ASTR-3415-001: Astrophysics
Solutions to Problem Set 2
Spring 2003

1. (20 pts) Calculate the rate at which the Earth's orbital size changes as a result of the combined mass-loss from the solar wind and thermonuclear reactions. (*Hint:* $L_\odot = 3.83 \times 10^{33}$ erg/s.)

   Solution: The luminosity of a star is the measure of how much energy a star is emitting per unit time Δt. Hence, we can write

   $$L_\odot = 3.83 \times 10^{33} \, \frac{\text{erg}}{\text{s}} = \frac{E}{\Delta t} = \frac{mc^2}{\Delta t} \, ,$$

   where m is the mass being converted to energy. The mass loss due to thermonuclear burning via sunlight can then be expressed as

   $$\begin{aligned} \dot{M}_\odot(\text{nuc}) &= \frac{m}{\Delta t} = \frac{L_\odot t}{c^2 \Delta t} = \frac{L_\odot}{c^2} = \frac{3.83 \times 10^{33} \text{ erg/s}}{(3.00 \times 10^{10} \text{ cm/s})^2} \\ &= 4.26 \times 10^{12} \text{ gm/s } \times \frac{1 M_\odot}{1.99 \times 10^{33} \text{ gm}} \\ &= 2.14 \times 10^{-21} M_\odot/\text{s } \times \frac{3.16 \times 10^7 \text{ s}}{1 \text{ yr}} \\ &= 6.75 \times 10^{-14} M_\odot/\text{yr} \end{aligned}$$

   In the notes, we are given that the mass-loss rate of the Sun by the solar wind is $\dot{M}_\odot(\text{wind}) = 10^{-14} M_\odot/\text{yr}$ (although not an exact value, it's good enough for this problem). Combining these two mass-loss rates gives

   $$\dot{M} = \dot{M}_\odot(\text{nuc}) + \dot{M}_\odot(\text{wind}) = 7.75 \times 10^{-14} M_\odot/\text{yr}.$$

   Now use the conservation of mechanical energy for the Earth's orbit:

   $$\begin{aligned} \text{KE}_b + \text{PE}_b &= \text{KE}_a + \text{PE}_a \\ \frac{1}{2} M_\oplus v_{\oplus b}^2 - \frac{G M_\oplus M_{\odot b}}{r_{\oplus b}} &= \frac{1}{2} M_\oplus v_{\oplus a}^2 - \frac{G M_\oplus M_{\odot a}}{r_{\oplus a}} \, , \end{aligned}$$

   where the 'b'-subscript represents a *before* time, the 'a'-subscript represents an *after* time, $M_\oplus$ is the mass of the Earth (which remains constant here), and $r_\oplus$ is the distance that the Earth is from the Sun.

   Now if we assume that Earth's orbit is circular, then we can set the gravitational force equal to the centripetal (*i.e.*, *center-seeking*) force:

   $$\begin{aligned} F_\text{cent} &= F_\text{grav} \\ \frac{M_\oplus v_\oplus^2}{r_\oplus} &= \frac{G M_\oplus M_\odot}{r_\oplus^2} \\ v_\oplus^2 &= \frac{G M_\odot}{r_\oplus} \, . \end{aligned}$$

Using this expression for the orbital velocity in the conservation of energy equation, we get

$$\begin{aligned}
\frac{1}{2}\frac{GM_\oplus M_{\odot b}}{r_{\oplus b}} - \frac{GM_\oplus M_{\odot b}}{r_{\oplus b}} &= \frac{1}{2}\frac{GM_\oplus M_{\odot a}}{r_{\oplus a}} - \frac{GM_\oplus M_{\odot a}}{r_{\oplus a}} \\
\frac{1}{2}\frac{GM_{\odot a}}{r_{\oplus a}} &= \frac{1}{2}\frac{GM_{\odot b}}{r_{\oplus b}} \\
\frac{M_{\odot a}}{M_{\odot b}} &= \frac{r_{\oplus a}}{r_{\oplus b}} \ .
\end{aligned}$$

If we divide both sides of this equation by Δt, we get

$$\frac{M_{\odot a}/\Delta t}{M_{\odot b}} = \frac{r_{\oplus a}/\Delta t}{r_{\oplus b}} \ .$$

Now let 'b' represent the current epoch of the Earth-Sun system and assume that the mass-loss rate remains constant in time, then the above equation can be written as

$$\frac{\dot{M}_\odot}{M_\odot} = \frac{\dot{r}}{1 \text{ AU}} \ ,$$

or

$$\dot{r} = \frac{7.75 \times 10^{-14} M_\odot/\text{yr}}{1 M_\odot} \text{ AU } = 7.75 \times 10^{-14} \text{ AU/yr} \ .$$

Since 1 AU $= 1.5 \times 10^{13}$ cm,

$$\boxed{\dot{r} = 1.2 \text{ cm/yr.}}$$

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2. (20 pts) We observe the Ca II K line of a chromospherically active star and see that the K_1 features are separated by 2.1 Å. If the star has a bolometric correction of –1.0 and a effective temperature of 3800 K, what is the luminosity of this star and what is its complete MK classification? If this star has a magnitude $V = 2.6$, how far away is it?

Solution: We are given the following parameters: $\Delta\lambda = 2.1$ Å, BC $= -1.0$, $T_{\text{eff}} =$ 3800 K, $V = 2.6$, and we are asked to find L, the spectral-luminosity class, and d. We first need to change the separation of the K_1 features from wavelength to velocity. To do this, we will use the Doppler effect:

$$\frac{\Delta\lambda}{\lambda_\circ} = \frac{v}{c} = \frac{\omega}{c} \ ,$$

where ω is the velocity parameter used in the Wilson-Bappu effect. For Ca II K, the line-center wavelength is $\lambda_\circ = 3933.66$ Å, as such

$$\omega = \frac{\Delta\lambda}{\lambda_\circ} c = \left(\frac{2.1 \text{ Å}}{3933.66 \text{ Å}} \right) 3.00 \times 10^5 \text{ km/s } = 160 \text{ km/s} \ .$$

Plugging this into the Wilson-Bappu relation we get

$$M_V = -14.9 \log \omega + 27.6 = -14.9 \log 160 + 27.6 = -5.25 \ .$$

In order to figure out luminosity, we need the absolute bolometric magnitude. This is found with the absolute visual magnitude and the bolometric correction:

$$M_{\rm bol} = M_V + BC = -5.25 - 1.0 = -6.25 \ .$$

The absolute bolometric magnitude is related to the luminosity by

$$M_{\rm bol} = 4.72 - 2.5 \log \left(\frac{L}{L_\odot} \right) \ .$$

Solving this equation for the luminosity gives

$$\begin{aligned} 2.5 \log \left(\frac{L}{L_\odot} \right) &= 4.72 - M_{\rm bol} = 4.72 - (-6.25) = 10.97 \\ \log \left(\frac{L}{L_\odot} \right) &= 4.388 \\ \frac{L}{L_\odot} &= 10^{4.388} = 2.44 \times 10^4 \ , \end{aligned}$$

or

$$\boxed{L = 2.44 \times 10^4 L_\odot \ .}$$

Such a large luminosity for such a cool star implies that this is a supergiant. Using Appendix E in the textbook, a supergiant star at $T_{\rm eff} = 3800$ K gives a spectral class of approximately K5 (I will also accept K7, M0, or M1). Appendix E shows that supergiant stars of spectral type K5 have luminosities around 38,000 $L_\odot$ and that K5 giants are 220 times brighter than the Sun. If we use Table II-2 from the notes, we can compare M_V with those tabulated and we see that this star is likely of luminosity class Ib. As such, the complete Morgan-Keenan spectral-luminosity class for this star is

$$\boxed{\text{K5 Ib} \ .}$$

To determine the distance, we will ignore interstellar absorption and use the standard distance modulus formula:

$$\begin{aligned} m - M = V - M_V &= 5 \log \left(\frac{d}{10 \text{ pc}} \right) \\ 5 \log \left(\frac{d}{10 \text{ pc}} \right) &= V - M_V = 2.6 - (-5.25) = 7.85 \\ \log \left(\frac{d}{10 \text{ pc}} \right) &= 1.57 \\ \frac{d}{10 \text{ pc}} &= 37.2 \ , \end{aligned}$$

or

$$\boxed{d = 372 \text{ pc} .}$$

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3. (10 pts) Problems 8.1 and 8.2, Page 251 in the Carroll and Ostlie textbook.

Solution (a) – **Problem 8.1:** Room temperature is typically taken to be 300 K = 27°C = 81°F. Using this value gives

$$\begin{aligned} k_B T &= 1.38 \times 10^{-16} \text{ erg/K } \times 300 \text{ K } = 4.14 \times 10^{-14} \text{ erg} \\ &= 4.14 \times 10^{-14} \text{ erg } \times \frac{1 \text{ eV}}{1.60 \times 10^{-12} \text{ erg}} = 2.59 \times 10^{-2} \text{ eV} \\ &= \frac{1}{38.6} \text{ eV } \approx \boxed{\frac{1}{40} \text{ eV} .} \end{aligned}$$

To determine the temperature from the 'eV' energies, set up the following ratio:

$$\begin{aligned} \frac{k_B T}{k_B T_{\text{room}}} &= \frac{E}{1/40 \text{ eV}} = \frac{40E}{\text{eV}} \\ T &= \frac{40E}{\text{eV}} T_{\text{room}} = \frac{40E}{\text{eV}} \cdot 300 \text{ K } = 12,000E \text{ (K/eV)} . \end{aligned}$$

Now converting the two energies into temperatures gives

$$\begin{aligned} T_1 &= 12,000(1.00 \text{ eV}) \text{ K/eV } = \boxed{12,000 \text{ K} .} \\ T_2 &= 12,000(13.6 \text{ eV}) \text{ K/eV } = \boxed{1.63 \times 10^5 \text{ K} .} \end{aligned}$$

Solution (b) – **Problem 8.2:** Here, we just need to convert erg into eV in Boltzmann's constant. Using the values listed in Appendix A we get

$$\begin{aligned} k_B &= 1.380658 \times 10^{-16} \text{ erg/K } \cdot \left(\frac{1 \text{ eV}}{1.60217733 \times 10^{-12} \text{ erg}} \right) \\ &= \boxed{8.6174 \times 10^{-5} \text{ eV/K} .} \end{aligned}$$

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4. (20 pts) Problem 8.5, Page 251 in the Carroll and Ostlie textbook.

Solution: This is just a Boltzmann equation problem:

$$\frac{n_j}{n_i} = \frac{g_j}{g_i} e^{-E_{ji}/k_\mathrm{B}T} ,$$

where the j and i subscripts correspond to the upper and lower levels, respectively, n is the number density and g is the statistical weight of the levels, and E_{ji} is the energy difference between the two levels. Solving Boltzmann's equation for T gives

$$\begin{aligned} e^{-E_{ji}/k_\mathrm{B}T} &= \frac{g_i n_j}{g_j n_i} \\ e^{E_{ji}/k_\mathrm{B}T} &= \frac{g_j n_i}{g_i n_j} \\ \frac{E_{ji}}{k_\mathrm{B}T} &= \ln\left(\frac{g_j n_i}{g_i n_j}\right) \\ T &= \frac{E_{ji}}{k_\mathrm{B}} \frac{1}{\ln(g_j n_i / g_i n_j)} . \end{aligned}$$

For this problem, $j = 2$ and $i = 1$, and the energy difference between these two states is $E_{21} = 13.6$ eV -3.40 eV $= 10.2$ eV from the data of Table 8.2 on page 231 of the textbook. Since this is hydrogen, the statistical weights are

$$\begin{aligned} g_n &= 2n^2 \\ g_2 &= 2 \cdot 2^2 = 8 \\ g_1 &= 2 \cdot 1^2 = 2 \end{aligned}$$

where we use n instead of i, j in hydrogen and the ratio of these two weights is $g_2/g_1 = 8/2 = 4$. Using these values in the temperature equation above, we get

$$\begin{aligned} T &= \frac{10.2 \text{ eV}}{8.6174 \times 10^{-5} \text{ eV/K}} \frac{1}{\ln(4n_1/n_2)} \\ &= 1.18 \times 10^5 \text{ K} \left(\frac{1}{1.386 \ln(n_1/n_2)}\right) = \frac{85,400 \text{ K}}{\ln(n_1/n_2)} . \end{aligned}$$

For the first ratio, $n_2/n_1 = 0.01$, so $n_1/n_2 = 100$, and for the second ratio, $n_2/n_1 = 0.1$, so $n_1/n_2 = 10$. Plugging these values into the equation above, we get

$$\begin{aligned} T_1 &= \frac{85,400 \text{ K}}{\ln(100)} = \boxed{19,800 \text{ K}} \\ T_2 &= \frac{85,400 \text{ K}}{\ln(10)} = \boxed{32,100 \text{ K}} \end{aligned}$$

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5. (20 pts) Problem 8.7, Page 251 in the Carroll and Ostlie textbook.

Solution: The equation for the partition function (following the notation of the textbook) is given by Eq. 8.5 on page 233 of the textbook:

$$Z = g_1 + \sum_{j=2}^{\infty} g_j\, e^{-(E_j - E_1)/k_{\rm B}T} \ .$$

For hydrogen, $g_n = 2n^2$ and $E_n = \dfrac{-13.6 \text{ eV}}{n^2}$, where n is used for the level ID in hydrogen instead of j. Keeping only the first 3 terms in the partition function summation, we get

$$Z = g_1 + g_2\, e^{-(E_2 - E_1)/k_{\rm B}T} + g_3\, e^{-(E_3 - E_1)/k_{\rm B}T} \ .$$

For $T = 10{,}000$ K, $k_{\rm B}T = (8.6174 \times 10^{-5} \text{ eV/K})(10^4 \text{ K}) = 0.86132$ eV. The statistical weights are

$$g_1 = 2 \cdot 1^2 = 2, \quad g_2 = 2 \cdot 2^2 = 8, \quad g_3 = 2 \cdot 3^2 = 18 \ ,$$

and the level energies are

$$E_1 = \tfrac{-13.6 \text{ eV}}{1^2} = -13.6 \text{ eV}, \quad E_2 = \tfrac{-13.6 \text{ eV}}{2^2} = -3.40 \text{ eV}, \quad E_3 = \tfrac{-13.6 \text{ eV}}{3^2} = -1.51 \text{ eV},$$

$$E_2 - E_1 = 10.2 \text{ eV}, \qquad E_3 - E_1 = 12.1 \text{ eV} \ .$$

Using these values in the 3-term partition function summation, we get

$$\begin{aligned} Z &= 2 + 8e^{-10.2 \text{ eV}/0.86132 \text{ eV}} + 18e^{-12.1 \text{ eV}/0.86132 \text{ eV}} \\ &= 2 + 5.76 \times 10^{-5} + 1.44 \times 10^{-5} \\ &= 2.000072 \approx 2 = g_1 \quad \surd \end{aligned}$$

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6. (30 pts) Problem 8.10, Page 252 in the Carroll and Ostlie textbook.

Solution (a): Whereas Problem #4 dealt with the Boltzmann equation, this problem deals with the Saha equation. For He, we are given

$$\chi_{\rm I} = \chi_1 = 24.6 \text{ eV}, \qquad \chi_{\rm II} = \chi_2 = 54.4 \text{ eV},$$

$$Z_{\rm I} = Z_1 = 1, \quad Z_{\rm II} = Z_2 = 2, \quad Z_{\rm III} = Z_3 = 1,$$

where χ is the ionization energy and Z is the partition function. The electron pressure is given as $P_e = 200$ dyne/cm^2. We now use Eq. 8.7 to find $N_{\rm II}/N_{\rm I}$ and $N_{\rm III}/N_{\rm II}$ for the temperatures of 5000 K, 15,000 K, and 25,000 K:

$$\frac{N_{i+1}}{N_i} = \frac{2k_{\rm B}T Z_{i+1}}{P_e Z_i} \left(\frac{2\pi m_e k_{\rm B}T}{h^2}\right)^{3/2} e^{-\chi_i/k_{\rm B}T} \ ,$$

where i is the stage of ionization. We start by evaluating this equation keeping T and Z as free parameters:

$$\frac{2k_{\rm B}}{P_e} = \frac{2 \cdot 1.38 \times 10^{-16}\ {\rm erg/K}}{200\ {\rm dyne/cm^2}} = 1.38 \times 10^{-18}\ {\rm cm^3}/K\ ,$$

$$\frac{2\pi m_e k_{\rm B}}{h^2} = \frac{2\pi(9.11 \times 10^{-28}\ {\rm gm})(1.38 \times 10^{-16}\ {\rm erg/K})}{(6.63 \times 10^{-27}\ {\rm erg\ s})^2} = 1.80 \times 10^{10}\ {\rm cm^{-2}\ K^{-1}}\ ,$$

$$\left(\frac{2\pi m_e k_{\rm B}}{h^2}\right)^{3/2} = \left(1.80 \times 10^{10}\ {\rm cm^{-2}\ K^{-1}}\right)^{3/2} = 2.41 \times 10^{15}\ {\rm cm^{-3}\ K^{-3/2}}\ .$$

Using the ionization energies for our two ratios, we get

$$e^{-\chi_1/k_{\rm B}T} = e^{-2.85\times10^5\ {\rm K}/T}, \qquad e^{-\chi_2/k_{\rm B}T} = e^{-6.31\times10^5\ {\rm K}/T}\ .$$

Combining all of these parameters together, we can write the Saha equation as

$$\frac{N_{i+1}}{N_i} = 3.33 \times 10^{-3}\ {\rm K}^{-5/2}\ \frac{Z_{i+1}}{Z_i}\, T^{5/2}\, e^{-\chi_i/k_{\rm B}T}\ .$$

Now for the number density ratio of the first ionization stage to the neutral stage we get:

$$\begin{aligned}\frac{N_{\rm II}}{N_I} = \frac{N_2}{N_1} &= 3.33 \times 10^{-3}\ {\rm K}^{-5/2}\ \frac{Z_2}{Z_1}\, T^{5/2}\, e^{-2.85\times10^5\ {\rm K}/T}\\ &= 3.33 \times 10^{-3}\ {\rm K}^{-5/2}\ \frac{2}{1}\, T^{5/2}\, e^{-2.85\times10^5\ {\rm K}/T}\\ &= 6.66 \times 10^{-3}\ {\rm K}^{-5/2}\ T^{5/2}\, e^{-2.85\times10^5\ {\rm K}/T}\ .\end{aligned}$$

We now tabulate this ratio for our 3 temperatures:

T	$N_{\rm II}/N_{\rm I}$
5000 K	2.07×10^{-18}
15,000 K	1.03
25,000 K	7370

Now for the number density ratio of the second ionization stage to the first ionization stage we get:

$$\begin{aligned}\frac{N_{\rm III}}{N_{II}} = \frac{N_3}{N_2} &= 3.33 \times 10^{-3}\ {\rm K}^{-5/2}\ \frac{Z_3}{Z_2}\, T^{5/2}\, e^{-6.32\times10^5\ {\rm K}/T}\\ &= 3.33 \times 10^{-3}\ {\rm K}^{-5/2}\ \frac{1}{2}\, T^{5/2}\, e^{-6.32\times10^5\ {\rm K}/T}\\ &= 1.67 \times 10^{-3}\ {\rm K}^{-5/2}\ T^{5/2}\, e^{-6.32\times10^5\ {\rm K}/T}\ .\end{aligned}$$

We now tabulate this ratio for our 3 temperatures:

T	$N_{\rm II}/N_{\rm I}$
5000 K	3.76×10^{-49}
15,000 K	2.32×10^{-11}
25,000 K	1.73×10^{-3}

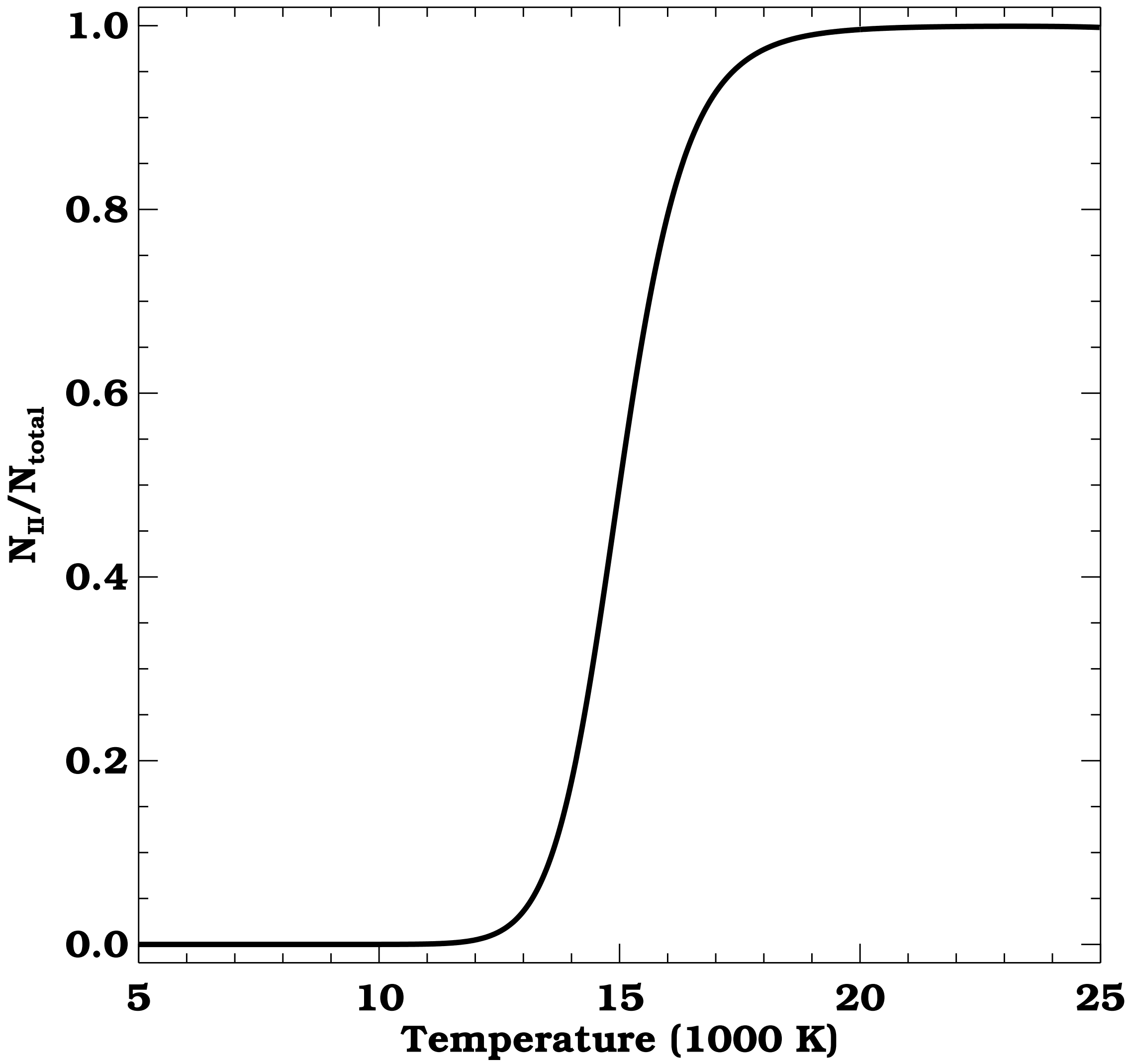
Solution (b): Let's start by working with the reciprocal of what we are trying to prove:

$$\frac{N_{\rm tot}}{N_{\rm II}} = \frac{N_{\rm I} + N_{\rm II} + N_{\rm III}}{N_{\rm II}} = \frac{N_{\rm I}}{N_{\rm II}} + \frac{N_{\rm II}}{N_{\rm II}} + \frac{N_{\rm III}}{N_{\rm II}}$$
$$= 1 + \frac{N_{\rm III}}{N_{\rm II}} + \frac{1}{(N_{\rm II}/N_{\rm I})} \, .$$

Now taking the reciprocal of this equation, we get

$$\boxed{\frac{N_{\rm II}}{N_{\rm tot}} = \left[1 + \frac{N_{\rm III}}{N_{\rm II}} + \frac{1}{(N_{\rm II}/N_{\rm I})}\right]^{-1} \, .}$$

Solution (c): For this portion of the problem, I have written an IDL procedure that makes use of the equations of Part (a) and Part (b) of this problem to produce a graph similar to that shown in Fig. 8.6 on page 235 in the textbook. A hardcopy of the IDL procedure (*i.e.*, hw3415ion.pro) is attached to this solution set (if you are retrieving these solutions from the web, click on the link for this file on the course web page). The plot is shown on the next page. The ratio $N_{\rm II}/N_{\rm tot} = 0.5$ at approximately 15,000 K as shown in the plot.



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7. (20 pts) Problem 9.4, Page 307 in the Carroll and Ostlie textbook.

Solution: Using the Planck function from Eq. (I-15) in the notes, the radiation pressure can be evaluated:

$$P_{\rm rad} = \frac{4\pi}{3c} \int_0^\infty \frac{2hc^2/\lambda^5}{e^{hc/\lambda k_{\rm B}T} - 1}\, d\lambda \ .$$

Let

$$x = \frac{hc}{\lambda k_{\rm B} T} \quad \Rightarrow \quad \lambda = \frac{hc}{x k_{\rm B} T}$$

$$\Downarrow$$

$$dx = -\frac{hc}{\lambda^2 k_{\rm B} T}\, d\lambda \quad \Rightarrow \quad d\lambda = -\lambda^2\, \frac{k_{\rm B}T}{hc}\, dx \ ,$$

then we can rewrite the numerator of the integrand as

$$\frac{2hc^2}{\lambda^5} = \frac{2hc^2}{(hc/xk_{\rm B}T)^5} = \frac{2x^5k_{\rm B}^5T^5}{h^4c^3} \ .$$

Now substituting this expression into the integral and replacing the $d\lambda$ differential, and replacing all additional λ terms with its x equivalents, we get (realizing that as λ goes from 0 to ∞, x goes from ∞ to 0)

$$\begin{aligned} P_{\rm rad} &= -\frac{4\pi}{3c}\int_{\infty}^{0}\frac{2k_{\rm B}^5T^5}{h^4c^3}\,x^5\,\frac{1}{e^x-1}\left(\frac{k_{\rm B}T}{hc}\right)\lambda^2\,dx \\ &= +\frac{4\pi}{3c}\cdot\frac{2k_{\rm B}^5T^5}{h^4c^3}\cdot\frac{k_{\rm B}T}{hc}\int_0^{\infty}\frac{x^5}{e^x-1}\left(\frac{hc}{xk_{\rm B}T}\right)^2\,dx \\ &= \frac{8\pi}{3}\cdot\frac{k_{\rm B}^6T^6}{h^5c^5}\cdot\frac{h^2c^2}{k_{\rm B}^2T^2}\int_0^{\infty}\frac{x^3}{e^x-1}\,dx \\ &= \frac{8\pi}{3}\cdot\frac{k_{\rm B}^4T^4}{h^3c^3}\int_0^{\infty}\frac{x^3}{e^x-1}\,dx \ . \end{aligned}$$

As we have seen on numerous occasions in this course, integrals of this form can be solved from

$$\int_0^{\infty}\frac{x^{n-1}}{e^x-1}\,dx = \Gamma(n)\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\right) \ ,$$

where the Gamma function can be determined from

$$\Gamma(n) = (n-1)!$$

if n is an integer. For our problem here, $n = 4$ and $\Gamma(4) = 3! = 6$. Also, the series we get from this integer takes the form

$$\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right) = \frac{\pi^4}{90} \ ,$$

as found in many mathematical handbooks. As such, the solution to our integral is

$$\int_0^{\infty}\frac{x^3}{e^x-1}\,dx = \Gamma(4)\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right) = 6\cdot\frac{\pi^4}{90} = \frac{\pi^4}{15} \ .$$

Using this in the radiation pressure integral equation, we get

$$P_{\rm rad} = \frac{8\pi}{3}\,\frac{\pi^4}{15}\,\frac{k_{\rm B}^4T^4}{h^3c^3} = \frac{2\pi^5k_{\rm B}^4}{15h^3c^2}\cdot\frac{4T^4}{3c} \ .$$

Now, the Stefan-Boltzmann constant is defined by

$$\sigma = \frac{2\pi^5k_{\rm B}^4}{15h^3c^2} \ ,$$

so the radiation pressure becomes

$$\boxed{P_{\rm rad} = \frac{4}{3}\frac{\sigma T^4}{c}\ .}$$

There also is another constant called the *radiation constant* 'a' which is related to the Stefan-Boltzmann constant by

$$a = \frac{4\sigma}{c}\ .$$

As such, we also can write the radiation pressure equation for a blackbody as

$$\boxed{P_{\rm rad} = \frac{1}{3}\, a\, T^4\ .}$$

Finally, the internal energy has already been derived in the textbook (see Eqs. 3.25 and 9.5) as $u = aT^4$, hence we can also write the blackbody radiation pressure as

$$\boxed{P_{\rm rad} = \frac{1}{3}\, u\ .}$$

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8. (20 pts) Problem 9.11, Page 308 in the Carroll and Ostlie textbook.

Solution (a): We are given that $\rho_c(\odot) = 162$ gm/cm^3 and that $\overline{k}_{\rm Ross} = 1.16$ cm^2/gm. Carroll and Ostlie call this the Rosseland opacity, but in fact it is the Rosseland mass absorption coefficient. The opacity κ [cm^{-1}] is related to the mass absorption coefficient by $\kappa = k\rho$, hence the Rosseland mean opacity at the center of the Sun is

$$\overline{\kappa}_{\rm Ross} = \overline{k}_{\rm Ross}\, \rho_c(\odot) = (1.16\ {\rm cm}^2/{\rm gm})(162\ {\rm gm/cm}^3) = 188\ {\rm cm}^{-1}\ .$$

The mean-free-path ℓ of a photon traveling through a gas (*i.e.*, the distance it travels before interacting with a gas particle) is just the reciprocal of the opacity:

$$\boxed{\ell = \frac{1}{\overline{\kappa}_{\rm Ross}} = \frac{1}{188\ {\rm cm}^{-1}} = 5.32 \times 10^{-3}\ {\rm cm} = 53.2\ \mu{\rm m}\ ,}$$

where μm is a micrometer (= micron).

Solution (b): The first thing we need to do is figure out how much time it takes for a photon to travel from one collision to another. We can simply use a 1-dimensional equation of motion to approximate this time per collision rate (see the equation on page 297 in the textbook):

$$\Delta t \approx \frac{\ell}{v} = \frac{\ell}{c} = \frac{5.32 \times 10^{-3}\ {\rm cm}}{3.00 \times 10^{10}\ {\rm cm/s}} = 1.77 \times 10^{-13}\ {\rm sec/collision}\ ,$$

since photons are traveling at the speed of light in between collisions.

Every time a photon scatters off of a gas particle during a collision, it can be re-emitted in any direction $\Longrightarrow$ the photon takes a *random walk* through the gas. The statistics for such a path shows that a particle will travel a distance d following Eq. 9.23 on page 277 in the textbook:

$$d = \ell \sqrt{N} \ ,$$

where ℓ is the mean-free-path, N is the number of collisions suffered, and d will be the total traveled distance. Since we are trying to get from the center of the Sun to the surface (*i.e.*, photosphere), $d = R_\odot = 6.96 \times 10^{10}$ cm. Hence, the total number of collisions that a photon makes on its trek from the center of the Sun to the surface is

$$N = \left(\frac{R_\odot}{\ell}\right)^2 = \left(\frac{6.96 \times 10^{10}\ \text{cm}}{5.32 \times 10^{-3}\ \text{cm}}\right)^2 = 1.71 \times 10^{26}\ \text{collisions} \ .$$

Finally, the total time it will take to travel a distance $d = R_\odot$ will just be the number of collisions times the average time for each collision to take place, hence

$$\begin{aligned} t_{\text{tot}} = N\,\Delta t &= (1.71 \times 10^{26}\ \text{collisions})(1.77 \times 10^{-13}\ \text{sec/collision}) = 3.03 \times 10^{13}\ \text{s} \\ &= \frac{3.03 \times 10^{13}\ \text{s}}{3.16 \times 10^{7}\ \text{s/yr}} = \boxed{9.60 \times 10^{5}\ \text{yr}} \end{aligned}$$

$\Longrightarrow$ nearly a million years! In this solution, we have assumed that the amount of time for a collision to take place is negligible with respect to Δt (*i.e.*, the time in between the collisions). This assumption is correct and if this were a graduate-level course in radiative transfer, I would have had you proved it!

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9. (20 pts) Problem 11.14, Page 432 in the Carroll and Ostlie textbook.

Solution: This was a typo. In fact, I wanted the student to tackle Problem 11.4 in the textbook, which requires that you do Problem 11.3 first. As such, this problem will be considered **extra-credit** and you have the option of doing either 11.3/11.4 or the above posted 11.14. All solutions are shown here.

Solution (11.3): As we did in Problem #6, we need to solve the Saha equation. We are given that $T = T_{\text{eff}} = 5770$ K for the Sun and that $P_e = 15$ dyne/cm^2. The comment about the Pauli Exclusion Principle was made to give us a clue to the partition function value for H^-. Since only one H^- state can exist in hydrogen due to PEP, the partition function must be equal to 1 (the extra electron can only have the opposite spin of the lower energy bound electron). Neutral hydrogen, with its single electron, can exist in either a spin "up" or "down" (with respect to the proton's spin) state, hence it has a partition function of 2. The extra electron in H^- can be knocked off (*i.e.*, ionized) with a 0.75 eV photon (see page 398 of the textbook). In the Saha equation, the lower ionization state will correspond to H^- and the upper ion state will correspond to H I. As such, here are our givens:

$$\begin{array}{lll} H^- : & \chi_1 = 0.75 \text{ eV} & Z_1 = 1 \\ \text{H I} : & \chi_2 = 13.6 \text{ eV} & Z_2 = 2 \\ \text{gas} : & T = 5770 \text{ K} & P_e = 15 \text{ dyne/cm}^2 \end{array}$$

With these, we can calculate the various terms in the Saha equation:

$$\frac{2k_B T Z_2}{P_e Z_1} = \frac{2 \cdot 1.38 \times 10^{-16} \text{ erg/K} \cdot 5770 \text{ K} \cdot 2}{15 \text{ dyne/cm}^2 \cdot 1} = 2.12 \times 10^{-13} \text{ cm}^3 ,$$

$$\frac{2\pi m_e k_B T}{h^2} = \frac{2\pi(9.11 \times 10^{-28} \text{ gm})(1.38 \times 10^{-16} \text{ erg/K})(5770 \text{ K})}{(6.63 \times 10^{-27} \text{ erg s})^2} = 1.04 \times 10^{14} \text{ cm}^{-2} ,$$

Using the ionization energy of H^-, we get

$$e^{-\chi_1/k_B T} = e^{-0.75 \text{ eV}/8.62\times10^{-5} \text{ eV/K} \cdot 5770 \text{ K}} = 0.22 .$$

Using these in the Saha equation, the ion number density ratio is

$$\begin{aligned} \frac{N_{\text{H I}}}{N_{\text{H}^-}} = \frac{N_2}{N_1} &= \frac{2k_B T Z_2}{P_e Z_1} \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\chi_1/k_B T} \\ &= (2.12 \times 10^{-13} \text{ cm}^3)\,(1.04 \times 10^{14} \text{ cm}^{-2})^{3/2}\,(0.22) \\ &= \boxed{4.9 \times 10^7 ,} \end{aligned}$$

For every one H^- ion, there are 49 million neutral hydrogen atoms in the solar atmosphere! Then why does H^- dominate all continuous opacity sources in the solar atmosphere? Because its absorption cross-section is truly enormous in comparison to the other opacity sources in the Sun's atmosphere. This is why H^- is also important in the internal structure of stars. In cool stars, where temperatures in the outer envelope are relatively cool (in comparison to hot stars), H^- is very efficient in blocking the flow of photons which sets up convective instability — a convection zone forms in these regions.

Solution (11.4): In the solution for Problem 11.3 in the textbook, we used the Saha equation to get the number density ratio of H I to H^-, we now will use the Boltzmann equation to figure out ratios of various levels within neutral hydrogen. Boltzmann's equation is

$$\frac{n_j}{n_i} = \frac{g_j}{g_i} e^{-E_{ji}/k_B T} ,$$

where j represents the higher energy state and i the lower state. Let's calculate the ratios of the first three excited states ($j = 2, 3, 4$) with respect to the ground state ($i = 1$) in hydrogen in the Sun's atmosphere (with $T = T_{\text{eff}} = 5770$ K). The statistical weights are found with $g_j = 2j^2$ and $E_{ji} = E_j - E_i$. The table below shows the values of these input parameters and the results from the Boltzmann equation (with $k_\mathrm{B} = 8.6174 \times 10^{-5}$ eV/K, and hence $k_\mathrm{B}T = 0.4972$ eV):

Level (j)	$\boldsymbol{g_j}$	$\boldsymbol{g_j/g_1}$	$\boldsymbol{E_j}$ **(eV)**	$\boldsymbol{E_j - E_1}$ **(eV)**	$\boldsymbol{n_j/n_1}$
1	2	1	-13.60	0.00	1.00
2	8	4	-3.40	10.20	4.93×10^{-9}
3	18	9	-1.51	12.09	2.48×10^{-10}
4	32	16	-0.850	12.75	1.10×10^{-11}

As can be seen from the table,

$$N_{\mathrm{H\ I}} = \sum_{i=1}^{\infty} n_i \approx n_1 \ ,$$

most of the neutral hydrogen is in the ground state on the order of 5 billion to one in comparison to the excited states. As such, set $N_{\mathrm{H\ I}} = n_1$ and compare this with the ratio found in Problem 11.3 in the textbook:

$$\begin{aligned} \frac{N_{\mathrm{H}^-}}{n_3} &= \frac{N_{\mathrm{H}^-}}{N_{\mathrm{H\ I}}} \cdot \frac{N_{\mathrm{H\ I}}}{n_3} = \frac{N_{\mathrm{H}^-}}{N_{\mathrm{H\ I}}} \cdot \frac{n_1}{n_3} = \left(\frac{N_{\mathrm{H\ I}}}{N_{\mathrm{H}^-}} \cdot \frac{n_3}{n_1} \right)^{-1} \\ &= \left[(4.97 \times 10^7)(2.48 \times 10^{-10}) \right]^{-1} = (1.23 \times 10^{-2})^{-1} \\ &= \boxed{81.1\ .} \end{aligned}$$

There are 81 H^- ions for every hydrogen atom in the $n = 3$ state, hence (assuming their absorption cross-sections are similar, which they are not, $\sigma_{\mathrm{H}^-} \gg \sigma_{3\kappa}$), H^- continua (both bound-free and free-free) will dominate the hydrogen Paschen continuum.

Solution (11.14): This question deals with magnetic pressure and asks us to derive

$$P_m = \frac{B^2}{8\pi}$$

from the following relations

$$u_m = \frac{B^2}{8\pi}, \qquad \text{and} \qquad W = \int P_m \, dV \ ,$$

where P_m is the magnetic pressure, u_m is the magnetic energy density, W is the work done by the magnetic field, and B is the magnetic field strength (note that these relations are only valid in the cgs unit system — there would be additional constants in the SI version).

This is actually very straight forward to prove. The energy density is defined as

$$u_m = \frac{d(\mathrm{PE})}{dV} \ ,$$

where PE is the potential energy of the magnetic field. A magnetic force field is conservative which means that the work W done on a particle by the field is independent of the path taken. For such force fields, $dW = d(\mathrm{PE})$. We can rewrite the work-integral equation as a differential equation:

$$\frac{dW}{dV} = P_m \ .$$

Using this with our definition of energy density we get

$$P_e = \frac{dW}{dV} = \frac{d(\mathrm{PE})}{dV} = u_m = \frac{B^2}{8\pi} \ . \quad \checkmark$$

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10. (20 pts) Problem 11.9, Page 434 in the Carroll and Ostlie textbook.

Solution (a): From the data in Example 11.2 in the textbook, $P = 5 \times 10^4$ dyne/cm^2, $\rho = 2.5 \times 10^{-7}$ gm/cm^3, and the acceleration due to gravity is $g = 2.8 \times 10^4$ cm/s^2 at the base of the solar photosphere. The pressure scale height is given by Eqs. (IV-28) and (IV-30) in the notes (Eqs. 10.62 and 10.63 on page 352 in the textbook):

$$\frac{1}{H_P} \equiv -\frac{1}{P}\frac{dP}{dr} \qquad \text{or} \qquad H_P = \frac{P}{\rho g}$$

for gas in hydrostatic equilibrium. Using our above mentioned photospheric values, we get

$$\begin{aligned} H_P &= \frac{5 \times 10^4 \ \mathrm{dyne/cm^2}}{(2.5 \times 10^{-7} \ \mathrm{gm/cm^3})(2.8 \times 10^4 \ \mathrm{gm/s^2})} = 7.1 \times 10^6 \ \mathrm{cm} \\ &= \boxed{71 \ \mathrm{km} \ ,} \end{aligned}$$

about 15% the thickness of the photosphere.

Solution (b): We are given the ratio of the mixing length to the pressure scale height is $\alpha = 2.2$. Based upon our solution from Part (a), the mixing length is

$$\ell = \alpha \, H_P = 2.2 \cdot 71 \ \mathrm{km} \ = 160 \ \mathrm{km}.$$

The average upward velocity measured for granules (*i.e.*, the convective velocity) is $\overline{v}_c \approx 0.4$ km/s. Using a simple 1-dimensional equation of motion, the time it would take for a blob of gas to move one mixing length is

$$t_c = \frac{\ell}{\overline{v}_c} = \frac{160 \ \mathrm{km}}{0.4 \ \mathrm{km/s}} = 400 \ \mathrm{s} = \boxed{\sim 7 \ \mathrm{min} \ .}$$

The characteristic lifetime of a granule ranges between 5 and 10 minutes which is consistent with this travel time.