ASTR-3415: Astrophysics Course Lecture Notes Section XI

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Abstract

These class notes are designed for use of the instructor and students of the course ASTR-3415: Astrophysics. This is the Version 1.2 edition of these notes.

XI. Gravitation, Cosmology, and the Big Bang

A. Olber's Paradox — Why does it get dark at night?

1. During the 19th and beginnings of the 20th centuries, the scientific viewpoint was that the Universe is unchanging (*static*), has no spatial boundaries (*infinite*), and had no beginning (*eternal*). If the Universe is static, infinite, and eternal, we should light in every direction we look.

   a) This observation is referred to as **Olber's Paradox**.

   b) Actually, Halley (of the comet fame), was the first to pose this question, Olber was the first to formally publish it in 1826.

   c) The inverse-square law of light cannot be used as a solution to Olber's Paradox for the following reasons:

      i) The luminosity at the surface of a star is given by $L = 4\pi R^2 F$, where R is the radius of the star (as given on page I-3 of the notes).

      ii) The flux of a star falls off as r^{-2} (see Eq. I-11), where r is the distance to the star: $f = (R/r)^2 F$.

      iii) For a uniform distribution of stars, each successive shell radially away from the Earth will contain $4\pi r^2 N$ stars, where N is the areal density [cm^2] of stars on a shell at surface area $4\pi r^2$.

      iv) The total flux that should arrive at Earth from all of the stars in the Universe is a convolution of these two quantity giving $f_{\text{obs-shell}} = (4\pi r^2 N)(f) =$

$(4\pi r^2 N)(R/r)^2 F = LN \Longrightarrow$ the sky should be ablaze with light for an infinitely large eternal Universe!

v) Even for a non-uniform distribution of stars, the inverse-square law will fail to account for the dark night sky for an infinite, eternal Universe.

d) Though not known when Olber's Paradox was published, interstellar absorption also cannot be used to explain the dark night sky.

i) Absorption from gas and dust in the ISM and IGM would heat the gas and dust.

ii) Over an infinite amount of time, this dust and gas would reach an equilibrium temperature that is equal to the effective temperature of the integrated light from the stars.

iii) Hence, the ISM and IGM would shine as bright as the surface of the stars contained in the Universe.

2. The solution to Olber's Paradox is that the Universe is expanding (hence not static) and is not eternal $\Longrightarrow$ Big Bang Theory — it had a beginning!

a) Light gets redshifted out of the visible band for stars and galaxies at large distances.

b) As we look out, we look back in time. We cannot look infinitely far out since, sooner or later, we will see the Big Bang.

3. How long ago did this happen? We will treat this question in a very simplified way initially: Calculate when all the galaxies were at the same position from Hubble's Law.

a) Since there is still some uncertainty to the value of Hubble's constant, we will use the following scale factor formalization:

$$\begin{aligned} H_\circ &= 100\, h \text{ km/s/Mpc } = 3.24 \times 10^{-18}\, h \text{ s}^{-1} \\ &= 1.02 \times 10^{-10}\, h \text{ yr}^{-1} , \end{aligned} \quad \text{(XI-1)}$$

where h (the Hubble constant scale factor) = 1.0 if $H_\circ$ = 100 km/s/Mpc and $h = 0.5$ if $H_\circ$ = 50 km/s/Mpc.

b) From Newtonian mechanics, the distance to a galaxy is $d = v_r\, t$, where v_r is the galaxy's radial velocity (assumed constant here), and t is the time since the galaxy started at the origin. But from Hubble's Law, $v_r = H_\circ\, d$, so

$$t_{\rm H} = \frac{d}{v_r} = \frac{d}{H_\circ\, d} = \frac{1}{H_\circ}$$

or

$$\boxed{t_{\rm H} = 9.78 \times 10^9\, h^{-1} \text{ yr} .} \quad \text{(XI-2)}$$

c) The Universe actually younger than that due to gravity slowing down the expansion over time. Hence the time given in Eq. (XI-2) corresponds to the maximum age of the Universe which is referred to as the **Hubble Time**.

i) If $h = 1.0$ ($H_\circ$ = 100 km/s/Mpc), then $t_{\rm H} = 9.78$ billion years.

ii) If $h = 0.5$ ($H_\circ$ = 50 km/s/Mpc), then $t_{\rm H} = 19.6$ billion years.

iii) The best value for Hubble's constant (as determined by HST and WMAP, see Eqs. X-14 and

X-15) is $h = 0.71$ ($H_\circ = 71$ km/s/Mpc), and $t_{\rm H} =$ 13.8 billion years.

d) If a galaxy is farther than $d_{\rm obs} = c\,t_{\rm H}$ (= 9.78, 13.8, or 19.6 billion light years, depending upon the value of $H_\circ$) away, we will never see it since light would not have had enough time to reach us $\Longrightarrow$ $d_{\rm obs}$ is the size of the **observable universe**.

B. Gravitation.

1. For Hubble's Law to be true, galaxies in the Universe have to be distributed *homogeneously* and *isotropically* (see below for definitions of these terms) on a large scale.

a) Hubble's Law is referred to as the observed *kinematic world model* of the Universe.

b) One question we will come back to later is: *How large is large?*

2. E.A. Milne and W.H McCrea (1934) extended this, at first purely kinematic model, so as to make it a **Newtonian cosmology**.

a) They investigated the motions (*i.e.*, trajectory) of a medium of *gas* particles, where the gas particles represent *galaxies* in the Universe (*i.e.*, the galaxies are treated as point particles).

b) These trajectories can be determine in accordance with Newtonian mechanics if one demands throughout that the distribution of gas particles are:

i) **Homogeneous**: The volume number density [$\rm cm^{-3}$] of galaxies is constant throughout the Universe.

ii) **Isotropic**: The areal number density [cm^{-2}] of galaxies on the sky from any point in the Universe is constant (note that isotropy results automatically for a homogeneous distribution).

iii) **Irrotational**: The Universe as a whole is not rotating about some axis.

c) Consider at time t a galaxy at distance $R(t)$, then according to Newton's law of gravitation, this galaxy is attracted by the mass within a sphere of radius R by

$$M = \frac{4\pi}{3} R^3 \rho(t), \qquad \text{(XI-3)}$$

where $\rho(t)$ is the mass-density at the instant of time considered.

d) Thus, the equation of motion of this galaxy (of mass m) is determined by setting the force of motion equal to the gravitational force:

$$\begin{aligned} F &= F_g \\ m\,\frac{d^2R}{dt^2} &= -\frac{G\,M\,m}{R^2} \end{aligned}$$

or

$$\frac{d^2R}{dt^2} + \frac{GM}{R^2} = 0, \qquad \text{(XI-4)}$$

where the mass M is constant.

e) Multiplying each term in Eq, (XI-4) by $\dot{R} = dR/dt$, it is then possible to easily integrate Eq. (XI-4) and obtain the *energy equation*:

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{GM}{R} = \kappa\ , \qquad \text{(XI-5)}$$

where κ is the integration constant, or

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi}{3} G\rho(t) + \frac{kc^2}{R^2} = 0 \,, \qquad \text{(XI-6)}$$

in which we have written $\kappa = -kc^2/2$ in anticipation of later comparison with relativistic calculations.

f) We can define the current **Hubble constant** as

$$H_\circ \equiv \dot{R}_\circ / R_\circ, \qquad \text{(XI-7)}$$

where we denote present time $t = t_\circ$ by a subscript $\circ$.

g) For a complete characterization of a model universe, we need, besides $H_\circ$, a second variable that describes the deceleration of the Universe due to its mass M. This is the so-called **deceleration parameter**:

$$q_\circ \equiv -\left(\frac{\ddot{R}_\circ}{R_\circ}\right) \Big/ \left(\frac{\dot{R}_\circ}{R_\circ}\right)^2 = -\frac{\ddot{R}_\circ}{R_\circ\, H_\circ^2} = \frac{4\pi G\rho_\circ}{3H_\circ^2} \qquad \text{(XI-8)}$$

using Eqs. (XI-3), (XI-4), and (XI-7).

h) This formula relates the acceleration $\ddot{R}_\circ$ to a uniform acceleration which would lead to the observed velocity $R_\circ H_\circ$ at distance $R_\circ$ in the Hubble time $t_{\rm H} = t_\circ = H_\circ^{-1}$, starting from zero velocity.

i) The solution of the above equations leads to world models which, from a starting point (singularity) of infinitely great density, either expands monotonically (total energy $M\kappa \geq 0$) or oscillates periodically between $R = 0$ and an $R_{\rm max}$ (if $\kappa < 0$). Static models are not possible within the framework of Eq. (XI-4).

j) Newton actually realized this which is why he wanted the Universe to be infinite in size.

3. Relativistic cosmologies are based on the **general theory of relativity** instead of Newtonian mechanics. Here we use a *spacetime* description of the Universe in cosmology $\Longrightarrow$ we lay a coordinate grid across the spacetime manifold.

a) An **event** is defined as a *point* in spacetime, specified by 4 coordinates x^i:

$$\begin{array}{ll} x^0 & : \; t - \text{ time coordinate} \\ \left\{ \begin{array}{l} x^1 \\ x^2 \\ x^3 \end{array} \right\} & : \; \vec{r} - \text{ spatial position coordinate} \end{array}$$

b) A **world line** is the locus of successive events in a particle's history.

i) Spatial distance in spacetime: ds.

ii) Separation in spacetime: $d\tau$.

4. In spacetime, the separation or *line element* is defined by the tensor equation:

$$d\tau^2 \equiv \sum_{i,j=0}^{3} g_{ij}\, dx^i\, dx^j \; , \qquad \text{(XI-9)}$$

meanwhile, the spatial distance is given by

$$ds^2 \equiv \sum_{\mu,\nu=1}^{3} g_{\mu\nu}\, dx^\mu\, dx^\nu \; , \qquad \text{(XI-10)}$$

where i, j, μ, and ν are summation labels and not exponents. The various **metrics** or configuration coefficients are given below.

a) **Minkowski metric**/spacetime (*i.e.*, special relativity):

$$g_{ij} = \eta_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{c^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{c^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{c^2} \end{pmatrix} \qquad \text{(XI-11)}$$

in a local inertial coordinate system

$$d\tau^2 = g_{ij}\, dx^i\, dx^j \quad \text{(XI-12)}$$
$$= dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \quad \text{(XI-13)}$$
$$= dt^2 - \frac{dr^2}{c^2} \,. \quad \text{(XI-14)}$$

i) $d\tau$ is invariant $\Longrightarrow$ another inertial observer O measuring $d\tau$ gets the same answer.

ii) Note that since $dr = v\, dt$, we can write

$$d\tau^2 = dt^2 - \frac{v^2\, dt^2}{c^2} \,, \quad \text{(XI-15)}$$

or

$$dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} \,, \quad \text{(XI-16)}$$

where $d\tau$ is the *proper time interval.* One recognizes this equation immediately as the time dilation equation (Eq. VIII-25) from special relativity.

iii) **Principle of Equivalence**: At every space-time point in an arbitrary gravitational field, it is possible to choose a locally inertial coordinate system, such that the laws of nature take the same form as in a non-accelerating coordinate system in the absence of gravity.

Frame f:

$$m\vec{a} = m\vec{g} + \vec{F}_{ng} \,, \quad \text{(XI-17)}$$

and in f', a frame with acceleration $\vec{g}$:

$$m(\vec{a} - \vec{g}) = \vec{F}_{ng} \quad \Longrightarrow \quad m\vec{a'} = \vec{F}_{ng} \,, \quad \text{(XI-18)}$$

where F_{ng} is a *non-gravitational* force. Hence, *locally* we can say that $g_{ij} = \eta_{ij}$.

b) A line element in spacetime with spherical symmetry has a metric tensor of

$$g_{ij} = \begin{pmatrix} e(r,t) & 0 & 0 & 0 \\ 0 & -\frac{f(r,t)}{c^2} & 0 & 0 \\ 0 & 0 & -\frac{r^2}{c^2} & 0 \\ 0 & 0 & 0 & -\frac{r^2 \sin^2\theta}{c^2} \end{pmatrix} \qquad \text{(XI-19)}$$

or

$$d\tau^2 = e(r,t)\, dt^2 - \frac{1}{c^2}\left[f(r,t)\, dr^2 + r^2\, d\theta^2 + r^2\, \sin^2\theta\, d\phi^2\right] , \qquad \text{(XI-20)}$$

where $e(r,t)$ and $f(r,t)$ are functions to be determined from the boundary conditions of the problem in question.

i) For a *static* line element in spherically symmetric spacetime (**Schwarzschild metric**),

$$\begin{aligned} e(r) &= 1 - \frac{2GM}{c^2 r} \\ &= 1 - \frac{r_s}{r} , \end{aligned} \qquad \text{(XI-21)}$$

where $r_s = 2GM/c^2$ is the **Schwarzschild radius** (see Eq. VIII-41). Also

$$f(r) = \frac{1}{1 - r_s/r} \qquad \text{(XI-22)}$$

in this metric.

ii) The line element in spacetime then becomes

$$\begin{aligned} d\tau^2 &= \left(1 - \frac{r_s}{r}\right) dt^2 - \\ &\quad \frac{1}{c^2}\left(\frac{dr^2}{1 - r_s/r} + r^2\, d\theta^2 + r^2\, \sin^2\theta\, d\phi^2\right) , \end{aligned} \qquad \text{(XI-23)}$$

which is our spacetime equation that we developed for a black hole (see Eqs. VIII-44 ans VIII-45).

c) The **Robertson-Walker metric** concerns a spherical-symmetric, homogeneous spacetime. Its metric tensor is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{R^2(t)}{c^2}\,\frac{1}{1-kr^2} & 0 & 0 \\ 0 & 0 & -\frac{R^2(t)}{c^2}\,r^2 & 0 \\ 0 & 0 & 0 & -\frac{R^2(t)}{c^2}\,r^2\,\sin^2\theta \end{pmatrix} \quad \text{(XI-24)}$$

or

$$d\tau^2 = dt^2 - \frac{R^2(t)}{c^2}\left(\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\,\sin^2\theta\,d\phi^2\right) , \quad \text{(XI-25)}$$

where r is the coordinate distance, $R(t)r$ is the metric distance, and $R(t)$ is known as the *scale factor* of the metric.

d) This Robertson-Walker metric spans a 3-space of constant curvature K:

$$K = K(t) = \frac{k}{R^2(t)} \quad \text{(XI-26)}$$

$$k = \begin{cases} +1 & : \text{ positive curvature} \\ 0 & : \text{ flat positive space (zero curvature)} \\ -1 & : \text{ negative curvature.} \end{cases}$$

5. On a curved surface, the shortest and longest paths (*i.e.*, the *extrema*) between two points in spacetime are called **Geodesics**.

a) On a spherical (*i.e.*, positive curvature) surface one can have both a minimum and a maximum geodesic between two points A and B.

b) On a flat (*i.e.*, zero curvature) or hyperbolic (*i.e.*, negative curvature), one can only have a minimum geodesic.

6. The **interval** between two events in spacetime, $\tau = \tau_{\rm AB}$, is defined by

$$\tau = \tau_{\rm AB} = \int_A^B d\tau \ , \tag{XI-27}$$

where $d\tau = \sqrt{d\tau^2}$ (from Eqs. XI-9, 12, 23, and 25) is the spacetime line element (*i.e.*, separation). Making use of Eq. (XI-9) we can calculate the spacetime interval from the definition of a geodesic:

$$\tau = \text{extremum} \left[\int_A^B \sqrt{d\tau^2} \right] \tag{XI-28}$$

$$= \text{extremum} \left[\int_A^B \sum_{i,j=0}^{3} \left(g_{ij}\, dx^i\, dx^j \right)^{1/2} \right] \tag{XI-29}$$

$$= \text{extremum} \left[\int_A^B \left[g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \right]^{1/2} d\tau \right] \ , \tag{XI-30}$$

where we have not included the summation sign in the last integral, but its inclusion is implied. The technique for finding the extremum is very complicated and beyond the scope of this class (it involves taking the derivative of the Lagrangian equation of the metric and setting this equal to zero). From any of the definitions written in Eqs. (XI-28, 29, or 30), we can define one of three outcomes for a given geodesic:

a) τ is *real* — events have **time-like** separations or time-like geodesics. Events can lie on a world line of a material particle, since material particles have time-like geodesics. In that case, $d\tau$ is the proper time between events.

b) τ is *imaginary* — **space-like** separations. The events cannot lie on a world line of a material particle. $c\sqrt{-d\tau^2}$ is the proper separation if we were at a particular event. This would be the regime one would be in if one could travel faster than light.

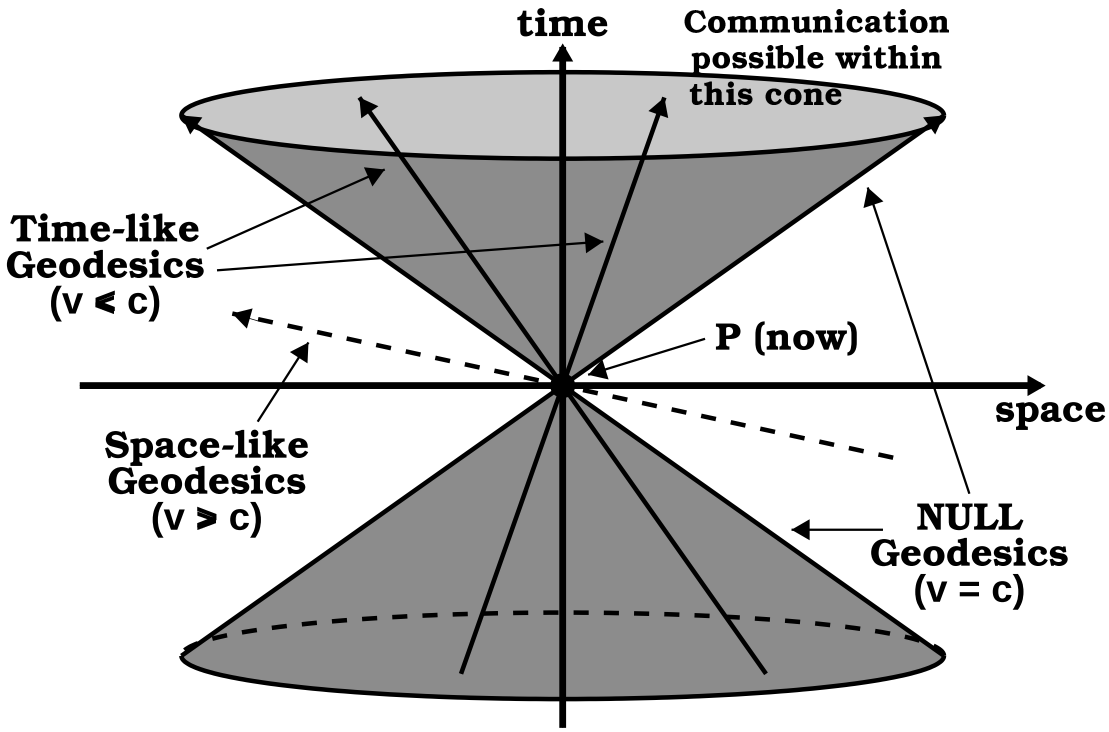

Figure XI–1: Light-cone diagram of event P occurring in spacetime.

c) $\tau = 0$ — **light-like** separation or **null geodesics** $\Longrightarrow$ this is the way light propagates. World lines of photons are null geodesics.

7. These various *separations* can be seen in Figure XI-1. Here we define the **light-cone**, which are double (hyper-) cones joining an event P to past and future events. Time-like geodesics all lie in the cone. Events and geodesics outside the cones (*i.e.*, space-like) can never communicate to events inside the cones (*i.e.*, our Universe).

8. In the general theory of relativity, Einstein used the concept of geodesics to derive the **Field Equations** of the Universe:

$$\mathcal{R}_{ij} - \frac{1}{2}\, g_{ij}\, \mathcal{R} = -\frac{8\pi G}{c^2}\, T_{ij} \ . \qquad \text{(XI-31)}$$

a) $\mathcal{R}_{ij}$ are the components of the **Riemann curvature tensor**, which are related to the scale factor of the Universe. Note that $\mathcal{R} = g^{ij}\,\mathcal{R}_{ij}$.

b) T_{ij} are the components of the **energy-momentum tensor**.

c) Essentially, the field equations are nothing more than the conservation of energy and momentum in a 4-D space-time.

9. The tensor field equations as shown in Eq. (XI-31) produce non-static metrics as was shown by de Sitter and others. Einstein didn't like this and added an additional constant term to keep the Universe static:

$$\mathcal{R}_{ij} - \frac{1}{2}\,g_{ij}\,\mathcal{R} + c^2\,\Lambda\,g_{ij} = -\frac{8\pi\,G}{c^2}\,T_{ij}\ . \qquad \text{(XI-32)}$$

a) Λ has been coined the **cosmological constant**.

b) This constant term, $c^2\,\Lambda\,g_{ij}$, effectively represents a *negative* gravity, in order to keep the Universe static. We will have more to say about this later in this section.

c) After Einstein heard of Hubble's results that the Universe is expanding, he claimed that adding this constant to his equations was the *biggest blunder of his life*!

d) But is Λ really zero? We will keep this constant in mind while developing the formalism of these equations.

e) At this point, we will go no further with the tensor form of the field equations since it's too advanced for this course. However, we can still develop a differential equation for the change in scale factor of the Universe from an algebraic definition of curvature. We will introduce this in the Structure of the Universe subsection.

C. The Big Bang Theory.

1. The Universe started in a hot, high density state, which began to expand and cool a Hubble Time ago $\Longrightarrow$ the **Big Bang Theory**.

a) The Big Bang occurred everywhere in space, not just at one location $\Longrightarrow$ we are in the Big Bang!

b) Galaxies were <u>not</u> thrown apart $\Longrightarrow$ the fabric of space itself is expanding and the galaxies move apart as a result as they ride along on the fabric of spacetime.

2. We shall be developing equations that describes the evolution of the Universe with time. In these equations, we will start with two postulates called the **Cosmological Principle**.

a) **Homogeneity** — matter is uniformly distributed in space on a very large scale ($d > 100$ Mpc).

b) **Isotropy** — the Universe looks the same in every direction.

3. In addition, two additional assumptions are typically included when describing the Big Bang:

a) **Universality** — physical laws and constants are the same everywhere in the Universe.

b) **Cosmological Redshifts** — redshifts are caused by the expansion of the Universe through the Doppler Effect.

4. We see the Big Bang fireball in every direction as microwave blackbody radiation:

3 K Cosmic Microwave Background (CMB) radiation

a) When this light was emitted some 300,000 years after the Big Bang when the Universe was around 3000 K.

b) As the Universe expanded, this light was redshifted until today it is microwave light ($z = 1100$!, the farthest quasar is at $z = 4.9$).

c) Penzias and Wilson discovered this background radiation in the early 1960's, confirming the theoretical predictions of the Big Bang Theory made by Dicke and Peebles. Penzias and Wilson later won a Nobel Prize for their discovery.

d) The COBE spacecraft was launch in the early 1990's to investigate this background radiation.

i) Found that the Universe radiates as a perfect blackbody (after the Earth's, Sun's, Milky Way's, amd Local Group's motions are subtracted) at a temp of 2.726 K.

ii) Small variations in the thermal distribution of this radiation on the sky on the order of 1 part in 100,000 (*i.e.*, the *intrinsic anisotropy*) show that by the time this radiation was emitted, inhomogeneities in the mass-energy of the Universe had begun which would later form the galaxies.

D. The Structure of the Universe.

1. We will now develop the equations that describe how the Universe has evolved in time. We will assume the *cosmological principle* as postulates to the global structure of the Universe which allows us to use the Robertson-Walker metric (see Eq. XI-24 and XI-25) to describe the geometry of spacetime.

2. Since the Universe is assumed homogeneous, isotropic, and irrotational, the θ and ϕ coordinates are constant and we can ignore them in our description of spacetime. The separation in events then becomes

$$d\tau^2 = dt^2 - R^2(t)\,\frac{dr^2}{c^2(1 - k\,r^2)} \ . \qquad \text{(XI-33)}$$

3. As described in Eq. (XI-26), the constant k is a measure of the global curvature K of spacetime. The curvature K can be expressed as a function of t and r with the **Gaussian curvature formula** for an orthogonal coordinate system:

$$K = \frac{1}{2g_{11}g_{22}}\left\{-\frac{\partial^2 g_{11}}{\partial(x^2)^2} - \frac{\partial^2 g_{22}}{\partial(x^1)^2} + \frac{1}{2g_{11}}\left[\frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1} + \left(\frac{\partial g_{11}}{\partial x^2}\right)^2\right] + \frac{1}{2g_{22}}\left[\frac{\partial g_{11}}{\partial x^2}\frac{\partial g_{22}}{\partial x^2} + \left(\frac{\partial g_{22}}{\partial x^1}\right)^2\right]\right\} \qquad \text{(XI-34)}$$

a) Note that for plane-polar coordinates, $K = 0 \Longrightarrow$ space is flat.

b) For a spherical surface, $K = 1/R^2$.

c) For *space-only* curvature, x^1 and x^2 are spatial coordinates, where $x^1 \perp x^2$ in Eq. (XI-33).

d) However, for spacetime curvature, $x^1 = t$ and $x^2 = r$ in this equation. Hence, $K = K(t)$ is a function of time.

4. Using the Robertson-Walker metric in this curvature formula gives

$$K(t) = -\frac{\ddot{R}(t)}{R(t)} \ . \qquad \text{(XI-35)}$$

a) Matter in the Universe is described by its mass density $\rho(t)$, and is the cause of the curvature of spacetime (though Λ can give curvature without matter). With this in mind, we can use a *dimensional analysis* type of argument to describe $K(t)$ in terms of a linear proportionality to this mass density $\rho(t)$. We guess at an equation:

$$K(t) = \alpha\, \rho(t)\, G^{\ell}\, c^{m} + \text{ constant}, \qquad \text{(XI-36)}$$

where the *constant* is inserted to allow for the possibility that empty spacetime ($\rho = 0$) might be curved.

b) It follows from Eq. (XI-35) that $K(t)$ must have dimensions of $(\text{time})^{-2}$. If α is dimensionless, this implies that the powers of ℓ and m to which G and c are raised are 1 and 0 respectively.

c) When taking the Newtonian limit (see below), $\alpha = 4\pi/3$.

d) A curved empty universe requires that the cosmological constant introduced in Eq. (XI-32) be nonzero. As such, we choose the constant in Eq. (XI-36) to be equal to $-\Lambda/3$ for consistency with the field equations. Therefore, the cosmological constant Λ has the same dimensions as $K(t)$, or $(\text{time})^{-2}$.

e) The curvature $K(t)$ now becomes

$$K(t) = \frac{4\pi \rho(t)\, G}{3} - \frac{\Lambda}{3}\ . \qquad \text{(XI-37)}$$

f) Equating this with Eq. (XI-35) gives the equation of motion for $R(t)$ as

$$\ddot{R}(t) = -\frac{4\pi \rho(t)\, G}{3}\, R(t) + \frac{\Lambda}{3}\, R(t)\ . \qquad \text{(XI-38)}$$

g) Precisely this same formula follows from Einstein's general field equations.

h) If $\Lambda > 0$, Λ acts like a negative density ρ. Since self-gravitation of matter acts to slow down the expansion of the Universe, a positive Λ must act to accelerate it. For this reason, $\Lambda R/3$ is sometimes called the **cosmic repulsion term**.

i) If $\Lambda = 0$, Eq. (XI-38) follows exactly from Newtonian mechanics.

i) To see this, consider the galaxies lying inside the comoving sphere with radial coordinate r.

ii) A galaxy on the surface of the sphere will accelerate inwards under the attraction of the mass within the sphere.

iii) If the galaxy has a mass m, Newton's second law gives

$$\begin{aligned} m\,a &= m\,r\,\ddot{R}(t) = F_{\rm gal} \\ &= -\frac{G\,m\,M}{[r\,R(t)]^2} \\ &= -\frac{Gm(4\pi/3)\rho(t)\,[rR(t)]^3}{[rR(t)]^2}\,, \end{aligned}$$

where $F_{\rm gal}$ is the gravitational force on the galaxy and M is the mass that lies inside sphere.

iv) Thus,

$$\ddot{R}(t) = -\frac{4\pi}{3}\,G\,\rho(t)\,R(t) = -\frac{4\pi}{3}\,G\,\rho\,R\,, \quad \text{(XI-39)}$$

hence our choice of α in Eq. (XI-36). Note that whenever cosmological parameters are listed without a subscript ID, the parameter is to be taken as a variable function of time.

j) Even though Newtonian mechanics seems to work quite well in describing the evolution of the Universe, general relativity still must be used since expansion velocities quickly exceed the speed of light if one looks out far enough under the assumptions of classical mechanics.

5. Since we don't know $\rho(t)$ throughout the entire history of the Universe, we need to eliminate this term from Eq. (XI-38). We will use the conservation of mass-energy here: Within a co-moving sphere the mass remains constant, where the volume is proportional to $R^3(t)$ [*i.e.*, $\rho(t)\,R^3(t) = \rho_\circ\,R_\circ^3 =$ constant], thus

$$\rho(t) = \frac{\rho(t_\circ)\,R^3(t_\circ)}{R^3(t)} = \frac{\rho_\circ\,R_\circ^3}{R^3(t)}\,, \qquad \text{(XI-40)}$$

where $\rho_\circ$ is the current mass density of the Universe (*i.e.*, a measurable quantity) and $R_\circ$ is the current scale factor of the Universe.

6. Eq. (XI-38) now becomes

$$\ddot{R}(t) = -\frac{4\pi\rho_\circ\,G\,R_\circ^3}{3R^2(t)} + \frac{\Lambda R(t)}{3}\,. \qquad \text{(XI-41)}$$

7. Multiplying this equation by $\dot{R}(t) = \dot{R}$ and integrating, we obtain

$$\begin{aligned}
\dot{R}\,\ddot{R} &= -\frac{4\pi\rho_\circ\,G\,R_\circ^3}{3R^2}\,\dot{R} + \frac{\Lambda R}{3}\,\dot{R} \\
\dot{R}\,\frac{d\dot{R}}{dt} &= -\frac{4\pi\rho_\circ\,G\,R_\circ^3}{3}\,R^{-2}\,\frac{dR}{dt} + \frac{\Lambda R}{3}\,\frac{dR}{dt} \\
\dot{R}\,d\dot{R} &= -\frac{4\pi\rho_\circ\,G\,R_\circ^3}{3}\,R^{-2}\,dR + \frac{\Lambda}{3}\,R\,dR \\
\int \dot{R}\,d\dot{R} &= -\frac{4\pi\rho_\circ\,G\,R_\circ^3}{3}\int R^{-2}\,dR + \frac{\Lambda}{3}\int R\,dR \\
\frac{1}{2}\dot{R}^2 &= -\frac{4\pi\rho_\circ\,G\,R_\circ^3}{3}\left(\frac{R^{-1}}{-1}\right) + \frac{\Lambda}{3}\,\frac{R^2}{2} + \text{ const.}
\end{aligned}$$

$$\dot{R}^2(t) = \frac{8\pi\rho_\circ G R_\circ^3}{3R(t)} + \frac{\Lambda}{3} R^2(t) + \text{ const.}$$
$$= \frac{8\pi}{3} G \rho(t) R^2(t) + \frac{\Lambda}{3} R^2(t) + \text{ const.}, \quad \text{(XI-42)}$$

where we have made use of mass-energy conservation $[i.e., \rho_\circ R_\circ^3 = \rho(t) R^3(t)]$ in the last step of this derivation. The *constant* must be proportional to c^2, since it is the only combination of the quantities c, G, and Λ which have the required dimensions of (velocity)^2. The field equations of general relativity give its exact value as $-kc^2$, where k is the curvature index of the Robertson-Walker metric given in Eq. (XI-26). Thus we have

$$\dot{R}^2(t) = -kc^2 + [8\pi G \rho(t) + \Lambda] \frac{R^2(t)}{3} . \quad \text{(XI-43)}$$

a) By setting $\Lambda = 0$, we can rewrite Eq. (XI-43) as

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8\pi}{3} G\rho\right] R^2 = -kc^2 , \quad \text{(XI-44)}$$

which is known as the **Friedmann equation**.

b) With the cosmological constant, Friedmann's equation becomes

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8\pi}{3} G\rho - \frac{1}{3}\Lambda\right] R^2 = -kc^2 , \quad \text{(XI-45)}$$

which I shall refer to as the *modified* Friedmann equation. Note that this is slightly different than the textbook's Eq. (27.66) since the textbook defines Λ in units of (length)$^{-2}$ whereas I have defined it in units of (time)$^{-2}$ ($i.e.$, $\Lambda_{\text{notes}} = \Lambda_{\text{textbook}} \, c^2$).

c) We will developing solutions to both forms of these Friedmann equations in the subsection titled Big Bang Models. The solutions will be described by various parameters as described below.

i) **Hubble's constant** is the proportionality constant between the rate of change of the scale factor (*i.e.*, the expansion velocity) to the scale factor $R(t)$ at the current time $t_\circ$ (see Eq. XI-7):

$$H_\circ \equiv \frac{\dot{R}(t_\circ)}{R(t_\circ)} = \frac{\dot{R}_\circ}{R_\circ} \ . \qquad \text{(XI-46)}$$

ii) The **deceleration parameter** was defined by Eq. (XI-8):

$$q_\circ \equiv -\frac{\ddot{R}(t_\circ)\, R(t_\circ)}{\dot{R}^2(t_\circ)} = -\frac{\ddot{R}(t_\circ)}{R(t_\circ)\, H_\circ^2} \ . \qquad \text{(XI-47)}$$

Note that for a universe with $\Lambda = 0$, we can use Eq. (XI-39) in Eq. (XI-47) to give the deceleration parameter as

$$q_\circ = \frac{4\pi G}{3} \frac{\rho_\circ}{H_\circ^2} \ . \qquad \text{(XI-48)}$$

iii) The cosmological constant can be determined from $\rho_\circ$, $H_\circ$, and $q_\circ$ by using Eq. (XI-47) in Eq. (XI-38) which gives

$$\Lambda = 4\pi\rho_\circ G - 3q_\circ H_\circ^2 \ . \qquad \text{(XI-49)}$$

iv) The curvature index also can be expressed in terms of $\rho_\circ$, $H_\circ$, and $q_\circ$ by using Eq. (XI-46) in Eq. (XI-43) which gives

$$k = \frac{R_\circ^2}{c^2} \left[4\pi G\rho_\circ - H_\circ^2\,(q_\circ + 1)\right] \ . \qquad \text{(XI-50)}$$

v) The quantities $\rho_\circ$ (current mass density), $q_\circ$ (current deceleration parameter), and $H_\circ$ are measurable, so that both Λ and k can be determined from these observable quantities.

vi) If the cosmological constant is zero, the **density parameter** is defined as

$$\Omega_\circ \equiv \frac{\rho_\circ}{\rho_c} \,, \qquad \text{(XI-51)}$$

where ρ_c is called the **critical density** (see §XI.E.b on the next page) of the current Universe (note that the textbook labels this as $\rho_{c,\circ}$) $\Longrightarrow$ a universe with $\Omega_\circ = 1$ has $k = 0 \Longrightarrow$ Euclidean (flat) space.

vii) For $\Lambda \neq 0$, the current density parameter has two components

$$\Omega_\circ = \Omega_m + \Omega_\Lambda \,, \qquad \text{(XI-52)}$$

where Ω_m is the density parameter due to matter (sometimes this is listed as Ω_b for baryon density) given by Eq. (XI-51) and Ω_Λ (sometimes given as $\Omega_{\rm vac}$, the vacuum energy density) is given by

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H_\circ^2} \,, \qquad \text{(XI-53)}$$

when Λ is expressed in units of (time)$^{-2}$ [when Λ is expressed in units of (length)$^{-2}$, $\Omega_\Lambda \equiv \frac{\Lambda c^2}{3H_\circ^2}$].

E. Observational Constraints in Modeling the Universe.

1. Current Mass Density:

a) What is the current mass density of the Universe? Are there enough galaxies (*i.e.*, mass) to stop the expansion? The mass density of the Universe indicates its geometry. Here, we will assume (for the time being) that $\Lambda = 0$.

i) If the Universe's mass density, $\rho_\circ$, is less than a critical density, $\rho < \rho_c$ $(0 < \Omega_\circ < 1, 0 < q_\circ < 1/2)$, gravity will not halt the expansion $\Longrightarrow$ the Universe will continue to expand forever, a so-called **Open Universe**.

ii) If $\rho_\circ > \rho_c$ ($\Omega_\circ > 1, q_\circ > 1/2$), gravity will halt the expansion and cause a contraction down to a *Big Crunch*! $\Longrightarrow$ a **Closed Universe**.

iii) The Universe may be able to rebound (Big Bang) and start over again in such a universe $\Longrightarrow$ an **Oscillating Universe**.

iv) If $\rho_\circ = \rho_c$ ($\Omega_\circ = 1, q_\circ = 1/2$), gravity will halt the expansion after an infinite amount of time $\Longrightarrow$ a **Flat Universe**.

v) If there were no matter in the Universe, then $\rho = \rho_\circ = 0$, $\Omega_\circ = 0$, and $q_\circ = 0$ — also an open universe (see Table XI-1).

b) The current critical density is given by the expression

$$\rho_c = \frac{3H_\circ^2}{8\pi G} = 1.88 \times 10^{-29}\, h^2 \text{ gm/cm}^3 \;, \qquad \text{(XI-54)}$$

which ranges from 4.70×10^{-30} gm/cm^3 for $H_\circ = 50$ km/s/Mpc to 1.88×10^{-29} gm/cm^3 for $H_\circ = 100$ km/s/Mpc. For the WMAP value for $H_\circ$, the current value of the critical density is 9.48×10^{-30} gm/cm^3.

c) We can now ask, what is $\rho_\circ$ (or $\Omega_\circ$) of the Universe? (Refer to Table 27.1 on page 1278 for additional information of this topic.)

i) Galaxy counting = amount of luminous matter: $\Omega_{\text{lum-gal}} \approx 0.01 \Longrightarrow 100$ times too small in order to

Table XI–1: **Structure of the Universe**

Type	**Geometry**	**Curvature**	**Density Parameter**	**Deceleration Parameter**	**Age**
Closed	Spherical	Positive	$\Omega_\circ > 1$	$q_\circ > \frac{1}{2}$	$t_\circ < \frac{2}{3}(1/H_\circ)$
Flat	Flat	Zero	$\Omega_\circ = 1$	$q_\circ = \frac{1}{2}$	$t_\circ = \frac{2}{3}(1/H_\circ)$
Open	Hyperbolic	Negative	$0 < \Omega_\circ < 1$	$0 < q_\circ < \frac{1}{2}$	$\frac{2}{3}(1/H_\circ) < t_\circ < 1/H_\circ$
No Matter	Hyperbolic	Negative	$\Omega_\circ = 0$	$q_\circ = 0$	$t_\circ = 1/H_\circ$

close Universe.

ii) Galaxy dynamics = amount of matter in galaxy clusters out to 30 Mpc (includes both bright and dark matter): $\Omega_{\rm gal} = 0.25 \pm 0.10 = 25\,\Omega_{\rm lum\text{-}gal} \Longrightarrow$ 4 times too small for closure.

iii) Deuterium (^{2}H) abundance (baryons): $\Omega_b = 0.07 \pm 0.04 = 7\,\Omega_{\rm lum\text{-}gal} = 0.3\,\Omega_{\rm gal} \Longrightarrow$ only 30% of dark matter is composed of baryons. We will discuss the reason for the current deuterium density giving the baryon density in the History of the Universe subsection below.

iv) Light (photons): $\rho_{\rm rad} = a\,T_{\rm rad}^4/c^2$ and $T_{\rm rad} =$ 2.73 K,

$$\begin{aligned} \rho_{\rm rad} &= 6.5 \times 10^{-34}\ {\rm gm/cm^3} \\ \Omega_{\rm rad} &= 7 \times 10^{-5} \\ \rho_{\rm rad} &\ll \rho_b \end{aligned}$$

$\Longrightarrow$ today we line in a *matter dominated universe!* Radiation dominated at earlier times (see History of the Universe subsection).

v) What is the identity of the non-baryonic matter that causes $\Omega_{\rm gal} = 0.25$?

- Neutrinos outnumber photons in the Universe by a factor of 10^9. Neutrinos were originally thought to be massless (like photons). However, one outcome of the theory of quantum chromodynamics suggest that neutrinos have mass and that they oscillate in state (*e.g.*, electron-neutrinos to muon-neutrinos to tau-neutrinos).

  - In the 1990s, Los Alamos have detected muon-neutrinos transmuting into electron-neutrinos. The amount of oscillations place a mass range of 0.5–5.0 eV for the neutrino.

  - Supernova 1987A that exploded in the LMC sent a 10-second burst of 19 electron anti-neutrinos that was detected by the various solar neutrino detectors around the world. The *neutrino event* preceded the first sightings of the supernova's light by 3 hours. Confirms supernova theory that it should take about 3 hours for the core-collapse shock to propagate to the stellar surface. The fact that the neutrinos arrived nearly as quick as the photons (allowing for the shock time delay) indicates that the mass of the neutrinos must be less than 3 eV.

  - Assuming a mass of 1 eV (1.8×10^{-33} gm) for a neutrino, gives a universal mass density that is over 10 times the critical density of the Universe!

- Other hypothetical non-baryonic matter has been speculated on, including weakly interactive particles (WIMPS) and massive magnetic monopoles. These strange particles have never been detected however.

vi) From all of the matter measurements (along with the best estimates of the non-baryonic mass), $\Omega_m \approx 1/4 - 1/3$.

2. Cosmological Constant:

a) Prior to the launch of the *Hubble Space Telescope*, the best estimate for the size of the cosmological constant, based upon the measured density (see below), $H_\circ$, and $q_\circ$, gave $\Lambda = (-0.91 \pm 0.93) \times 10^{-20}\,\text{yrs}^{-2}$. The negative value for Λ indicates a *cosmic attraction* on top of gravity! However, recently the value of Λ has been scrutinized in a much more detailed way.

b) Modern field theory now associates this term with the energy density of the vacuum (hence the $\Omega_{\rm vac}$ mentioned above). If the cosmological constant today comprises most of the energy density of the Universe, then the extrapolated age of the Universe is much larger than the current maximum age given by Hubble's Law.

c) Adding a cosmological constant term to the inflationary model, an extension of the Big Bang Theory (see below), leads to a model that appears to be consistent with the observed large-scale distribution of galaxies and clusters, with COBE's measurements of the cosmic microwave background fluctuations, and with the observed properties of X-ray clusters.

d) A few groups of astronomers has ascertained (with large uncertainties) from the brightness of Type Ia supernovae at high redshift that the Universe is currently expanding faster today than it was 5 to 7 billion years ago.

i) This acceleration, called a de Sitter universe (see next section), is predicted from general relativity (and Newtonian cosmology) for $\Lambda > 0$.

ii) The supernova data implies that $\Omega_\Lambda \approx 2/3 - 3/4$, giving $\Omega_\circ \approx 1 \Longrightarrow$ a flat universe!

iii) This supernova data given an age for the Universe at 14.2±1.7 Gyr which is consistent with the WMAP value.

e) The **Compton Gamma Ray Observatory (CGRO)** carried an instrument called the *Burst and Transient Source Experiment* (BATSE) which monitored gamma ray bursters during its 8 year lifetime. BATSE ascertained that these gamma ray bursts are cosmological, and in 1994, also showed evidence that the Universe is expanding faster at present times than in the past.

3. Hubble's Constant:

a) As discussed at the beginning of this section, the *HST Key Project* reports this parameter to be 72±8 km/s/Mpc (see Freeman *et al.* 2001, *Astrophysical Journal*, **553**, 47).

b) The WMAP mission determined $H_\circ = 71 \pm 4$ km/s/Mpc from the cosmic microwave background radiation.

4. **Deceleration Parameter**:

   a) This is perhaps the most difficult parameter of cosmology to measure — though it can be determined from the other parameters previously mentioned in Eq. (XI-49).

   b) Figure 27.18 shows the difficulty with such a measurement from a plot of redshift versus visual magnitude.

   c) One of the most accurate techniques in the determination of $q_\circ$ from an angular size of extragalactic radio sources as a function of redshift. These measurements give a value of 0.1 for the deceleration parameter.

F. Big Bang Modeling.

1. The solution to the field equations indicate the geometry of the Universe. By knowing the value of $\Omega_\circ$, we will know the overall shape (*i.e.*, **curvature**) of the Universe in 4 dimensions and know the final fate of the Universe as shown in Figure XI-2.

   a) A *positive* curvature is a spherical space. As an analogy, assume a 2-D surface being bent into a third dimension in the shape of a sphere's surface. The Universe is said to be **closed**, if spacetime has a curvature where $k = +1$.

   b) A *negative* curvature is a hyperbolic space. With our above analogy, assume a 2-D surface being bent into a third dimension in the shape of a saddle. The Universe is said to be **open**, if spacetime has a curvature where $k = -1$.

   c) A *flat* curvature is a Euclidean space. The 2-D analogy is simply a flat, plane surface that is not curved into a third dimension. The Universe is said to be **flat**, if spacetime has a curvature where $k = 0$.

Figure XI–2: The three possible geometries that spacetime can have.

2. Solution to the Friedmann Equation ($\Lambda = 0$).

a) **The Integral Equation Solution.**

i) The easiest way to solve the Friedmann equation (see Eq. XI-44) is to isolate the dt differential on one side of the equation and integrate the dt side and the dR side.

ii) All of the Friedmann universes has a 'big bang' origin: $R = 0$ at $t = 0$. As such, Eq. (XI-44) can be written as the following integral:

$$t = \int_0^{R(t)} \frac{dR}{\sqrt{\frac{8\pi G \rho_\circ R_\circ^3}{3R} - kc^2}} \; . \qquad \text{(XI-55)}$$

iii) If we define

$$\begin{aligned} R_m \equiv \frac{8\pi G \rho_\circ r_\circ^3}{3c^2} &= \frac{2q_\circ c}{H_\circ |q_\circ - 1|^{3/2}} \quad (k \neq 0) \\ &= \frac{R_\circ^3 H_\circ^2}{c^2} \quad (k = 0) \; , \end{aligned} \qquad \text{(XI-56)}$$

we can simplify Eq. (XI-55) to

$$t = \int_0^{R(t)} \frac{dR}{c\sqrt{(R_m/R) - k}} \; . \qquad \text{(XI-57)}$$

iv) This equation has the following solutions for the three different values of k:

$$t = \begin{cases} \frac{R_m}{c} \sin^{-1}\left\{\sqrt{\frac{R}{R_m} - \sqrt{\left[\frac{R}{R_m}\left(1 - \frac{R}{R_m}\right)\right]}}\right\} & (k = +1) \\ \frac{2R_m}{3c}\left(\frac{R}{R_m}\right)^{3/2} & (k = 0) \\ \frac{R_m}{c}\sqrt{\left[\frac{R}{R_m}\left(1 + \frac{R}{R_m}\right)\right] - \sinh^{-1}\left(\sqrt{\frac{R}{R_m}}\right)} & (k = -1) \; . \end{cases} \qquad \text{(XI-58)}$$

Figure XI–3: Histories of Friedmann model universes ($\Lambda = 0$). Note that the present size of the Universe (hence, present time) is indicated by $R = 1$.

v) One can now easily see why the $k = -1$ case is called a *hyperbolic* universe.

vi) Figure XI-3 shows the histories of these three types of universes.

b) Closed universes ($\boldsymbol{k = +1, \Omega_\circ > 1}$).

i) From the first equation in Eq. (XI-56), the scale factor would reach a maximum size R_m (assuming $q_\circ \approx 1$) for our Universe (assuming our Universe is closed) at

$$R_m \approx \frac{2c}{H_\circ} \; . \qquad \text{(XI-59)}$$

Using the WMAP value for Hubble's constant, this maximum would be $R_m = 2.6 \times 10^{28}$ cm $= 2.8 \times 10^{10}$ ly $= 8.5 \times 10^9$ pc (8.5 Gpc).

ii) By setting $\dot{R} = 0$ in the Friedmann equation, this maximum size would be reached in time t_m given by

$$t_m = \frac{\pi R_m}{2c} \, , \qquad \text{(XI-60)}$$

or 4.4×10^{10} yr (44 Gyr) after the Big Bang. Since the Universe is currently about 14 Gyr old, we would still have another 30 Gyr before the collapse would begin.

iii) At this point (*i.e.*, t_m), such a universe starts to collapse back down to a 'big crunch' at $t = 2t_m$. The Big Crunch for our Universe would occur 88 Gyr after the Big Bang or 74 Gyr into the future — we have nothing to worry about!

iv) We also can calculate the current age of the Universe $t_\circ$ by setting $R = R_\circ$ in Eq. (XI-58) for $k = +1$:

$$t_\circ = \frac{2q_\circ}{H_\circ |2q_\circ - 1|^{3/2}} \sin^{-1}\left[\sqrt{\frac{2q_\circ - 1}{q_\circ}} - \frac{1}{2q_\circ}\sqrt{2q_\circ - 1}\right] \, . \qquad \text{(XI-61)}$$

Once again, using $q_\circ = 1$ with the WMAP value for $H_\circ$ gives $t_\circ = 2.8 \times 10^{10}$ yr (28 Gyr) which is a factor of 2 bigger than the best estimate for the current age of the Universe (14 Gyr) — obviously if the Universe is closed, $q_\circ < 1$ to give a more reasonable value for $t_\circ$.

v) Since the solution for $k = +1$ presented in Eq. (XI-58) is fairly complicated, one can introduce a *link* parameter (call it x) between R and t and write this one equation into two *parametric* equations (here we will give two separate equations for each parametric equation [$R(x)$ and $t(x)$] by making use of the density parameter):

$$R = \frac{4\pi G\rho_\circ}{3kc^2}\,(1-\cos x) \qquad \text{(XI-62)}$$

$$= \frac{1}{2}\,\frac{\Omega_\circ}{\Omega_\circ - 1}\,(1-\cos x) \qquad \text{(XI-63)}$$

and

$$t = \frac{4\pi G\rho_\circ}{3k^{3/2}c^3}\,(x-\sin x) \qquad \text{(XI-64)}$$

$$= \frac{1}{2H_\circ}\,\frac{\Omega_\circ}{(\Omega_\circ - 1)^{3/2}}\,(x-\sin x)\,, \qquad \text{(XI-65)}$$

where $x \geq 0$ has no specific meaning associated with it other than linking R to t.

c) **Flat universe ($\boldsymbol{k = 0, \Omega_\circ = 1}$).**

i) With such a universe, the expansion velocity would just reach zero after an infinite amount of time has passed.

ii) We can easily invert Eq. (XI-58) for $k = 0$ to derive $R(t)$ (after realizing that $\rho_\circ = \rho_c$ for such a universe):

$$R = (6\pi G\rho_c)^{1/3}\,t^{2/3} \qquad \text{(XI-66)}$$

$$= \left(\frac{3}{2}\right)^{2/3}\left(\frac{t}{t_{\rm H}}\right)^{2/3}. \qquad \text{(XI-67)}$$

iii) Taking the time derivative of this equation gives

$$\dot{R} = \frac{2}{3}\,(6\pi G\rho_c)^{1/3}\,t^{-1/3}\,. \qquad \text{(XI-68)}$$

As can be seen from this equation, when $t \to \infty$, $\dot{R} \to 0$ as previously stated.

iv) Once again we can calculate the current age of the Universe $t_\circ$ by setting $R = R_\circ$ in Eq. (XI-58) for $k = 0$:

$$t_\circ = \frac{2R_m}{3c}\left(\frac{R_\circ}{R_m}\right)^{3/2} = \frac{2}{3H_\circ} = \frac{2}{3}\,t_{\rm H} \ . \qquad \text{(XI-69)}$$

For the WMAP value of $H_\circ$, the current age of the Universe is 9.2×10^9 yr (9.2 Gyr). Since the oldest globular star clusters are on the order of 12-13 Gyr old, our Universe cannot have both $k = 0$ and $\Lambda = 0$.

d) Open universes ($\boldsymbol{k = -1, 0 < \Omega_\circ < 1}$).

i) With such a universe, the expansion velocity remains greater than zero after an infinite amount of time has passed.

ii) For such an open universe, the current age of the Universe $t_\circ$ is found by setting $R = R_\circ$ in Eq. (XI-58) for $k = -1$. This gives the following solution:

$$\begin{aligned} t_\circ &= \frac{2q_\circ}{H_\circ|2q_\circ - 1|^{3/2}}\left[\frac{\sqrt{|2q_\circ - 1|}}{2q_\circ} - \right. \\ &\qquad \left. \sinh^{-1}\sqrt{\frac{|2q_\circ - 1|}{2q_\circ}}\right] \qquad \text{(XI-70)} \\ &\approx \frac{1}{H_\circ} = t_{\rm H} \ . \end{aligned}$$

Using the WMAP value for $H_\circ$ gives $t_\circ = 1.4 \times 10^{10}$ yr (14 Gyr) which is consistent with the age of the oldest stars seen in the Milky Way.

iii) Since the solution for $k = -1$ presented in Eq. (XI-58) is fairly complicated (like the case for the closed universe), one can introduce a *link* parameter (call it x) between R and t and write this one equation into two *parametric* equations (here we will give two separate equations for each parametric equation [$R(x)$ and $t(x)$] by making use of the density parameter):

$$R = \frac{4\pi G\rho_\circ}{3|k|c^2}\,(\cosh x - 1) \qquad \text{(XI-71)}$$
$$= \frac{1}{2}\,\frac{\Omega_\circ}{1-\Omega_\circ}\,(\cosh x - 1) \qquad \text{(XI-72)}$$

and

$$t = \frac{4\pi G\rho_\circ}{3|k|^{3/2}c^3}\,(\sinh x - x) \qquad \text{(XI-73)}$$
$$= \frac{1}{2H_\circ}\,\frac{\Omega_\circ}{(1-\Omega_\circ)^{3/2}}\,(\sinh x - x)\,, \qquad \text{(XI-74)}$$

where x has no specific meaning associated with it other than linking R to t.

iv) Since $\cosh x = (e^x + e^{-x})/2 \geq 1$ and $\sinh x = (e^x - e^{-x})/2 \geq x$, Eqs. (XI-71) through (XI-74) show that R increases monotonically with t.

3. Solution to the Modified Friedmann Equation ($\Lambda \neq 0$ — for simplicity, we will only investigate flat universes with $k = 0$).

a) The Integral Equation Solution.

i) The modified Friedmann equation given in Eq. (XI-45) can be analytically solved in the general case in terms of *elliptic functions*, but the result is not easy to understand. As such, we will limit our discussion to those $\Lambda \neq 0$ universes where $k = 0$ $(\Omega_\circ = 1) \Longrightarrow$ flat universes.

ii) An examination of the modified Friedmann equation shows that such universes start from condensed $(R = 0)$ states.

iii) For such cases, Eq. (XI-45) can be solved by isolating the time differential on one side of the equation and the scale factor terms on the other side. We then have (setting $t_{\rm init} = 0$ when $R = 0$)

$$t = \int_0^{R(t)} \frac{dR}{\sqrt{\frac{8\pi G\rho_\circ R_\circ^3}{3R} + \frac{\Lambda R^2}{3}}} \ . \qquad \text{(XI-75)}$$

iv) This equation has three sets of solutions based upon the value of Λ:

$$t = \begin{cases} \frac{2}{\sqrt{3\Lambda}} \sinh^{-1}\left[\left(\frac{R}{R_\circ}\right)^{3/2} \sqrt{\frac{\Lambda}{8\pi G\rho_\circ}}\right] & (\Lambda > 0) \\ \left(\frac{R}{R_\circ}\right)^{3/2} (6\pi G\rho_\circ)^{-1/2} & (\Lambda = 0) \\ \frac{2}{\sqrt{3|\Lambda|}} \sin^{-1}\left[\left(\frac{R}{R_\circ}\right)^{3/2} \sqrt{\frac{|\Lambda|}{8\pi G\rho_\circ}}\right] & (\Lambda < 0) \ . \end{cases} \qquad \text{(XI-76)}$$

v) Unlike the three cases where $\Lambda = 0$, these three equations can be easily inverted to give

$$R(t) = \begin{cases} R_\circ \left(\frac{8\pi G\rho_\circ}{\Lambda}\right)^{1/3} \sinh\left(\frac{1}{2}\sqrt{3\Lambda}\, t^{2/3}\right) & (\Lambda > 0) \\ R_\circ \,(6\pi G\rho_\circ)^{1/3}\, t^{2/3} & (\Lambda = 0) \\ R_\circ \left(\frac{8\pi G\rho_\circ}{|\Lambda|}\right)^{1/3} \left[\sin\left(\frac{1}{2}\sqrt{3|\Lambda|}\, t\right)\right]^{2/3} & (\Lambda < 0) \ . \end{cases} \qquad \text{(XI-77)}$$

vi) Figure XI-4 shows the histories of these three types of universes.

Figure XI–4: Histories of flat ($k = 0$) modified Friedmann model universes. Note that the present size of the Universe (hence, present time) is indicated by $R = 1$.

b) Closed universes ($\boldsymbol{\Lambda < 0}$).

i) If the cosmological constant is less than zero, then this constant acts to assist gravity in slowing down the expansion of the Big Bang. As such, such a universe will expand to a maximum size then recollapse due to the sine term in Eq. (XI-77) for $\Lambda < 0$.

ii) The maximum size will be reached when $dR/dt = 0$ when $\Lambda < 0$ in Eq. (XI-77) giving

$$\frac{dR}{dt} = \frac{1}{3} R_\circ \left(\frac{8\pi G\rho_\circ}{|\Lambda|}\right)^{1/3} \sqrt{3|\Lambda|}\, \frac{\cos\left(\frac{1}{2}\sqrt{3|\Lambda|}t\right)}{\left[\sin\left(\frac{1}{2}\sqrt{3|\Lambda|}t\right)\right]^{1/3}} = 0\ , \tag{XI-78}$$

which will occur when

$$\cos\left(\frac{1}{2}\sqrt{3|\Lambda|}t\right) = 0\ . \tag{XI-79}$$

Since we are looking for the first maximum here, the above equation gives

$$\frac{1}{2}\sqrt{3|\Lambda|}\,t = \frac{\pi}{2}$$
$$t = t_m = \frac{\pi}{\sqrt{3|\Lambda|}} \ . \qquad \text{(XI-80)}$$

Using $|\Omega_\Lambda| = 2/3$, Eq. (XI-53) gives $|\Lambda| = 1.0 \times 10^{-20}$ yr^{-2} and $t_m = 1.8 \times 10^{10}$ yr (18 Gyr) $\Longrightarrow$ only 4 billion years until the Universe starts to recollapse if our Universe has $\Lambda = -1.0 \times 10^{-20}$ yr^{-2} and $k = 0$!

iii) The size of such a universe at t_m is given by

$$R = R_m = \left(\frac{8\pi G\rho_\circ}{|\Lambda|}\right)^{1/3} R_\circ = \left(\frac{\Omega_m}{|\Omega_\Lambda|}\right)^{1/3} R_\circ \ . \qquad \text{(XI-81)}$$

For this example, we have assumed that $\Omega_\Lambda = -2/3$, a flat universe implies that $\Omega_m = 5/3$ to give $\Omega_\circ = 1$. As such, the maximum size that such a universe would reach after time t_m is $R_m = 2.5\,R_\circ$, or two and a half times the current size of the Universe.

iv) The current age of such a universe is obtained from setting $R = R_\circ$ in Eq. (XI-76) which gives

$$t_\circ = \frac{2}{\sqrt{3|\Lambda|}} \sin^{-1}\sqrt{\frac{|\Lambda|}{8\pi G\rho_\circ}} \qquad \text{(XI-82)}$$
$$= \frac{2}{\sqrt{3|\Lambda|}} \sin^{-1}\sqrt{\frac{|\Omega_\Lambda|}{\Omega_m}} \ . \qquad \text{(XI-83)}$$

For our values of Λ, Ω_Λ, and Ω_m above, this gives an age of the Universe as 7.9×10^9 yr (7.9 Gyr), which is not old enough to account for the oldest stars in the Milky Way.

c) **Flat universe ($\boldsymbol{\Lambda = 0, k = 0}$).**

i) The solution to the modified Friedmann equation in this case is identical to the solution of the normal Friedmann equation.

ii) As such, Eqs. (XI-66) and (XI-67) describes the history of this type of universe, $R(t)$.

iii) The current age of such a universe given by Eq. (XI-69). For the parameters found for our current Universe, this age is 9.2 Gyr, once again, insufficient to account for the oldest stars in the Galaxy.

d) **Open (de Sitter type) universes ($\boldsymbol{\Lambda > 0}$).**

i) If the cosmological constant is greater than zero, then the universe is curved even if empty and this curvature imparts a negative pressure that opposes gravity. In such a universe, the universe can at first decelerate, then go through a saddle phase (see Fig. XI-4), then start to accelerate.

ii) Since our Universe seems to be accelerating and is thought to have a flat curvature, this type of universe may best describe ours.

iii) The time and size that such a universe crosses over from deceleration to acceleration will occur when d^2R/dt^2 goes from negative values to positive values, hence, when $d^2R/dt^2 = 0$. Using Eq.

(XI-41), we see that this saddle point occurs at

$$\begin{aligned}
\ddot{R} &= -\frac{4\pi\rho_\circ G R_\circ^3}{3R^2} + \frac{\Lambda R}{3} = 0 \\
\frac{\Lambda R}{3} &= \frac{4\pi\rho_\circ G R_\circ^3}{3R^2} \\
R^3 &= \frac{4\pi\rho_\circ G}{\Lambda} R_\circ^3 \\
R = R_s &= \left(\frac{4\pi\rho_\circ G}{\Lambda}\right)^{1/3} R_\circ \qquad \text{(XI-84)} \\
&= \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} R_\circ \, . \qquad \text{(XI-85)}
\end{aligned}$$

For our Universe, we have determined that $\Omega_\Lambda \approx 2/3$ and $\Omega_m \approx 1/3$, which gives the size of the Universe when it went from a deceleration phase to an acceleration phase (the saddle point) of $R_s = 0.79\, R_\circ$ — when the Universe was 80% of its current size.

iv) We also can determine the time this cross-over took place by plugging this value for R into the first equation of Eq. (XI-76):

$$\begin{aligned}
t_s &= \frac{2}{\sqrt{3\Lambda}} \sinh^{-1}\left[\left(\frac{R_s}{R_\circ}\right)^{3/2} \sqrt{\frac{\Lambda}{8\pi G\rho_\circ}}\right] \\
&= \frac{2}{\sqrt{3\Lambda}} \sinh^{-1}\left[\sqrt{\frac{4\pi G\rho_\circ}{\Lambda}} \sqrt{\frac{\Lambda}{8\pi G\rho_\circ}}\right] \\
&= \frac{2}{\sqrt{3\Lambda}} \sinh^{-1}\left(\sqrt{\frac{1}{2}}\right) \, . \qquad \text{(XI-86)}
\end{aligned}$$

A table of hyperbolic trigonometric identities shows that

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \, .$$

Using this for our inverse hyperbolic sine above, we get

$$\begin{aligned}\sinh^{-1}\left(\sqrt{\frac{1}{2}}\right) &= \ln\left(\frac{1}{2} + \sqrt{\frac{1}{2} + 1}\right) \\ &= \ln\left(\sqrt{\frac{1}{2} + \frac{3}{2}}\right) = \ln\left(\frac{2}{\sqrt{2}}\right) \\ &= \ln\left(\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\right) \\ &= \ln\sqrt{2}\ .\end{aligned}$$

Finally, plugging this into Eq. (XI-86), we get the saddle cross-over time of

$$t_s = \frac{2\ln\sqrt{2}}{\sqrt{3\Lambda}}\ . \qquad \text{(XI-87)}$$

Using $\Lambda = 1.0 \times 10^{-20}$ yr^{-2} (which results from $\Omega_\Lambda = 2/3$ and $H_\circ = 71$ km/s/Mpc), we see that this deceleration to acceleration cross-over time occurred at $t_s = 4.0 \times 10^9$ yr after the Big Bang, or about 10 billion years ago!

v) The current age of the Universe can be obtained by setting $R = R_\circ$ in the first equation of Eq. (XI-76). Doing this we get

$$t_\circ = \frac{2}{\sqrt{3\Lambda}} \sinh^{-1}\sqrt{\frac{\Lambda}{8\pi G\rho_\circ}} \qquad \text{(XI-88)}$$

$$= \frac{2}{\sqrt{3\Lambda}} \sinh^{-1}\sqrt{\frac{\Omega_\Lambda}{\Omega_m}}\ . \qquad \text{(XI-89)}$$

- If we use $\Omega_\Lambda = 2/3$ ($\Lambda = 1.0 \times 10^{-20}$ yr^{-2}) and $\Omega_m = 1/3$, that gives the current age of the Universe as $t_\circ = 1.3 \times 10^{10}$ yr (13 Gyr) $\Longrightarrow$ consistent with the oldest stars in the Galaxy.

- If we use $\Omega_\Lambda = 3/4$ ($\Lambda = 1.1 \times 10^{-20}$ yr^{-2}) and $\Omega_m = 1/4$, that gives the current age of the Universe as $t_\circ = 1.4 \times 10^{10}$ yr (14 Gyr) $\Longrightarrow$ once again, consistent with the oldest stars in the Galaxy.

vi) As can be seen, such a de Sitter-type universe is consistent with the current observational constraints for the Universe, which means the Universe will continue to accelerate to a cold and dark death when the stellar furnaces finally burn themselves out.

G. History of the Universe.

1. Singularity, the Big Bang Itself!

$$t = 0, \quad D = 0, \quad \rho = \rho_{\rm rad} \rightarrow \infty, \quad T = T_{\rm rad} \rightarrow \infty \ .$$

a) If the Universe is closed, then a finite amount of mass-energy is located in a zero volume (like a black hole singularity).

b) If the Universe is open or flat, then the Universe has an infinite total amount of mass-energy located in an infinite volume at this stage.

c) We currently have no physics that can describe the history and events occurring in the Universe at this point. Perhaps if quantum mechanics and general relativity are ever combined (*i.e.*, **quantum gravity**), we will have a physical theory that can describe the Universe here and explain why the Big Bang ever occurred.

d) In the header lists for each of these eras, t represents time since the Big Bang, D the diameter of the Universe at time t, ρ is the mass-energy density, T is the temperature, and later, z corresponds to the redshift. Each of these values are listed at the beginning and ending of each era.

2. Quantum Era.

$$0 < t < 10^{-43} \text{ sec} \quad = \quad t_{\rm P} = \text{ Planck Time,}$$
$$0 < D < 10^{-33} \text{ cm} \quad = \quad \ell_{\rm P} = \text{ Planck Length,}$$
$$\rho = \rho_{\rm rad} > 10^{90} \text{ gm/cm}^3, \qquad T = T_{\rm rad} > 10^{32} \text{ K} .$$

a) The earliest time that can be addressed by current physical theory is the **Planck time**:

$$t_{\rm P} \equiv \sqrt{\frac{\hbar G}{c^5}} = 5.39 \times 10^{-44} \text{ s} , \qquad \text{(XI-90)}$$

where $\hbar$ is the angular Planck's constant ($\hbar = h/2\pi$).

b) In a Planck time, the speed of light crosses a distance called the **Planck length**:

$$\ell_{\rm P} = t_{\rm P}\, c \equiv \sqrt{\frac{\hbar G}{c^3}} = 1.62 \times 10^{-33} \text{ cm} . \qquad \text{(XI-91)}$$

c) The uncertainty principle from quantum mechanics tells us that the uncertainty in a particle's momentum times the uncertainty in position must be greater that $\hbar$. If we use the Schwarzschild radius of the early Universe as the position uncertainty, we can use conservation of energy (see page 1308 in the textbook) to describe the **Planck mass**:

$$m_{\rm P} \equiv \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-5} \text{ g} . \qquad \text{(XI-92)}$$

The Planck mass can be interpreted as the minimum mass that any primordial black holes can have if created in the Big Bang.

d) Physics will not be understood in this Quantum Era until quantum effects are successfully included in gravity as described previously.

e) During this time, it is speculated that all forces, including gravity, act as one force $\Longrightarrow$ the **Theory of Everything**.

3. GUT (Grand Unified Theory) Era.

$$\begin{aligned} 10^{-43}\ \text{sec} &< \quad t \quad < 10^{-34}\ \text{sec} \\ 10^{-33}\ \text{cm} &< \quad D \quad < 10^{-24}\ \text{cm} \\ 10^{90}\ \text{gm/cm}^3 &< \quad \rho_{\text{rad}} \quad < 10^{72}\ \text{gm/cm}^3 \\ 10^{32}\ \text{K} &< \quad T \quad < 10^{27}\ \text{K}\ . \end{aligned}$$

a) At the beginning of this era, gravity breaks from the unified force, following the equations of general relativity, and gets progressively weaker (see Fig. 28.7 on page 1310 in the textbook). This **symmetry breaking** acts like a phase transition of the Universe.

b) During this time, the strong, weak, and electromagnetic forces act as one as described by the **Grand Unified Theory**.

c) Temperature is so high that only field particles exist: gravitons, weakons (*i.e.*, intermediate vector bosons), photons, and gluons (see Fig. IV-1 on page IV-27 in these notes).

d) At the end of this era, the strong force decouples from the **electroweak** force $\Longrightarrow$ the Universe goes through another phase transition.

4. **Inflationary Era.**

$$\begin{aligned} 10^{-34}\ \text{sec} &< \quad t \quad < 10^{-32}\ \text{sec} \\ 10^{-24}\ \text{cm} &< \quad D \quad < 330\ \text{cm} \\ 10^{72}\ \text{gm/cm}^3 &< \rho_{\text{rad}} < 10^{37}\ \text{gm/cm}^3 \\ 10^{27}\ \text{K} &< \quad T \quad < 10^{18}\ \text{K}\ . \end{aligned}$$

a) The electroweak-decoupling phase transition causes the Universe to expand exponentially from 10^{-24} cm to 330 cm (10^{28} cm today).

b) During this time, all baryonic matter is created from the primeval *soup* of field particles $\Longrightarrow$ individual **quarks** and **antiquarks** are made $\Longrightarrow$ matter (and antimatter) arise from the field energy particles via $E = mc^2$.

i) Field particles all have integer spins $\Longrightarrow$ they are bosons.

ii) The GUT predicts that baryon number conservation and charge & partity (CP) conservation can be violated occasionally.

iii) These CP violations can cause slight asymmetries in decay rates of a given boson decaying to more stable particles:

- The *kaon* K can decay to a *pion* π via either

$$\begin{aligned} K &\rightarrow \pi^- + e^+ + \nu_e \\ K &\rightarrow \pi^+ + e^- + \overline{\nu}_e\ . \end{aligned}$$

- The first of these two reactions occurs slightly (but measurably) more frequently than the second.

- As such, it is possible to get slight asymmetries between matter and antimatter over time as these particles are made out of the field particle soup.

c) During this time, the temperatures are two high for the strong force to connect the quarks together to make baryons and anti-baryons.

d) Due to CP violation, for every 30 million antiquarks, there are 30 million + 1 quarks by the end of this era.

e) The Universe resumes a linear expansion at the end of this era.

5. Quark Era.

$$\begin{aligned} 10^{-32}\text{ sec} &< \quad t \quad < 10^{-6}\text{ sec} \\ 330\text{ cm} &< \quad D \quad < 10^{9}\text{ cm } = 4D_{\oplus} \\ 10^{37}\text{ gm/cm}^3 &< \quad \rho_{\rm rad} \quad < 10^{17}\text{ gm/cm}^3 \\ 10^{18}\text{ K} &< \quad T \quad < 10^{13}\text{ K}\ . \end{aligned}$$

a) Forces between quarks act strangely: The farther away they get from each other, the stronger the force exerted (opposite of the direction of gravity). At the beginning of this era, the temperature is so high that quark motions can overcome this force and hence are <u>not</u> bound with each other.

b) At $t = 10^{-12}$ sec, $T = 10^{15}$ K, $\rho = 10^{24}$ gm/cm^3, the electromagnetic and weak forces decouple $\Longrightarrow$ when this occurs, leptons start to form.

c) This era ends when the temperature is cool enough for quarks to form bound states ($T = 10^{13}$ K, $\rho = 10^{17}$

gm/cm^3) and become hadrons.

6. Hadronic Era.

$$\begin{aligned} 10^{-6}\text{ sec} < &\ t\ < 1\text{ sec} \\ 10^{9}\text{ cm} < &\ D\ < 10^{14}\text{ cm } = 15\text{ AU} \\ 10^{17}\text{ gm/cm}^3 < &\ \rho_{\rm rad}\ < 10^{5}\text{ gm/cm}^3 \\ 10^{13}\text{ K} < &\ T\ < 10^{10}\text{ K}\ . \end{aligned}$$

a) During this time, proton-antiproton pairs constantly annihilate and refrom $\Longrightarrow$ hadrons are said to be in thermal equilibrium with photons (*i.e.*, the radiation field).

b) At the end of this era, $T = 10^{10}$ K, $\rho = 10^5$ gm/cm^3, the energy density is too low to produce proton-antiproton pairs. These particles annihilate one last time — except there is a *slight* asymmetry between protons and antiprotons, for every 10^9 antiprotons there are $10^9 + 1$ protons (once again, due to CP violation). The remaining protons have nothing to annihilate with and remain.

7. Lepton Era.

$$\begin{aligned} 1\text{ sec} < &\ t\ < 1\text{ min} \\ 10^{14}\text{ cm} < &\ D\ < 10^{15}\text{ cm } = 150\text{ AU} \\ 10^{5}\text{ gm/cm}^3 < &\ \rho_{\rm rad}\ < 10^{4}\text{ gm/cm}^3 \\ 10^{10}\text{ K} < &\ T\ < 6\times 10^{9}\text{ K}\ . \end{aligned}$$

a) Electron-positron pairs are still in equilibrium.

b) At the end of this era, $T = 6 \times 10^9$ K, $\rho = 10^4$ gm/cm^3, the energy density becomes too low to make new electron-positron pairs $\Longrightarrow$ an excess of electrons is left over due to asymmetries in the intermediate vector boson decays.

8. Nucleosynthesis Era.

$$\begin{aligned} 1 \text{ min} &< t < 3 \text{ min} \\ 10^{15} \text{ cm} &< D < 10^{19} \text{ cm} = 10 \text{ ly} \\ 10^{4} \text{ gm/cm}^3 &< \rho_{\rm rad} < 10^{-8} \text{ gm/cm}^3 \\ 6 \times 10^{9} \text{ K} &< T < 10^{7} \text{ K} . \end{aligned}$$

a) Temperatures and densities exist such that H fuses into ^{2}H (deuterium).

$$\begin{aligned} {}^1\text{H} + {}^1\text{H} &\longrightarrow {}^2\text{H} + e^+ + \nu \\ n + {}^1\text{H} &\longrightarrow {}^2\text{H} + \gamma . \end{aligned}$$

b) Temperatures are too high to fuse ^{2}H $\Longrightarrow$ it photodissociates as fast as it forms. He (helium) cannot be created — even though the temperature is high enough $\Longrightarrow$ the **deuterium bottleneck**.

c) When $T < 10^9$ K ($t \approx 100$ sec, 1.5 min), ^{2}H no longer dissociates and fuses immediately into He $\Longrightarrow$ however, the Universe is now too cool to fuse He $\rightarrow$ C (carbon) and O (oxygen) via the triple-α process.

d) At $t = 3$ min, $T < 10^7$ K, $\rho < 10^{-8}$ gm/cm^3, and He production ceases. Since the last stages of the p-p chain are not as efficient as the first stage, an excess number of ^{2}H (deuterium) is left over (and some ^{7}Li [lithium-7] from the branch reactions of the full p-p chain) $\Longrightarrow$ the amount left over depends critically on the density of the Universe at that time (see Fig. 28.1 on page 1293 in your textbook). Since deuterium and lithium-7 are easily destroyed in the interior of stars, all of the ^{2}H and ^{7}Li we currently see in the Universe arose during this Nucleosynthesis Era.

e) The future of the Universe is now set $\Longrightarrow$ all within the first 3 minutes!

9. Radiation Era.

$$\begin{aligned} 3 \text{ min} &< \; t \; < 1000 \text{ yr} \\ 10^9 &< \; z \; < 2200 \\ 10^{19} \text{ cm} &< \; D \; < 10^{21} \text{ cm} \; = 1000 \text{ ly} \; = 300 \text{ pc} \\ 10^{-8} \text{ gm/cm}^3 &< \; \rho \; < 10^{-15} \text{ gm/cm}^3 \\ 10^7 \text{ K} &< \; T \; < 10^5 \text{ K} \, . \end{aligned}$$

a) In this era, $\rho_{\rm rad} > \rho_{\rm matter}$.

b) The temperature is still greater than 10^5 K which keeps hydrogen ionized.

c) Since H is ionized, there are an abundance of free electrons which effectively *blocks* the flow of radiation (*i.e.*, photons) $\Longrightarrow$ Compton scattering and Thompson scattering.

d) The Universe is completely *opaque* during this time.

e) At the end of this era, $\rho_{\rm m} = \rho_{\rm baryon} = \rho_B = \rho_{\rm rad}$ (here we are assuming that all of the Universe's mass is in the form of baryons). With this information, we can now determine the total number of baryon particles in the Universe and the total mass of the baryons at this point. Both the total number and total mass remains constant from this point forward as the Universe expands.

i) The total mass of all the baryons present at the end of this era is equal to

$$M_B = V \, \rho_B = \frac{4\pi}{3} \, D^3 \, \rho_b \; . \qquad \text{(XI-93)}$$

ii) The baryon mass density can now be obtained with

$$\rho_B = \rho_{\rm rad} = \frac{aT^4}{c^2} \ . \qquad \text{(XI-94)}$$

Using the values for T listed above ($a = 7.566 \times 10^{-15}$ erg/cm^3/K^4), we get $\rho_b = 8.4 \times 10^{-16}$ gm/cm^3.

iii) Using the diameter of the Universe at the end of this era listed above, we get the total mass of the baryons as $M_B = 3.5 \times 10^{48}$ grams $= 2 \times 10^{15} M_\odot$.

iv) At this point in the Universe, 90% of all the particles are hydrogen nuclei (protons) and 10% are helium nuclei (2 protons + 2 neutrons). With this information, we can determine the total number of baryons that were created from the Big Bang:

$$\begin{aligned} n_B &= 0.90 \, \frac{M_B}{m_{\rm H}} + 0.10 \, \frac{M_B}{m_{\rm He}} = \left(\frac{0.9}{m_{\rm H}} + \frac{0.1}{m_{\rm He}} \right) M_B \\ &= 1.9 \times 10^{72} \text{ baryons.} \end{aligned}$$

f) We know that the COBE and WMAP microwave maps of the sky show temperature (hence density) variations (*i.e.*, fluctuations) on the cosmic microwave background. We now ask the question that if these density fluctuations existed during the radiation era, will they become unstable and collapse under their own weight?

i) To answer this, we just need to calculate the Jeans' mass of the Universe during this era.

ii) Kolb and Turner, in their *The Early Universe* textbook, give the Jeans' mass of baryons for an expanding universe during the radiation era as

$$M_{B-J}^r = 5.4 \times 10^{18} \, \Omega_B h^2 T_{\rm eV}^{-3} \ M_\odot \ , \qquad \text{(XI-95)}$$

where $T_{\rm eV}$ is the temperature of the radiation field expressed in units of eV (1 eV $= 1.1605 \times 10^4$ K).

iii) If we use the temperature calculated for the cross-over time and use a temperature slightly higher than this (say 1.50×10^5 K $= 12.9$ eV) for the temperature of the radiation field just prior to the radiation-dominated to matter-dominated cross-over, setting $\Omega_B = \Omega_\circ$ gives a Jeans' mass of

$$M^r_{B-J} = \frac{5.4 \times 10^{18}\ M_\odot}{(12.9)^3} = 2.5 \times 10^{15}\ M_\odot\ . \qquad \text{(XI-96)}$$

iv) As can be seen, $M_B < M^r_{B-J}$, so no gravitational instabilities will grow and any density fluctuations that exist will not result in a collapse.

10. Matter Era.

$$\begin{array}{rcll} 1000 \text{ yr} & < & t & < \text{ present } = 15 \text{ Gyr} \\ 2200 < & & z & < 0 \\ 10^{21} \text{ cm} & < & D & < 10^{28} \text{ cm } = 15 \text{ Gly } = 4.6 \text{ Gpc} \\ 10^{-15} \text{ gm/cm}^3 & < & \rho & < 3 \times 10^{-30} \text{ gm/cm}^3 \\ 10^5 \text{ K} & < & T & < 2.7 \text{ K}\ . \end{array}$$

a) Here we are assuming that *present* is at $t = 15 \times 10^9$ (15 billion) years.

b) Matter begins to dominate radiation in this era: $\rho_{\rm matter} > \rho_{\rm rad}$.

c) H (hydrogen) becomes completely neutral when $T < 3000$ K ($t = 300,000$ years, $z = 1100$) $\Longrightarrow$ the Universe becomes transparent to light since the opacity from electron scattering drops to zero!

i) We see this epoch today as the 2.7 K background radiation $\Longrightarrow$ visible light when emitted at $z = 1100$, redshifted today ($z = 0$) to microwave radiation.

ii) This time is called the **recombination time** of the Universe.

iii) This is what we are seeing when we observe the 2.7 K background.

d) Inhomogeneities in the matter and radiation fields begin to grow due to gravitational instabilities just after recombination.

i) Kolb and Turner, in their *The Early Universe* textbook, also give the Jeans' mass of baryons for an expanding universe during the time just after recombination as

$$M_{B-J}^{m} = 1.3 \times 10^5 \left(\Omega_B h^2\right)^{-1/2} \left(\frac{z}{1100}\right) M_\odot \ . \qquad \text{(XI-97)}$$

ii) Since $z < 1100$ past the recombination time, $M_{B-J}^{m} < 1.3 \times 10^5 \ M_\odot$ for the remainder of the matter era. As such, $M_B \gg M_{B-J}^{m} \rightarrow$ the mass of baryons will become gravitationally unstable which will cause density fluctuations to grow and collapse to occur. Note that due to this large inequality, the collapse occurs very rapidly.

e) Galaxies begin to "light-up" as the density inhomogeneities collapse to a high enough value that star formation begins.

i) It is at this time that the Population III stars (*i.e.*, no metalicity) begin to form in earnest from the IGM and ISM.

ii) This occurs at approximately $z = 20, t = 1 \times 10^8$ years after the Big Bang.

f) Galaxies begin to cluster at $z \approx 10, t = 3 \times 10^8$ years.

g) The first Population II (low metalicity) stars form in our Galaxy out of material expelled from the Population III stars at $z \approx 4.5, t = 8 \times 10^8$ years after the Big Bang.

h) Quasars become active and Population II stellar formation rates begin to drop in the Milky Way at $z \approx 3, t = 1 \times 10^9$ years.

i) Population I stars begin to form in Milky Way at $z \approx 1, t = 4 \times 10^9$ years.

j) The Sun and solar system form when $z = 0.02$ and $t = 10 \times 10^9$ years after the Big Bang.

k) Life begins on Earth around $t = 11 \times 10^9$ years after the Big Bang, and the first primates arise at $t = 14.96 \times 10^9$ years after the Big Bang.

l) Presently the Universe is at of age of 15×10^9 (15 billion) years (*i.e.*, $z = 0$).