ASTR-3415: Astrophysics
Course Lecture Notes
Section VIII

Dr. Donald G. Luttermoser
East Tennessee State University

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Abstract

These class notes are designed for use of the instructor and students of the course ASTR-3415: Astrophysics. This is the Version 1.2 edition of these notes.

VIII. The Theory of Relativity and Stellar Corpses

A. White Dwarf Stars.

1. Stars will end up in one of 3 states: white dwarf, neutron star, or black hole. Stars that are initially $\sim$4 $M_\odot$ or less on the main sequence will wind up as a white dwarf.

2. As the carbon-oxygen core continues to collapse after He-core burning, the outer envelope of the star continues to expand away, helped along by strong stellar winds.

   a) The outer envelope detaches itself from the collapsing core and becomes a shell of material surrounding the core.

   b) When the shell be comes thin enough so that the hot core can be seen through the shell, UV photons emitted from the core fluoresce the expanding shell $\Longrightarrow$ a **planetary nebula** forms (see pages VII-7 through VII-13 of the last section of the notes for details).

   c) This shell will continue to expand, getting thinner and thinner as it gets larger until it begins to intermingle with the surrounding ISM.

   d) Since the outer envelope of the star was enhanced with C, N, and O from the nuclear reactions due the shell burning phases, the ISM gets enhanced with these elements.

   e) Future stars that will be born out of the ISM gas will thus have higher abundances of C, N, and O due to this previous epoch of stellar evolution.

3. The core continues to collapse until the free electrons in the core are forced so close together that the **Pauli Exclusion Principle** takes effect — the object is now stable due to **degenerate electron pressure** $\Longrightarrow$ the core is now called a **white dwarf** (WD).

a) In classical thermodynamics and statistical mechanics, the position and velocity of particles is dictated by the **Maxwell-Boltzmann statistics**. Here, the velocity distribution of the electrons or the *equation of state* of the gas is derived by calculating the distribution of particles in *phase space* according to the rules of probability theory $\Longrightarrow$ the 6-dimensional element $\Delta\Omega$ of phase space is formed by the product of the familiar volume element $\Delta V = \Delta x\, \Delta y\, \Delta z$ in position space and the element in momentum space $\Delta p_x\, \Delta p_y\, \Delta p_z$.

b) However, for an electron gas, the applicability of this procedure is restricted by the Pauli Exclusion Principle, which requires that one quantum state or one quantum cell in phase space of size $\Delta\Omega = h^3$ (h = Planck's constant) may be occupied by at most one electron of each spin direction, that is, by two electrons altogether. **Fermi-Dirac statistics** take this into account.

c) In the interior of a WD the material is so compressed due to the enormous pressure and comparatively low temperature that all cells h^3 in phase space, up to a certain threshold energy $E_{\rm F}$, the **Fermi energy**, are completely filled.

4. The equation of state of the degenerate electron gas can easily be calculated. Note that proton gas which is also present does not become degenerate until much higher densities, so its pressure can be neglected.

a) Let there be n_e electrons/cm^3 (*i.e.*, the electron density) in a volume V, giving a total of $n_e V$ electrons in the volume. In momentum space p_x, p_y, p_z these electrons uniformly fill a sphere whose radius is the maximum momentum $p_\circ$ (or the threshold energy, the so-called Fermi energy $E_\circ = E_{\rm F} = p_\circ^2/2m_e$).

b) We therefore have a volume $4\pi/3 p_\circ^3 V$ in phase space and with 2 electrons per phase cell of size h^3, we obtain the relation

$$n_e V = \frac{2}{h^3} V \frac{4\pi}{3} p_\circ^3 \qquad \text{(VIII-1)}$$

$$p_\circ = \sqrt{2m_e E_\circ} = h \left(\frac{3n_e}{8\pi}\right)^{1/3} . \qquad \text{(VIII-2)}$$

c) Pressure can be expressed by the mean energy per electron $\overline{E}$ with

$$P = \frac{2}{3} n_e \overline{E} , \qquad \text{(VIII-3)}$$

where

$$\overline{E} = \frac{\int_0^{p_\circ} E \, 4\pi p^2 dp}{\int_0^{p_\circ} 4\pi p^2 dp} = \frac{3}{5} \frac{p_\circ^2}{2m_e} = \frac{3}{5} E_\circ . \qquad \text{(VIII-4)}$$

d) So the equation of state of a completely degenerate electron gas is

$$P = \frac{2}{5} n_e E_\circ = \frac{8\pi h^2}{15 m_e} \left(\frac{3n_e}{8\pi}\right)^{5/3} \qquad \text{(VIII-5)}$$

$\Longrightarrow$ pressure is completely independent of temperature! P only depends upon density for a degenerate gas.

e) The electron number density n_e can be related to the mass density ρ by

$$n_e = \left(\frac{\#\text{ electrons}}{\text{nucleon}}\right) \left(\frac{\#\text{ nucleons}}{\text{volume}}\right) = \left(\frac{Z}{A}\right) \frac{\rho}{m_{\rm H}} , \qquad \text{(VIII-6)}$$

where Z and A are the number of protons and nucleons, respectively, in the white dwarf's nuclei, and $m_{\rm H}$ is the mass of the hydrogen atom.

f) Thus the Fermi energy is proportional to the 2/3 power of the density.

5. Using this equation of state ($P \propto \rho^{5/3}$) along with HSE (*i.e.*, $P \propto \rho GM/R$), and the definition of density ($\rho \propto M/R^3$), one can show a mass-radius relation for white dwarfs as

$$R \propto M^{-1/3} \qquad \text{(VIII-7)}$$

$\Longrightarrow$ the more massive a white dwarf, the smaller it is.

6. Using the blackbody radiation law (*i.e.*, $L = 4\pi R^2 \sigma T_{\rm eff}^4$), one can determine a theoretical color-magnitude (*e.g.*, H-R) diagram that lies on the line

$$L \propto M^{-2/3} T_{\rm eff}^4 \,. \qquad \text{(VIII-8)}$$

a) Comparisons with observations yields the surprising result that most white dwarfs have masses of about 0.6 $M_\odot$.

b) They are all about the size of the Earth giving an enormous density for these objects ($\rho = 10^9$ kg/m^3) $\Longrightarrow$ 1 teaspoon of WD material on the surface of the Earth would weight 5.5 tons, as much as an elephant!

7. In 1931, S. Chandrasekhar included relativistic effects in degenerate matter and showed under these cases that the equation of state for degenerate matter follows $P \propto \rho^{4/3}$ $\Longrightarrow$ the material is more easily compressed (see §15.4 in your textbook).

a) More massive stellar cores will have relativistic degenerate electrons.

b) The results of Chandrasekhar's work showed that *relativistic* degenerate electron pressure is

$$P = \frac{c}{4}\left(\frac{3h^3}{8\pi}\right)^{1/3} n_e^{4/3} = \frac{c}{4}\left(\frac{3h^3}{8\pi}\right)^{1/3}\left(\frac{Z}{A}\frac{\rho}{m_{\rm H}}\right)^{4/3} , \qquad \text{(VIII-9)}$$

making use of Eq. (VIII-6).

c) Equations of state where pressure P is a function of density alone (*e.g.*, N or ρ), that is, independent of temperature, are referred to as a **polytropic equation of state** and the objects are called **polytropes**.

i) A differential equation can be set up from the HSE condition using polytropes. Such a differential equation is called the **Lane-Emden Equation**.

ii) Degenerate gas is a polytropic equation of state, and as such, Chandrasekhar in 1931 (*Astrophysical Journal*, **74**, 81) and 1935 (*Mon. Not. Royal Astro. Soc.*, **95**, 226 [Paper I] and *Mon. Not. Royal Astro. Soc.*, **95**, 207 [Paper II]) solved the Lane-Emden equation for the relativistic electron pressure equation of state and found that

$$M_c = \left(\frac{2}{\mu_e}\right)^2 1.4587\, M_\odot , \qquad \text{(VIII-10)}$$

where $\mu_e = A/Z$ is the mean electron molecular weight, A is the number of nucleons (protons plus neutrons) and Z is the number of protons in the white dwarf.

iii) For a helium white dwarf, $\mu_e = 2.00$ so $M_c = 1.46\, M_\odot$.

iv) For a carbon-oxygen white dwarf, $\mu_e = 2.01$ so $M_c = 1.44\, M_\odot$.

d) If the mass of a WD exceeds $M_c \approx 1.44\ M_\odot$, then the electron degeneracy pressure will be too weak to counterbalance gravity! The stellar remnant will continue its collapse with catastrophic results!

e) This mass limit of $M_c \approx 1.44\ M_\odot$ is called the **Chandrasekhar mass limit**.

f) All white dwarfs have $M < M_c$.

8. White dwarfs have the following spectral classifications. The first letter 'D' denotes degenerate and the second letter denotes the the primary spectral type in the optical spectrum.

a) **DA**: Only hydrogen Balmer lines are seen — no helium or metal lines present.

b) **DB**: Only He I lines are seen — no hydrogen or metal lines present.

c) **DC**: Continuous spectrum, no lines deeper than 5% in any part of the electromagnetic spectrum.

d) **DO**: He II strong, He I and H lines present.

e) **DZ**: Metal lines only, no H or He lines.

f) **DQ**: Carbon features, either atomic or molecular in any part of the electromagnetic spectrum.

9. The nearest (and brightest) white dwarfs to the solar system, each a companion to a brighter, and better known, main se-

Table VIII–1: Three of the Nearest and Brightest White Dwarf Stars

Name	**Sirius B**	**Procyon B**	**40 Eri B**
Mass ($M_\odot$)	1.034 ± 0.026	0.602 ± 0.015	0.501 ± 0.011
Radius ($R_\odot$ & $R_\oplus$)	$0.0084 \pm 0.00025 R_\odot$ $0.92 \pm 0.02 R_\oplus$	$0.01234 \pm 0.00032 R_\odot$ $1.347 \pm 0.034 R_\oplus$	$0.0136 \pm 0.0002 R_\odot$ $1.48 \pm 0.03 R_\oplus$
Effective Temperature (K)	$24,790 \pm 100$	$7,740 \pm 50$	$16,700 \pm 300$
Distance (parsec)	2.637 ± 0.011	3.498 ± 0.011	5.044 ± 0.213
Spectral Type	DA2	DQZ	DA4
V	8.44 ± 0.06	10.82 ± 0.03	9.50 ± 0.02
M_V	11.33 ± 0.06	13.10 ± 0.04	10.99 ± 0.11

quence star are shown in Table VIII-1 (primarily based on Table 3 from Provencal *et al.* 2002, *Astrophysical Journal*, **568**, 324).

B. Neutron Stars.

1. Stars more massive than 8 $M_\odot$ on the main sequence will be able to successively burn heavier and heavier elements as their core collapses.

2. The final type of stellar burning to go on in a massive core is Si (silicon) burning. The ash from this burning is Fe (iron).

   a) Fe is the most stable of all the chemical elements $\Longrightarrow$ you can neither get energy out of an Fe nucleus from either **nuclear fusion** (*i.e.*, bringing lighter elements together to form heavier ones) or **nuclear fission** (*i.e.*, the breaking apart of heavier elements into lighter ones).

   b) Once an Fe core forms, its collapse cannot be halted by further nuclear reactions.

c) The Fe core become degenerate as it collapses but is too massive for the electron degeneracy pressure to hold up the weight of the star.

d) The details of this final stage of stellar evolution for massive stars are described in §VII in the notes on pages VII-15 through VII-17.

3. The electrons interact with protons in an inverse β-decay as this final core collapse takes place for a massive star:

$$e^- + p \longrightarrow n + \nu,$$

forming a stellar core of *pure neutrons*!

4. Once the neutrons are formed, they stop the collapse of the core via the strong nuclear force $\Longrightarrow$ neutron degeneracy pressure holds up the weight of the core $\Longrightarrow$ a **neutron star** (NS=ns) is born!

a) This halting of the collapse of the core is rather sudden and causes the core to bounce and rebound a bit. This bounce sets up a shock wave which propagates outward and blows apart the outer envelope of the star in a **supernova** explosion!

b) A tremendous amount of energy is released during this explosion over a fraction of a second, increasing the luminosity of the star by a factor of 10^8!

c) However this luminosity increase only corresponds to 1% of the total energy released, most of the energy comes out in the form of neutrinos!

5. Like white dwarfs stars, the degenerate neutron equation of state forces the radii of neutron stars to scale as $M^{-1/3}$:

$$R_{\rm ns} \approx \frac{(18\pi)^{2/3}}{10} \frac{\hbar^2}{GM_{\rm ns}^{1/3}} \left(\frac{1}{m_{\rm H}}\right)^{8/3} . \qquad \text{(VIII-11)}$$

6. Although there is no limit on the minimum mass that a neutron star can possess, one often takes this minimum mass as the maximum mass a white dwarf can have — the Chandrasekhar mass: 1.44 $M_\odot$. Since the neutron star is composed of solid neutrons, the total number of neutrons in a neutron star ia $N_n = M_{\rm ns}/m_n \approx 10^{57}$ neutrons!

7. All stars rotate while on the main sequence. As the core of a star collapses, it spins faster and faster due to the conservation of angular momentum.

a) By the time a stellar core reaches NS size (*e.g.*, $R_{\rm ns} = 10 - 30$ km, the size of a city), it is spinning hundreds to thousands of times a second!

b) **Pulsars** are rapidly spinning neutron stars whose magnetic axis is not aligned with its polar axis.

c) Pulsars flash very rapidly as a result due to the *lighthouse effect* of the beamed light coming from the magnetic axes sweeping in the line-of-sight towards Earth.

d) Pulsars were first discovered in radio waves in 1967 $\Longrightarrow$ first thought to be signals from extraterrestrial intelligence — called LGM's for *Little Green Men.*

8. The small size and high mass ($1.4 M_\odot < M_{\rm ns} < 3 M_\odot$) of NS give them an exceedingly large density $\Longrightarrow \rho = 10^{17}$ kg/m^3, one teaspoon of NS material on Earth would weight over 500 million tons!

9. Neutron stars also have a maximum mass, if the mass of a stellar remnant exceeds $3M_{\odot}$, not even neutron degeneracy can hold up the stellar core. It collapses further to a black hole!

C. Einstein's Theory of Relativity.

1. In the late 19th century, 2 physicists, Michelson and Morley set out to measure the Earth's motion through a then hypothesized **Ether** which filled the Universe.

a) The existence of the Ether was speculated on in order to explain how light was able to propagate across the Universe $\Longrightarrow$ water waves need water to propagate, sound needs air to propagate, hence light waves must require the Ether to propagate.

b) The Michelson-Morley experiment did not detect any evidence that the Ether existed — either the Ether moves with the Earth or it did not exist!

c) Their experiment also showed that the speed of light remained the same no matter what direction they were looking in with respect to the Earth's motion.

d) The theory of electromagnetism shows us that light (*i.e.*, photons) can self-propagate without any need for a medium.

2. Einstein was stimulated by these results and used it to prove that there was no Ether through his development of the **theory of relativity**. The theory of relativity is separated into 2 parts. Part 1, the **special theory**, was developed in 1905 and is based upon 2 postulates:

a) **The Principle of Relativity**: The laws of physics are the same in all inertial (*i.e.*, non-accelerating) reference systems (or frames).

b) **The Constancy of the Speed of Light**: The speed of light in a vacuum is constant ($c = 2.997925 \times 10^{10}$ cm/s) and independent of the motion of the observer or the motion of the light source.

3. The full derivation of the math described here is shown on pages 96 through 118 in your textbook. I will summarize them here. Let a reference frame at rest be designated with unprimed variables and a frame moving at constant velocity u will have variables marked with a prime.

a) If light transverses an element of path length $dr = (dx^2 + dy^2 + dz^2)^{1/2}$ in a time element dt with vacuum light speed c, the independence of a photon's velocity on a reference frame requires that

$$dx^2 + dy^2 + dz^2 = (ct)^2 \qquad \text{(VIII-12)}$$

$$dx'^2 + dy'^2 + dz'^2 = (ct')^2 \qquad \text{(VIII-13)}$$

b) Now let's define the differential ds with

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = 0 \qquad \text{(VIII-14)}$$

$\Longrightarrow$ ds is known as an *event-line element* in a four-dimensional space-time manifold (3 spatial coordinates and a fourth coordinate ct (= light path)).

c) In the theory of special relativity, this four-dimensional line element ds^2 must remain invariant in all inertial frames. Let's rewrite Eq. (VIII-14) as a summation of all positive terms, then

$$ds^2 = dx^2 + dy^2 + dz^2 + d(ict)^2, \qquad \text{(VIII-15)}$$

where $i = \sqrt{-1}$. Hence the transformation $x, y, z \longrightarrow x', y', z'$ is not necessarily valid in special relativity as it is in Newtonian mechanics. Instead, one must include time in the transformation: $x, y, z, t \longrightarrow x', y', z', t'$.

d) This so-called **Lorentz transformation** is none other than a rotation in a four-dimensional space x, y, z, ict, in which by definition the line element ds is invariant.

e) Assume an object is moving with a velocity v in a straight line (assume in the x direction). Then according to Newtonian mechanics, which is based upon the **Galilean transformation** equations, to transform from the moving frame (designated with a prime) to the unmoving frame, we have

$$x' = x - ut \quad \text{(VIII-16)}$$
$$y' = y \quad \text{(VIII-17)}$$
$$z' = z \quad \text{(VIII-18)}$$
$$t' = t. \quad \text{(VIII-19)}$$

f) With the invariance equation above, Einstein showed that coordinate conversion must be handled with the Lorentz transformation equations:

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} \quad \text{(VIII-20)}$$
$$y' = y \quad \text{(VIII-21)}$$
$$z' = z \quad \text{(VIII-22)}$$
$$t' = \frac{t - xu/c^2}{\sqrt{1 - u^2/c^2}}. \quad \text{(VIII-23)}$$

Note that when $u \ll c$, then the Lorentz transformations reduce down to the Galilean transformations and Newtonian mechanics is recovered.

g) With the above transformations, it is convenient to write the velocity radical term as

$$\gamma \equiv \frac{1}{\sqrt{1 - u^2/c^2}} \;, \qquad \text{(VIII-24)}$$

where γ is called the **Lorentz factor**. Roughly speaking relativity differs from Newtonian mechanics by 1% ($\gamma = 1.01$) when $u/c \simeq 1/7$ and by 10% ($\gamma = 1.10$) when $u/c \simeq 5/12$.

h) The fundamental difference between the two transformations is that in the Lorentz transformations we realize that the communication signals between the two frames, which are needed to synchronize the clocks of a reference frame, propagate at a finite speed c, while in the Galilean transformations, we naively assume the velocity of these signals to be infinite.

4. This special theory essentially rewrites Newton's laws of motion. It also showed that mass and energy are interrelated with each other through the famous equation $E = mc^2$. The Lorentz transformations above lead to 3 important findings:

a) **Time Dilation**: A time interval between two events occurring at the same place in some frame of reference is *longer* by a factor $1/\sqrt{1 - u^2/c^2}$ when viewed from a frame moving (designated with a prime below) relative to the first frame and, consequently, in which the two events are spatially separated:

$$\Delta t_{\rm moving} = \frac{\Delta t_{\rm rest}}{\sqrt{1 - u^2/c^2}} \;. \qquad \text{(VIII-25)}$$

b) **Length Contraction**: A measured rod is *shorter* by a factor $\sqrt{1 - u^2/c^2}$ compared to its length in a frame in which it is at rest, when it is observed from a frame in which it is moving *parallel* to its own length following

$$L_{\rm moving} = L_{\rm rest}\,\sqrt{1 - u^2/c^2}\ . \qquad \text{(VIII-26)}$$

c) **Mass Increase without Bound**: Mass also increases for the object that is traveling close to the speed of light! Einstein showed that

$$m = \frac{m_\circ}{\sqrt{1 - u^2/c^2}}, \qquad \text{(VIII-27)}$$

where $m_\circ$ is the rest mass of the object. Note that as $u \rightarrow c$, $m \rightarrow \infty$! Since there is not an infinite amount of energy to be had in the Universe, it is impossible to push a mass from sublight speeds to speeds greater than light!

5. Time dilation has an important effect on the Doppler effect as derived on pages 107 through 112 in your textbook. If the Lorentz factor is $\gamma \gtrsim 1.01$, the Doppler equation starts to follow the **relativistic Doppler shift**.

a) For the wavelength of light λ, let's define the **redshift parameter** z as

$$z = \frac{\lambda_{\rm obs} - \lambda_{\rm rest}}{\lambda_{\rm rest}} = \frac{\lambda - \lambda_\circ}{\lambda_\circ} = \frac{\Delta\lambda}{\lambda_\circ}\ . \qquad \text{(VIII-28)}$$

b) At <u>non-relativistic</u> speeds, the Doppler formula is

$$z = \frac{v_r}{c}\ , \qquad \text{(VIII-29)}$$

where v_r is the radial velocity and c is the speed of light in the vacuum.

c) At <u>relativistic speeds</u>, the Doppler formula becomes

$$z = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}} - 1\ , \qquad \text{(VIII-30)}$$

or in terms of wavelength,

$$\lambda = \lambda_\circ \sqrt{\frac{1 + v_r/c}{1 - v_r/c}} \ . \qquad \text{(VIII-31)}$$

d) Note that we can also write the time dilation formula using redshift as

$$z + 1 = \frac{\Delta t_{\rm obs}}{\Delta t_{\rm rest}} \ . \qquad \text{(VIII-32)}$$

Example VIII–1. Show that the relativistic Doppler effect reduces to the standard form at low velocities.

For $v_r \ll c$, $1 - v_r/c \approx 1$, then we can Taylor expand the radical in Eq. (VIII-30) giving

$$z = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}} - 1 \approx \sqrt{1 + v_r/c} - 1 \approx 1 + v_r/c - 1 = v_r/c \ ,$$

which is the standard form of the Doppler equation.

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6. These effects have been observed in the laboratory with subatomic particles!

7. Whereas the special theory deals with non-accelerating motion, part 2 of the theory, the **general theory of relativity** (developed in 1916) deals with accelerating objects in a gravitational field — it essentially rewrites Newton's Universal Law of Gravitation.

8. Whereas Newton envisioned gravity as some *magical* force, Einstein describes gravity as a curvature in space and time (called the **space-time continuum**). We will save much of the discussion of the general theory for the end of the course when we discuss *cosmology*. However for now:

a) The general theory is based upon the **principle of equivalence**. You cannot distinguish between acceleration in a gravitational field and accelerating due to a mechanical force (*i.e.*, $F = ma$) in a gravity-free environment.

b) Space and time are interrelated — you cannot have one without the other!

c) The Universe can be considered to consist of a *fabric* of space-time (*i.e.*, the vacuum) with pockets of matter that exist within this fabric, where this matter *bends* the fabric of space-time. The more mass density, the greater the bending or **warping** of space-time (see Figure VIII-1).

9. According to K. Schwarzschild in 1916, the gravitational field of a mass M can be represented using the metric

$$ds^2 = \frac{dr^2}{1 - r_s/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \left(1 - \frac{r_s}{r}\right) c^2\, dt^2, \quad \text{(VIII-33)}$$

in which, as usual, r, θ, ϕ denote spatial polar coordinates and t is the time.

a) The constant of integration

$$r_s = \frac{2GM}{c^2}, \qquad \text{(VIII-34)}$$

is called the **gravitational radius** or the **Schwarzschild radius** of the mass M.

b) For the Sun ($M = 1\ M_\odot$), $r_s = 2.96$ km.

c) The size of r_s determines the departures from the Euclidean metric of empty space.

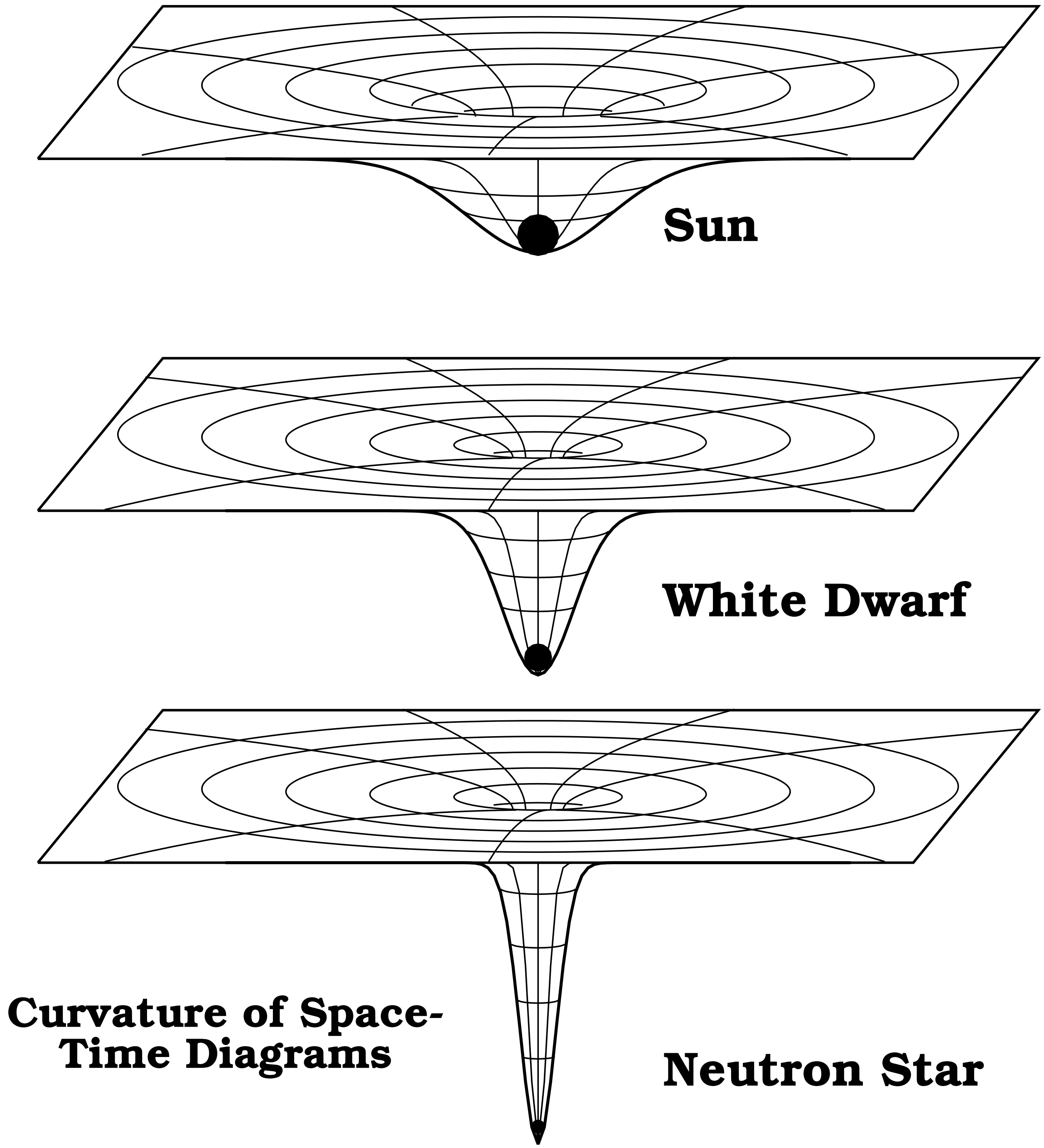

Figure VIII–1: The curvature of space-time gets greater and greater as a stellar core gets denser and denser as it collapses.

10. Equation (VIII-33) dictates that objects (and photons) follow **geodesics** (that is, shortest paths) in space-time.

a) To put it simply, general relativity reveal that objects still follow straight line paths as Newton said in his first law of motion, but in a gravitational field, those *straight* lines are now curved since space-time is warped — the object's motion curves around a massive body.

b) Even light bends around massive objects $\Longrightarrow$ gravitational lenses.

11. Starlight bending around the Sun during a solar eclipse in the 1920's proved the validity of general relativity.

D. Black Holes.

1. As a stellar core collapses, it gets denser and denser, and the escape velocity from the surface of the star goes increases. First, let's look at the conservation of energy from Newtonian mechanics for a body of mass m moving in the gravitational field from rest ($v = 0$) at the surface ($h = 0$) to a height h above the surface (marked with a '$\circ$' subscript) of a much larger body of mass M and radius R:

$$\mathrm{TE} = (\mathrm{KE} + \mathrm{PE})_\circ = (\mathrm{KE} + \mathrm{PE})_h = \text{ constant}, \quad \text{(VIII-35)}$$

where TE is the total energy,

$$\mathrm{KE} = \frac{1}{2}mv^2 \ , \quad \text{(VIII-36)}$$

is the kinetic energy, and

$$\mathrm{PE} = -\frac{GMm}{R} \ , \quad \text{(VIII-37)}$$

is the gravitational potential energy.

a) The conservation of energy becomes

$$\frac{mv^2}{2} - \frac{GMm}{R} = -\frac{GMm}{(R+h)}. \qquad \text{(VIII-38)}$$

b) We can solve this for h such that

$$h = (v^2/2g) \left[\frac{R}{R - (v^2/2g)} \right], \qquad \text{(VIII-39)}$$

where $g = GM/R^2$ is the surface gravity of mass M.

c) One can see from this that as $v^2/2g \rightarrow R, h \rightarrow \infty$ — the mass m escapes the gravitational field of M. When this occurs, the velocity required to send $h \rightarrow \infty$ is called the **escape velocity**:

$$v_{\rm esc} = \sqrt{\frac{2GM}{R}}\,. \qquad \text{(VIII-40)}$$

2. The escape velocity for various objects:

a) At the Earth's surface: $R = R_\oplus, M = M_\oplus$, so $v_{\rm esc} =$ 11.2 km/s.

b) At the Sun's photosphere: $R = R_\odot, M = M_\odot$, so $v_{\rm esc} =$ 620 km/s.

c) At a WD surface: $R = R_\oplus, M = M_\odot$, so $v_{\rm esc} = 6500$ km/s $= 0.02\ c$.

d) At a NS surface: $R = 30$ km, $M = 2M_\odot$, so $v_{\rm esc} =$ $230,000$ km/s $= 0.77\ c$.

3. A **black hole** (BH) is defined when an object reaches a size such that its escape velocity equals the speed of light. The collapse continues past this $v_{esc} = c$ size until the BH becomes infinitely small ($R \rightarrow 0$) $\Longrightarrow$ it collapses to a **singularity**.

Example VIII–2. How big would the Sun have to be in order to become a black hole?

$$v_{\rm esc} = c, \qquad M = M_\odot \qquad v_{\rm esc}^2 = \frac{2GM}{R}$$

$$R = \frac{2GM}{c^2} = 2.96 \times 10^3 \text{ m} = 2.96 \text{ km}$$

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a) The region around a black hole where $v_{\rm esc} = c$ is called the **event horizon**.

b) This region has a radius equal to the **Schwarzschild radius**. This radius scales with stellar mass with the relation

$$r_s = 2.96 \left(\frac{M}{M_\odot}\right) \text{ km.} \qquad \text{(VIII-41)}$$

c) Once again, this radius does not define the radius of the BH itself for it has zero volume.

4. Stellar cores will become black holes only if $M_c > 3M_\odot$. No known force in nature will prevent its collapse.

5. Of course as the core collapses, its mass density increases without bound and we enter a regime where Newton's Universal Law of Gravity begins to break down and general relativity must be used to describe the gravitational field.

6. Actually, as $R \rightarrow 0$, we reach a point where the general theory is no longer valid either $\Longrightarrow$ we must await the theory of quantum gravity to be fully developed. General relativity breaks down when

$$R \leq \ell_{\rm P} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-33} \text{ cm} \qquad \text{(VIII-42)}$$

and for time intervals less than

$$\tau_{\rm P} \leq \ell_{\rm P}/c = \sqrt{\frac{\hbar G}{c^5}} = 5.4 \times 10^{-44} \text{ sec}, \qquad \text{(VIII-43)}$$

where $\hbar = h/2\pi$. The quantities $\ell_{\rm P}$ and $\tau_{\rm P}$ are called the **Planck length** and the **Planck time**, respectively. We will have more to say about these quantities when we discuss the Big Bang.

7. Anything (*i.e.*, matter or light) that gets inside the event horizon will never be heard from again in our space-time continuum. But what happens to objects as they approach the event horizon? Let's assume we are observing a particle falling into a BH at a large distance r_∞ and that time passes at rate t_∞ at this position.

a) As the particle approaches the BH, it experiences normal Newtonian gravity when $r_\infty > r \gg r_s$.

b) As $r \rightarrow r_s$ (the strong field regime), general relativity has an enormous effect on the particle.

c) The interval between ticks for a clock in the frame of the particle, dt, is given by

$$dt = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} dt_\infty = \left(1 - \frac{r_s}{r}\right)^{1/2} dt_\infty \, . \qquad \text{(VIII-44)}$$

$\Longrightarrow$ clocks appear to run slow near a BH when observed from a large distance.

d) A standard ruler of length dr_∞ measured at point r_∞ would appear to be of length

$$dr = \frac{dr_\infty}{\left(1 - \frac{2GM}{rc^2}\right)^{1/2}} = \frac{dr_\infty}{\left(1 - \frac{r_s}{r}\right)^{1/2}} \qquad \text{(VIII-45)}$$

at a position r in the vicinity of a BH.

e) A result of these two equations is that the position of a particle released at position $r_\circ$ in a BH's gravitational field follows

$$r \simeq r_s + r_\circ \, e^{-ct_\infty/2r_s} \qquad \text{(VIII-46)}$$

as seen by a distant observer $\Longrightarrow$ the particle never reaches the event horizon as view by an external observer!

8. Another interesting thing occurs as a result of this slowing down of clocks — the frequencies of spectral lines get smaller (*i.e.*, the wavelengths get longer) $\Longrightarrow$ **gravitational redshift**.

$$\frac{\Delta\lambda}{\lambda_\circ} = \frac{\lambda_\infty - \lambda_\circ}{\lambda_\circ} = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} - 1 = \left(1 - \frac{r_s}{r}\right)^{-1/2} - 1 \, , \qquad \text{(VIII-47)}$$

where $\lambda_\circ$ is the characteristic (*i.e.*, rest) wavelength of the line emitted at r and λ_∞ is the apparent wavelength seen at r_∞ (note that $\Delta\lambda/\lambda \simeq 0.3$ for a neutron star).

9. Although we won't describe the details of the curvature of space-time here, we can present a general overview. Instead of our 3-dimensional universe, assume that we live in a 2-dimensional universe. Then according to the general theory, mass bends this 2-dimensional universe into a 3rd dimension. BH's bend space-time to such an extent that the BH actually *rips* a hole in the *fabric* of the Universe and reconnects with a distance part of the Universe or connects to a parallel universe in a different space-time continuum (see Figure VIII-2).

a) A connecting tunnel forms in space-time called an **Einstein-Rosen bridge** and also called a **wormhole**.

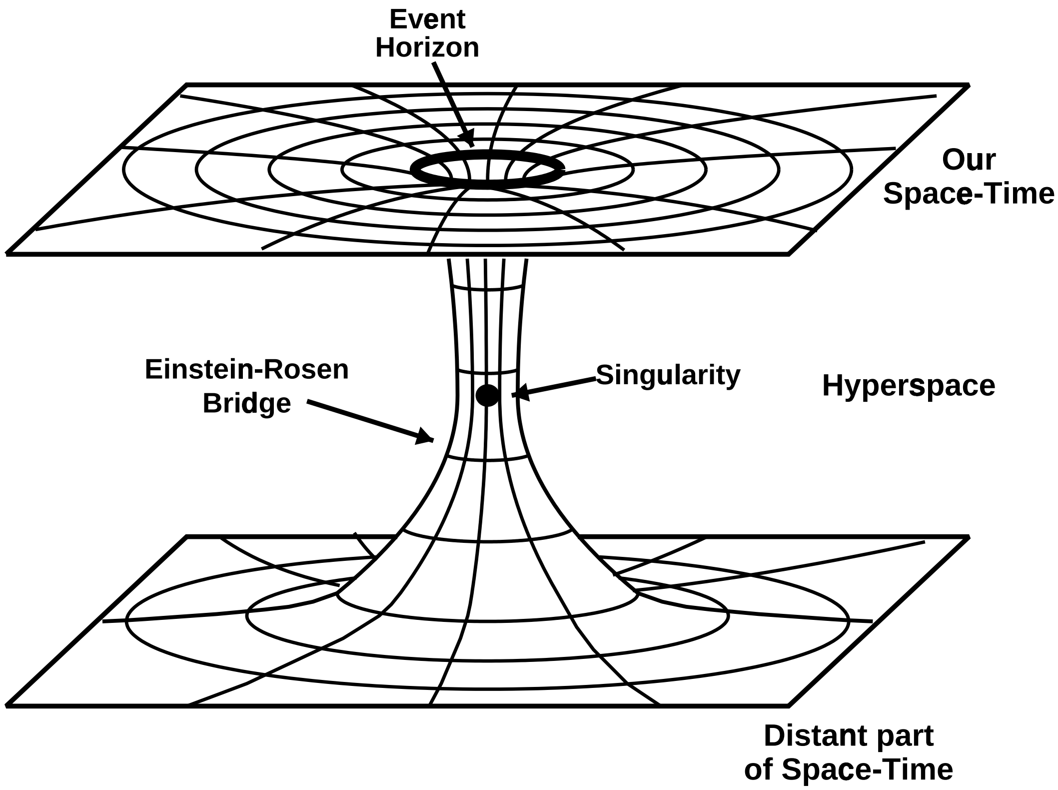

Figure VIII–2: The space-time continuum in the vicinity of a black hole.

b) Our 3-dimensional Universe bends into a 4th dimension, called **hyperspace**.

c) Tidal forces increase without bound as you approach the singularity $\Longrightarrow$ anything with size, even nuclear particles, get ripped apart from these tidal forces as matter falls into the wormhole.

10. The only way to detect a BH is to observe the effect one has on a companion star in a binary star system (see Figure VII-18). As material is stripped away from a normal star by the BH, it spirals down to the BH and heats up to very high temps (due to friction). Before crossing the event horizon, the gas heats to such a high temp that it emits X-rays. Black candidates must satisfy the following observational requirements:

a) An X-ray source in a binary star system whose X-rays vary in brightness over the time period of seconds (hence emitting region must be very small in size).

b) The unseen companion of the spectroscopic binary has a mass greater than 3 $M_\odot$.

c) The best candidates are the ones where the mass of the unseen companion is greater than the mass of the visible star.

11. The best stellar BH candidates are:

a) Cyg X-1 (optical component: 30 $M_\odot$, X-ray component: 7 $M_\odot$).

b) LMC X-3 (optical component: 5 $M_\odot$, X-ray component: 10 $M_\odot$).

c) V404 Cyg (optical component: 1 $M_\odot$, X-ray component: 12 $M_\odot$) — this is the best candidate.

d) SS 433 (BH with jets?).

12. Black holes can also result from non-stellar evolutionary processes. The Big Bang may have created an enormous number of **mini-black holes** and the center of large galaxies may contain **supermassive black holes** (1 million to 1 billion solar masses in size)!