ASTR-3415-001: Astrophysics
Solutions to Problem Set 1
Spring 2003

1. (40 pts total) Wien's Displacement Law.

   a) (24 pts) Derive Wien's Displacement Law from Planck's Law. Show **all** steps in your solution! (Hint: Note that at stellar temperatures, the peak flux of a blackbody occurs at either ultraviolet, optical, or near-infrared wavelengths. As such, $hc/\lambda \gg kT$, so $e^{hc/\lambda kT} \gg 1$. As such, you can simplify your derivative by ignoring the '1' in the denominator of the Planck function when deriving your solution. Note that even though this solution represents an approximation, it still gives the correct constant as compared to the exact solution derived numerically.)

Solution (a): Planck's Law:

$$B_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} \; .$$

At UV, optical, and near-IR wavelengths for typical stellar temperatures, $hc/\lambda \gg k_B T$, so $e^{hc/\lambda k_B T} \gg 1$, hence Planck's Law becomes

$$B_\lambda = \frac{2hc^2}{\lambda^5} \exp(-hc/\lambda k_B T) \; ,$$

and we can use this approximation to derive Wien's Law. $\lambda_{\rm max}$ occurs where $dB_\lambda/d\lambda = 0$, so

$$\begin{aligned} \frac{dB_\lambda}{d\lambda} = 2hc^2 \left(\frac{1}{\lambda^5} \frac{d}{d\lambda} e^{-hc/\lambda k_B T} + e^{-hc/\lambda k_B T} \frac{d}{d\lambda} \lambda^{-5} \right) &= 0 \\ \lambda^{-5} \left(\frac{hc}{\lambda^2 k_B T} \, e^{-hc/\lambda k_B T} \right) - 5\lambda^{-6} \, e^{-hc/\lambda k_B T} &= 0 \end{aligned}$$

or

$$\begin{aligned} \lambda^{-5} \left(\frac{hc}{\lambda^2 k_B T} \, e^{-hc/\lambda k_B T} \right) &= 5\lambda^{-6} \, e^{-hc/\lambda k_B T} \\ \frac{hc}{\lambda^7 k_B T} &= \frac{5}{\lambda^6} \qquad (\text{note } \lambda = \lambda_{\rm max} \text{ here}) \; . \end{aligned}$$

Finally,

$$\lambda_{\rm max} = \frac{hc}{5 k_B T} = \frac{(6.662620 \times 10^{-27} \text{ erg s})\,(2.9979 \times 10^{10} \text{ cm/s})}{5\,(1.38062 \times 10^{-16} \text{ erg/K})\, T}$$

or

$$\boxed{\lambda_{\rm max} = \frac{0.2898 \text{ cm K}}{T}} \quad \text{in cgs units.}$$

In units of Ångstroms (1 Å = 10^{-8} cm):

$$\boxed{\lambda_{\rm max} = \frac{2.898\times 10^{7}\ \text{Å K}}{T}}\,.$$

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b) (16 pts) Calculate the wavelength (in Å) of peak light emission for the stars Spica (B1 V, $T_{\rm eff}$ = 25,400 K), Vega (A0 V, $T_{\rm eff}$ = 9600 K), the Sun (G2 V, $T_{\rm eff}$ = 5770 K), and UU Aur (N2 II, $T_{\rm eff}$ = 2500 K). In what band of the electromagnetic spectrum (*i.e.*, X-ray, ultraviolet, etc.) does each of these maximum emissions correspond?

Solution (b):

$$\begin{aligned}
\text{Spica:}\quad \lambda_{\rm max} &= \frac{2.898\times 10^{7}\ \text{Å K}}{25,400\ \text{K}} = \boxed{1141\ \text{Å}\qquad \text{(FUV = far-UV)}}\,.\\
\text{Vega:}\quad \lambda_{\rm max} &= \frac{2.898\times 10^{7}\ \text{Å K}}{9600\ \text{K}} = \boxed{3019\ \text{Å}\qquad \text{(NUV = near-UV)}}\,.\\
\text{Sol:}\quad \lambda_{\rm max} &= \frac{2.898\times 10^{7}\ \text{Å K}}{5770\ \text{K}} = \boxed{5023\ \text{Å}\qquad \text{(yellow optical)}}\,.\\
\text{UU Aur:}\quad \lambda_{\rm max} &= \frac{2.898\times 10^{7}\ \text{Å K}}{2500\ \text{K}} = \boxed{11{,}590\ \text{Å}\qquad \text{(NIR = near-IR)}}\,.
\end{aligned}$$

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2. (25 pts) Derive the Stefan-Boltzmann Law from Planck's Law. Show **<u>all</u>** steps in your solution! (Hint: You will need get the integral in the functional form of $\int_0^\infty \frac{x^3}{e^x-1}$ and use the relation:

$$\int_0^\infty \frac{x^{n-1}}{e^x-1}\,dx = \Gamma(n)\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\right).$$

Finally, note that the series

$$\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right) = \frac{\pi^4}{90}$$

and that $\Gamma(n) = (n-1)!$, where $\Gamma(n)$ is the Gamma Function. Unlike the solution to Wien's Displacement Law, where you take a derivative over λ, for the Stefan-Boltzmann Law you will have to integrate over ν to get the integral of the Planck function in the form mentioned above.)

Solution: Planck's Law:

$$B_\nu = \frac{2h\nu^3}{c^2}\,\frac{1}{e^{h\nu/k_BT}-1}\ .$$

Derive Stefan-Boltzmann's Law:

$$\begin{aligned}
\mathcal{F} = \pi\, B = \pi \int_0^\infty B_\nu\, d\nu &= \pi \int_0^\infty \frac{2h\nu^3}{c^2}\,\frac{1}{e^{h\nu/k_BT}-1}\, d\nu \\
\mathcal{F} &= \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3}{e^{h\nu/k_BT}-1}\, d\nu\ .
\end{aligned}$$

Let $y = \dfrac{h\nu}{k_BT}$, then $dy = \dfrac{h}{k_BT}\, d\nu$, or

$$\nu = \frac{k_BT}{h}\, y \qquad d\nu = \frac{k_BT}{h}\, dy\ ,$$

so

$$\begin{aligned}
\mathcal{F} &= \frac{2\pi h}{c^2} \int_0^\infty \frac{(k_BT/h)^3\ y^3}{e^y-1} \left(\frac{k_BT}{h}\right) dy \\
&= \frac{2\pi h}{c^2} \left(\frac{k_BT}{h}\right)^4 \int_0^\infty \frac{y^3}{e^y-1}\, dy\ .
\end{aligned}$$

As given to you in the question on the previous page

$$\int_0^\infty \frac{x^{n-1}}{e^x-1}\, dx = \Gamma(n)\, \left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \cdots\right)\ ,$$

which is from Schaum's *Mathematical Handbook*, page 98, Eq. 15.80.

For our case, $n = 4$ and $\Gamma(n) = (n-1)!$, so

$$\Gamma(4) = (4-1)! = 3! = 3 \cdot 2 \cdot 1 = 6$$

and

$$\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right) = \frac{\pi^4}{90}\ ,$$

which is Eq. 19.20 from Schaum's *Mathematical Handbook*.

So,

$$\begin{aligned}
\mathcal{F} &= \frac{2\pi k_B^4 T^4}{h^3c^2} \int_0^\infty \frac{y^3}{e^y-1}\, dy \\
&= \frac{2\pi k_B^4 T^4}{h^3c^2}\, (6) \left(\frac{\pi^4}{90}\right) = \frac{2\pi k_B^4 T^4}{h^3c^2}\, \frac{\pi^4}{15} \\
&= \frac{2\pi^5 k_B^4}{15h^3c^2}\, T^4\ .
\end{aligned}$$

Finally, let

$$\sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2} = \frac{2\pi^5 (1.38062 \times 10^{-16} \text{ erg K}^{-1})^4}{15(6.62620 \times 10^{-27} \text{ erg s})^3 (2.9979 \times 10^{10} \text{ cm/s})^2}$$
$$= 5.6697 \times 10^{-5} \text{ erg/cm}^2\text{/s/K}^4 \; ,$$

hence

$$\boxed{\mathcal{F} = \sigma T^4} \; .$$

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3. (20 pts) Prove for an isotropic radiation field that $F_\nu = 0$.

Solution: From Eq. (I-5) of the course notes,

$$\pi F_\nu = \int_0^{2\pi} \int_{-1}^{1} I_\nu(\mu, \phi)\, \mu \, d\mu \, d\phi \; .$$

If the radiation field is isotropic, I_ν is independent of θ (hence $\mu = \cos\theta$) and ϕ, so

$$\pi F_\nu = I_\nu \int_0^{2\pi} \int_{-1}^{1} \mu \, d\mu \, d\phi \; .$$

Note that

$$\int_0^{2\pi} d\phi = \phi|_0^{2\pi} = 2\pi \; ,$$

so

$$\pi F_\nu = 2\pi I_\nu \int_{-1}^{1} \mu \, d\mu$$

and

$$\int_{-1}^{1} \mu \, d\mu = \frac{1}{2}\mu^2\Big|_{-1}^{1} = \frac{1}{2}\left[1^2 - (-1)^2\right] = \frac{1}{2}(1-1) = 0 \; .$$

So, $\pi F_\nu = 2\pi I_\nu \cdot 0 = 0$,

$$\therefore \qquad \boxed{F_\nu = 0} \; .$$

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4. (10 pts) At what distance from a 100-watt light bulb is the radiant flux equal to the solar constant?

Solution: The solar constant is $f_\odot = 1.360 \times 10^6$ erg/s/cm^2 as given in Example 3.2 on page 67 in your textbook. The bulb produces light at a power of 100 W, where 1 W = 1 J/s. Since 1 J (Joule) = 10^7 erg, this light bulb has power $P_B = 10^9$ erg/s.

By looking at the units, it is obvious that power and luminosity L are one in the same thing (see Example 3.2 in the textbook). Integrated flux is related to luminosity via:

$$f_B = \frac{L_B}{4\pi r^2} = \frac{P_B}{4\pi r^2} \ .$$

Here, we want the distance r where $f_B = f_\odot$, so

$$\begin{aligned} f_\odot &= f_B = \frac{P_B}{4\pi r^2} \\ r^2 &= \frac{P_B}{4\pi f_\odot} = \frac{10^9 \text{ erg/s}}{4\pi \times 1.360 \times 10^6 \text{ erg/s/cm}^2} \\ &= 58.5 \text{ cm}^2 \end{aligned}$$

or

$$\boxed{r = 7.65 \text{ cm}} \ .$$

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5. (35 pts) Consider a model of a star consisting of a spherical blackbody with a surface temperature of 28,000 K and a radius of 5.16×10^{11} cm. Let this model star be located at a distance of 180 pc from Earth. Determine the following for the star: (a) Luminosity. (b) Absolute bolometric magnitude. (c) Apparent bolometric magnitude. (d) Distance modulus. (e) Radiant flux at the star's surface. (f) Radiant flux at the Earth's surface (actually at the top of the Earth's atmosphere and compare this to the solar constant). (g) Peak wavelength λ_{max}. This is a model of the star Dschubba, the center star in the head of the constellation Scorpius.

To answer these questions, let's first write down some information about the Sun: $R_\odot = 6.9598 \times 10^{10}$ cm, $T_\odot = 5770$ K, $L_\odot = 3.8268 \times 10^{33}$ erg/s, $M_{\text{bol}}(\odot)$ and the solar constant is $f_\odot = 1.360 \times 10^6$ erg/s/cm^2.

Solution (a): The luminosity of a is just the amount of integrated flux integrated over the complete surface of the star. Assuming the star emits light as a spherical blackbody, we can set up a ratio with respect to the Sun:

$$\begin{aligned} L &= (4\pi R^2)(\sigma T^4) \\ \frac{L}{L_\odot} &= \frac{4\pi R^2}{4\pi R_\odot^2} \frac{\sigma T^4}{\sigma T_\odot^4} \\ &= \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4 \\ &= \left(\frac{5.16 \times 10^{11} \text{ cm}}{6.9598 \times 10^{10} \text{ cm}}\right)^2 \left(\frac{28,000 \text{ K}}{5770 \text{ K}}\right)^4 = (7.414)^2 (4.853)^4 = 3.048 \times 10^4 \\ L &= 3.048 \times 10^4 \ L_\odot = \boxed{1.2 \times 10^{38} \text{ erg/s}} \end{aligned}$$

Solution (b): The bolometric magnitude is related to the luminosity by

$$\begin{aligned} M_{\rm bol} - M_{\rm bol}(\odot) &= -2.5\log\left(\frac{L}{L_\odot}\right) = -2.5\log\left(\frac{1.2\times10^{38}\ \rm erg/s}{3.8268\times10^{33}\ \rm erg/s}\right) \\ &= -2.5\log(3.048\times10^{4}) = -11.2 \\ M_{\rm bol} &= -11.2 + M_{\rm bol}(\odot) = -11.2 + 4.76 = \boxed{-6.45} \end{aligned}$$

Solution (c,d): We need to answer the question to (d) before we can answer the (c) question. The distance modulus is

$$m_{\rm bol} - M_{\rm bol} = 5\log d - 5 = 5\log(180) - 5 = \boxed{6.28}$$

From this and the answer in (b), we get the apparent bolometric magnitude of

$$m_{\rm bol} = 6.28 + M_{\rm bol} = 6.28 - 6.45 = \boxed{-0.17}$$

Solution (e): The radiant flux at the stellar surface is given by

$$F = \frac{L}{4\pi R^2} = \frac{1.166\times10^{38}\ \rm erg/s}{4\pi(5.15^{11}\ \rm cm)^2} = \boxed{3.48\times10^{13}\ \rm erg/s/cm^2}$$

Note that we also could have determined this from the Stefan-Boltzmann Law:

$$F = \sigma T^4 = 5.67\times10^{5}\ \rm erg/s/cm^2/K^4\ (28,000\ K)^4 = \boxed{3.48\times10^{13}\ \rm erg/s/cm^2}$$

Solution (f): To determine the radiant flux of this star at the top of the Earth's atmosphere, we need to convert the distance to the star from parsecs to centimeters: $r = 180$ pc $\times 3.0856\times10^{18}$ cm/pc $= 5.55\times10^{20}$ cm. Thus

$$f = \frac{L}{4\pi r^2} = \frac{1.166\times10^{38}\ \rm erg/s}{4\pi(5.55\times10^{20}\ \rm cm)^2} = \boxed{3.01\times10^{-5}\ \rm erg/s/cm^2}$$

Comparing this to the solar constant, we get

$$\frac{f}{f_\odot} = \frac{3.01\times10^{-5}\ \rm erg/s/cm^2}{1.360\times10^{6}\ \rm erg/s/cm^2} = \boxed{2.21\times10^{-11}}$$

Solution (g): Using Wien's Displacement Law we get:

$$\lambda_{\rm max} = \frac{0.290 \text{ cm K}}{T} = \frac{0.290 \text{ cm K}}{28,000 \text{ K}} = 1.036 \times 10^{-5} \text{ cm} \ = \boxed{1036 \text{ Å}}$$

which is in the ultraviolet.

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6. (20 pts) (a) Calculate the energies and wavelengths of all possible photons that are emitted when the electron cascades from the $n = 3$ to the $n = 1$ orbit of the hydrogen atom. (b) Find the shortest wavelength photon emitted by the downward electron transition in the Lyman, Balmer, and Paschen series. In which of the electromagnetic spectrum are these wavelengths found?

Solution (a): The transitions involved are 3→1, 3→2, and 2→1. For this problem, we will need $h = 6.6262 \times 10^{-27}$ erg s and $c = 2.9979 \times 10^{10}$ cm/s. Also, since we are dealing with hydrogen, $Z = 1$ and the Rydberg constant is 109,677.6 cm^{-1}. We will use the conversion factors of 1 cm = 10^8 Å and 1 eV = 1.602×10^{-12} erg. We will need Rydberg's formula (expressed in wavenumbers),

$$\frac{1}{\lambda_{mn}} = R_A Z^2 \left(\frac{1}{n^2} - \frac{1}{m^2} \right) \ , \quad m > n \ ,$$

and the Planck photon energy equation,

$$E_{mn} = \frac{hc}{\lambda_{mn}} \ .$$

For our 3 transitions, we get

$$\begin{aligned} 3 \rightarrow 1: \ \frac{1}{\lambda_{31}} &= (109,677.6 \text{ cm}^{-1}) \left(\frac{1}{1} - \frac{1}{9} \right) & \text{Lyman-}\beta \\ &= 97,491.2 \text{ cm}^{-1} \\ \lambda_{31} &= 1.0257 \times 10^{-5} \text{ cm} \ = & \boxed{1025.7 \text{ Å}} \\ E_{31} &= \frac{(6.6262 \times 10^{-27} \text{ erg s})(2.9979 \times 10^{10} \text{ cm/s})}{1.0257 \times 10^{-5} \text{ cm}} \\ &= 1.937 \times 10^{-11} \text{ erg} = & \boxed{12.1 \text{ eV}} \\ 2 \rightarrow 1: \ \frac{1}{\lambda_{21}} &= (109,677.6 \text{ cm}^{-1}) \left(\frac{1}{1} - \frac{1}{4} \right) & \text{Lyman-}\alpha \\ &= 82,258.2 \text{ cm}^{-1} \\ \lambda_{21} &= 1.2157 \times 10^{-5} \text{ cm} \ = & \boxed{1215.7 \text{ Å}} \\ E_{21} &= \frac{(6.6262 \times 10^{-27} \text{ erg s})(2.9979 \times 10^{10} \text{ cm/s})}{1.2157 \times 10^{-5} \text{ cm}} \\ &= 1.634 \times 10^{-11} \text{ erg} = & \boxed{10.2 \text{ eV}} \end{aligned}$$

$$
\begin{aligned}
3 \to 2: \quad \frac{1}{\lambda_{32}} &= (109,677.6 \text{ cm}^{-1}) \left(\frac{1}{4} - \frac{1}{9}\right) && \text{H}\alpha \\
&= 15,233.0 \text{ cm}^{-1} \\
\lambda_{32} &= 6.5647 \times 10^{-5} \text{ cm} \; = && \boxed{6564.7 \text{ Å}} \\
E_{32} &= \frac{(6.6262 \times 10^{-27} \text{ erg s})(2.9979 \times 10^{10} \text{ cm/s})}{6.3647 \times 10^{-5} \text{ cm}} \\
&= 3.026 \times 10^{-12} \text{ erg} = && \boxed{1.89 \text{ eV}}
\end{aligned}
$$

Solution (b): The ionization wavelength or series limit wavelength can be found from the Rydberg formula above by letting $m \to \infty$. Then we can write

$$
\lambda_n = \frac{n^2}{R_A} = \frac{n^2}{109,677.6} \text{ cm } = 911.76 \text{ Å } \times n^2 \; .
$$

As such,

$$
\begin{aligned}
\lambda_1 &= 911.76 \text{ Å } \times 1^2 = \boxed{911.76 \text{ Å}} && \text{Lyman limit} \to \text{UV} \\
\lambda_2 &= 911.76 \text{ Å } \times 2^2 = \boxed{3647.1 \text{ Å}} && \text{Balmer limit} \to \text{UV/visual border} \\
\lambda_3 &= 911.76 \text{ Å } \times 3^2 = \boxed{8205.9 \text{ Å}} && \text{Paschen limit} \to \text{infrared}
\end{aligned}
$$
