}Si + ^{4}He	$\longrightarrow$	^{32}S + γ	6.948	2500
^{32}S + ^{4}He	$\longrightarrow$	^{36}Ar + γ	6.645	3500
^{12}C + ^{12}C	$\longrightarrow$	^{24}Mg + γ	13.930	800
^{16}O + ^{16}O	$\longrightarrow$	^{32}S + γ	16.539	2000

a) Various silicon burning reactions can occur at temperatures exceeding 3×10^9 K. Silicon burning produces the iron (Fe) group elements.

b) Once Fe is formed, reactions that produce heavier elements are all endothemic and have a tough time forming via the standard thermonuclear burning. Such elements, and the elements not built upon α-particles, are created via the **r-** (for rapid neutron capture) and **s-** (for slow neutron capture) **processes**. These processes will be discussed in the supernovae section of the course notes (*i.e.*, §VII).

c) The reaction times of this heavy element nucleosynthesis will be discussed in the stellar evolution sections of the notes.

E. Modeling the Interiors of Stars

1. During epochs when stellar evolution is not changing the structure of the star rapidly over time, we can model the interior of stars with the **equations of stellar structure**. Consider a single, nonmagnetic, nonrotating (hence, spherically symmetric) star. We can describe P, T, L_r, and M_r as functions of r

by four differential equations that describe:

a) Hydrostatic Equilibrium (Eq. IV-14):

$$\frac{dP}{dr} = -\frac{G\rho M_r}{r^2} \ . \qquad \text{(IV-53)}$$

b) Mass Conservation (Eq. IV-2):

$$\frac{dM_r}{dr} = 4\pi\rho r^2 \ . \qquad \text{(IV-54)}$$

c) Energy Conservation (Eq. IV-36):

$$\frac{dL_r}{dr} = 4\pi\rho\varepsilon r^2 \ . \qquad \text{(IV-55)}$$

d) Energy Transport:

i) Radiative for $\left|\frac{dT}{dr}\right|_{\rm rad} < \left|\frac{dT}{dr}\right|_{\rm ad}$ (Eq. IV-35):

$$\frac{dT}{dr} = -\frac{3k\rho}{4acT^3}\,\frac{L_r}{4\pi r^2} \ . \qquad \text{(IV-56)}$$

ii) Convective for $\left|\frac{dT}{dr}\right|_{\rm rad} > \left|\frac{dT}{dr}\right|_{\rm ad}$ (Eq. IV-31):

$$\begin{aligned} \frac{dT}{dr} = \left.\frac{dT}{dr}\right|_{\rm ad} &= \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\,\frac{dP}{dr} \\ &= \left(1-\frac{1}{\gamma}\right)\frac{\mu m_{\rm H}}{k_B}\,\frac{GM_r}{r^2} \qquad \text{(IV-57)} \end{aligned}$$

iii) Note that the convection criterion can also be checked with

$$\frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma-1} \ , \qquad \text{(IV-58)}$$

where γ is the ratio of specific heats given by Eq. (III-26). If this relation is valid, convection will carry the energy.

2. These 4 ordinary differential equations readily can be solved by setting the following boundary conditions:

a) At the stellar core:

$$\left.\begin{array}{rcl} M_r & = & 0 \\ L_r & = & 0 \end{array}\right\} \quad \text{when } r = 0 \ . \qquad \text{(IV-59)}$$

b) At the stellar surface:

$$\left.\begin{array}{rcl} P & = & 0 \\ T & = & 0 \end{array}\right\} \quad \text{when } r = R_\star \ . \qquad \text{(IV-60)}$$

3. In order to solve these 4 ODEs, we need 3 sets of equations that describe the gas characteristics:

a) Equation of state:

$$P = P(\rho, T, \ \text{composition}) \ . \qquad \text{(IV-61)}$$

Examples: ideal gas law (Eq. IV-23), radiation pressure (Eq. IV-15), degenerate electron pressure (needed for white dwarf stars).

b) Opacity (a *mean* opacity is calculated $\Longrightarrow$ **Rosseland mean opacity**):

$$k = k(P, T, \ \text{composition}) \ . \qquad \text{(IV-62)}$$

Examples: electron scattering, free-free opacities, bound-free opacities (all of these are combined to calculate the mean opacity). Remember, 'k' is the mass-absorption coefficient = κ (opacity) / ρ (mass density).

c) Energy Sources and Sinks:

$$\varepsilon = \varepsilon(P, T, \ \text{composition}) \ . \qquad \text{(IV-63)}$$

Examples: nuclear energy sources (*e.g.*, Eqs. IV-49, 50, 52), neutrino losses.

4. Modeling the interiors of stars has lead to the following theorem:

> The mass and composition of a star uniquelt determines its radius, luminosity, and internal structure, as well as its subsequent evolution.

This is referred to as the **Vogt-Russell theorem**.

5. The final ingredient in modeling the internal structure of stars is that stars evolve "quasistatically" by slowly changing their composition through nuclear burning.

6. Everything we understand about stars, their evolution, and the chemical evolution of the Universe results from such modeling.