ASTR-3415-001: Astrophysics
Solutions to Exam 1
Spring 2003

1. (40 points total) Assume a gas has an opacity that is constant at all wavelengths and has the functional form

$$\kappa = A \left(\frac{z}{z_\circ}\right)^{3/2} ,$$

where $A = 3.45 \times 10^{11}$ cm^{-1} is the opacity at the reference depth $z_\circ = 5.26 \times 10^3$ km and z increases into the gas as viewed from outside. (Note that z increases as τ increases.)

(a) [10 points] What is the optical depth of the gas at $z_\circ$?

Solution (a): Note that in Eq. (I-18) in your notes,

$$d\tau_\nu = -\kappa_\nu \, ds ,$$

where τ_ν increases in the opposite direction to s. However in this problem, you are told that τ_ν and z increase in the same direction (*i.e.*, inward), as such

$$d\tau_\nu = \kappa_\nu \, dz .$$

Now, since κ is independent of wavelength (hence frequency), so is the optical depth. As such,

$$d\tau = \kappa \, dz = A \left(\frac{z}{z_\circ}\right)^{3/2} dz .$$

To determine the optical depth at $z_\circ$, we just integrate both sides of the equation from the top of the gas, where both $\tau = 0$ and $z = 0$, and integrate to $z_\circ$ and solve for τ at that position. Then

$$\begin{aligned} \int_0^\tau d\tau &= \int_0^{z_\circ} A \left(\frac{z}{z_\circ}\right)^{3/2} dz \\ \tau - 0 &= \frac{A}{z_\circ^{3/2}} \int_0^{z_\circ} z^{3/2} \, dz \\ \tau &= \frac{A}{z_\circ^{3/2}} \, \frac{2}{5} \, z^{5/2}\Big|_0^{z_\circ} = \frac{2A}{5} \, \frac{z_\circ^{5/2}}{z_\circ^{3/2}} = \frac{2A}{5} \, z_\circ \end{aligned}$$

We need to convert $z_\circ$ from km to cm, so $z_\circ = 5.26 \times 10^3 \text{ km} \times 10^5 \text{ cm/km} = 5.26 \times 10^8$ cm. Using this and the value for A (which is the opacity at $z_\circ$), we get

$$\tau = 0.4 \, (3.45 \times 10^{11} \text{ cm}^{-1}) \, (5.26 \times 10^8 \text{ cm}) = \boxed{7.26 \times 10^{19}} ,$$

which is an enormous optical depth (hence we are really in the optically thick regime).

(b) [10 points] At what depth (both in cm and in terms of $z_\circ$) does $\tau = 1$?

Solution (b): Once again, we use the optical depth definition equation but this time we integrate τ from 0 to 1 and now solve for z.

$$\begin{aligned}
\int_0^1 d\tau &= \int_0^z A \left(\frac{z}{z_\circ}\right)^{3/2} dz \\
1 - 0 &= \frac{A}{z_\circ^{3/2}} \int_0^z z^{3/2}\, dz \\
1 &= \frac{A}{z_\circ^{3/2}} \frac{2}{5}\, z^{5/2}\Big|_0^z = \frac{2A}{5} \frac{z^{5/2}}{z_\circ^{3/2}} \\
z^{5/2} &= \frac{5 z_\circ^{3/2}}{2A} \\
z &= \left(\frac{5 z_\circ^{3/2}}{2A}\right)^{2/5} = \left(\frac{5(5.26 \times 10^8 \text{ cm})^{3/2}}{2(3.45 \times 10^{11} \text{ cm}^{-1})}\right)^{2/5} \\
z &= \left(87.4 \text{ cm}^{5/2}\right)^{2/5} = \boxed{5.98 \text{ cm}} .
\end{aligned}$$

As such, optical depth unity is reached virtually immediately as we look into the gas — *this fog is thick a pea soup!*

Now to convert this into units of $z_\circ$,

$$z = \frac{5.98 \text{ cm}}{5.26 \times 10^8 \text{ cm}}\, z_\circ = \boxed{1.14 \times 10^{-8}\, z_\circ} .$$

(c) [10 points] Assume the gas is in thermal equilibrium at a temperature of 10,000 K, what is the emissivity (a number is requested here) for the gas at $z = z_\circ$ and $\lambda = 5000$ Å?

Solution (c): The emissivity ϵ_ν is related to the opacity κ_ν and source function S_ν by

$$\epsilon_\nu = \kappa_\nu\, S_\nu \ .$$

In thermal equilibrium, the source function is equal to the Planck function, so

$$\epsilon_\nu = \kappa_\nu\, B_\nu \ ,$$

as given in Eq. (I-23) in your notes. Since we need frequency and not wavelength here, we must first convert λ to ν:

$$\nu = \frac{c}{\lambda} = \frac{2.997925 \times 10^{10} \text{ cm/s}}{5000 \text{ Å } \times 10^{-8} \text{ cm/Å}} = 5.996 \times 10^{14} \text{ Hz.}$$

Now, for a gas at 10,000 K, the Planck function is equal to

$$\begin{aligned}
B_\nu &= \frac{2h\nu^3/c^2}{e^{h\nu/k_BT}-1} \\
&= \frac{2(6.6262\times10^{-27}\text{ erg s})(5.996\times10^{14}\text{ s}^{-1})^3/(2.997925\times10^{10}\text{ cm/s})^2}{e^{(6.6262\times10^{-27}\text{ erg s }\times 5.996\times10^{14}\text{ s}^{-1})/(1.3806\times10^{-16}\text{ erg/K }\times10^4\text{ K})}-1} \\
&= \frac{3.1784\times10^{-3}\text{ erg/s/cm}^2\text{/Hz/sr}}{e^{2.8777}-1} = \frac{3.1784\times10^{-3}\text{ erg/s/cm}^2\text{/Hz/sr}}{16.774} \\
&= 1.895\times10^{-4}\text{ erg/s/cm}^2\text{/Hz/sr.}
\end{aligned}$$

At $z = z_\circ$, the opacity is

$$\kappa = A\left(\frac{z_\circ}{z_\circ}\right)^{3/2} = A = 3.45\times10^{11}\text{ cm}^{-1},$$

so the emissivity is

$$\begin{aligned}
\epsilon_\nu &= \kappa_\nu B_\nu = (3.45\times10^{-11}\text{ cm}^{-1})(1.895\times10^{-4}\text{ erg/s/cm}^2\text{/Hz/sr}) \\
&= \boxed{6.538\times10^{7}\text{ erg/s/cm}^3\text{/Hz/sr}}\ .
\end{aligned}$$

(d) [10 points] What is the value of the source function (again a number is requested) at this temperature and wavelength? Define *source function* in words.

Solution (d): As shown above, the source function is just the Planck function for gas in thermal equilibrium, hence

$$\boxed{S_\nu = B_\nu = 1.895\times10^{-4}\text{ erg/s/cm}^2\text{/Hz/sr}}\ .$$

I will accept any one of the following two definitions of the source function:

- The ratio of the emissivity to the opacity.
- The ratio of the number of photon creation events to the number of photon destruction events.

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2. (60 pts) Three terms (see Table I-2 on page I-21 in your notes) exist in an atomic model. Each term is in a subshell that is less than half filled. Term 1 has the following characteristics: $S = 0$, $L = 0$ and even parity; Term 2 has $S = 0$, $L = 1$, and odd parity; finally, Term 3 has $S = 1$, $L = 2$, and odd parity. The partition function for the ion of this atom is equal to 1 and the partition function of the atom itself is equal to 4 for a gas at the Sun's effective temperature. The ionization energy of the atom is 8.67 eV.

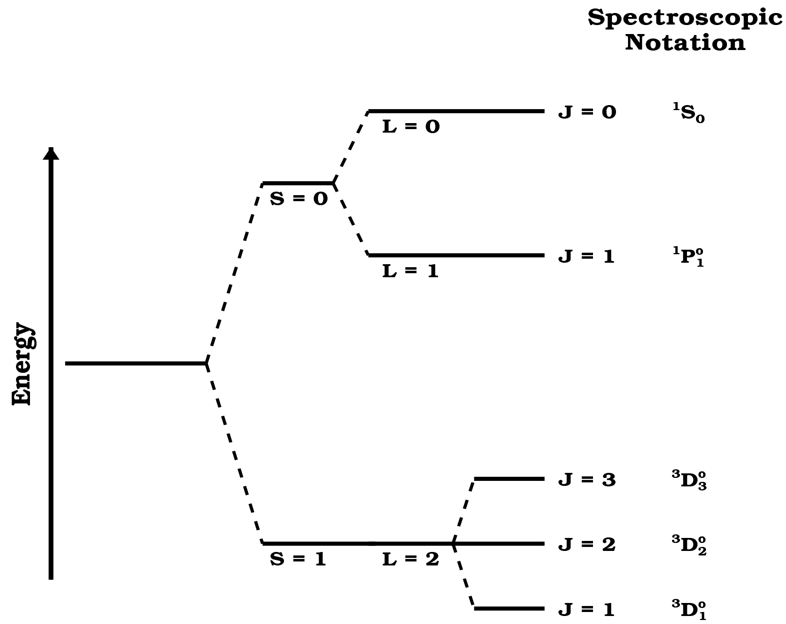
(a) [15 points] Write out the spectroscopic notation for all 5 levels (including J values) arising from these terms.

Solution (a): As shown in Table I-2 of your notes, a "Term" is a group of levels specified by L and $S \Longrightarrow$ lines that arise out of the same term are said to belong to the same multiplet. A term will have the number of levels as described by the number of J quantum vectors given by $|L - S| \leq J \leq L + S$. The spectroscopic notation is determined from ${}^{(2S+1)}L_J$. We will address each term separately:

- Term 1: Even parity, $S = 0$, $(2S + 1 = 1)$, $L = 0$, (an 'S' state), $|L - S| = 0$, $L + S = 0$, hence $J = 0$. $\therefore$ $\boxed{{}^1\mathrm{S}_0}$.
- Term 2: Odd parity, $S = 0$, $(2S + 1 = 1)$, $L = 1$, (a 'P' state), $|L - S| = 1$, $L + S = 1$, hence $J = 1$. $\therefore$ $\boxed{{}^1\mathrm{P}^\circ_1}$.
- Term 3: Odd parity, $S = 1$, $(2S + 1 = 3)$, $L = 2$, (a 'D' state), $|L - S| = 1$, $L + S = 3$, hence $J = 1, 2, 3$. $\therefore$ $\boxed{{}^3\mathrm{D}^\circ_1 \,,\, {}^3\mathrm{D}^\circ_2 \,,\, {}^3\mathrm{D}^\circ_3}$.

(b) [15 points] Draw an energy level diagram similar to the one in Figure I-6 on page I-20 of your course notes and locate each level with respect to the other levels using Hund's Rule.

Solution (b): We are told from the start that the subshell that each term arises from is less than half filled. As such, Hund's Rules are (1) the higher the S-value, the lower the energy, (2) the higher the L-value (for this S-value), the lower the energy, and (3) the higher the J-value, the higher then energy. The figure on the next page shows the energy diagram that results from these rules for these terms.



(c) [15 points] Determine which of the 7 transitions from these levels are allowed, semi-forbidden, and forbidden. List the reasons for why they are considered as such. (Note that in addition to what is given in your notes, forbidden transitions also occur if either ΔS and ΔJ or ΔL and ΔJ are violated.)

Solution (c): The allowed transitions are ones that obey all of the Russell-Saunders selection rules. Violation of these rules will result in either semi-forbidden or forbidden transitions. The table below tallies the results for all the transitions of this model.

Type	**Transition**	**LS Rules Violated**
Allowed	$^1S_0 \rightarrow {}^1P^o_1$	all rules obeyed
Semi-forbidden	none	no transitions are semi-forbidden
Forbidden	$^1P^o_1 \rightarrow {}^3D^o_1$	no change in parity
	$^1P^o_1 \rightarrow {}^3D^o_2$	no change in parity
	$^1P^o_1 \rightarrow {}^3D^o_3$	parity, ΔS and ΔJ
	$^1S_0 \rightarrow {}^3D^o_1$	ΔS and ΔL
	$^1S_0 \rightarrow {}^3D^o_2$	ΔS and ΔL
	$^1S_0 \rightarrow {}^3D^o_3$	ΔS and ΔL

(d) [15 points] If the ratio of the ion number density to the neutral atom number density is 4.56×10^{-2} at the continuum formation depth in the Sun's atmosphere, what is the electron density at this depth?

Solution (d): This is just an ionization equilibrium question where we use the Saha equation as given in Eq. (I-30) of your notes. Since we have a neutral atom, $k = 1$, so the Saha equation becomes

$$\frac{N_2}{N_1} n_e = 2 \frac{u_2}{u_1} \frac{(2\pi m k_B T)^{3/2}}{h^3} e^{-\chi_1/k_B T} ,$$

where $u_1 = 4$ is the partition function value for the neutral atom (as given in the original question), $u_2 = 1$ is the partition function of the ion, and finally, $\chi_1 = 8.67$ eV $\times 1.602 \times 10^{-12}$ erg/eV $= 1.39 \times 10^{-11}$ erg. At the continuum formation depth, the Sun's temperature is just its effective temperature (5770 K), so $k_B T = (1.3806 \times 10^{-16}\,\mathrm{erg/K})(5770\,\mathrm{K}) = 7.966 \times 10^{-13}$ erg. Using the following values for our constants, m (the mass of the electron) $= 9.10956 \times 10^{-28}$ gm, $k_B = 1.3806 \times 10^{-16}$ erg/K, and $h = 6.6262 \times 10^{-27}$ erg s, we can solve the Saha equation for this temperature and ionization energy:

$$
\begin{aligned}
\frac{N_2}{N_1} n_e &= 2\frac{1}{4}\frac{[2\pi(9.10956 \times 10^{-28}\ \mathrm{gm})(1.3806 \times 10^{-16}\ \mathrm{erg/K})(5770\ \mathrm{K})]^{3/2}}{(6.6262 \times 10^{-27}\ \mathrm{erg\ s})^3} \times \\
&\qquad e^{-(1.39\times 10^{-11}\ \mathrm{erg})/(7.966\times 10^{-13}\ \mathrm{erg})} \\
&= \frac{1}{2}\frac{(4.560 \times 10^{-39}\ \mathrm{erg\ gm})^{3/2}}{2.9093 \times 10^{-79}\ \mathrm{erg^3 s^3}} e^{-17.449} \\
&= 0.5\,\frac{3.079 \times 10^{-58}\ \mathrm{erg^{3/2} gm^{3/2}}}{2.9093 \times 10^{-79}\ \mathrm{erg^3 s^3}} \times 2.6420 \times 10^{-8} \\
&= 1.3979 \times 10^{13}\ \mathrm{erg^{-3/2} gm^{3/2} s^{-3}} = 1.3979 \times 10^{13}\ (\mathrm{gm\ cm^2\ s^{-2}})^{-3/2}\mathrm{gm^{3/2} s^{-3}} \\
&= 1.3979 \times 10^{13}\ \mathrm{cm^{-3}}\ .
\end{aligned}
$$

We are told that $N_2/N_1 = 4.56 \times 10^{-2}$. Using this fact in the above equation, we can solve for the electron density n_e:

$$
\begin{aligned}
n_e &= \frac{1.3979 \times 10^{13}\ \mathrm{cm^{-3}}}{N_2/N_1} = \frac{1.3979 \times 10^{13}\ \mathrm{cm^{-3}}}{4.56 \times 10^{-2}} \\
&= \boxed{3.066 \times 10^{14}\ \mathrm{cm^{-3}}}\ .
\end{aligned}
$$