ASTR-3415-001: Astrophysics
Solutions to Problem Set 3
Spring 2003

1. (10 pts) Problem 10.1, Page 375 in the Carroll and Ostlie textbook. Note that Carroll and Ostlie's $\overline{\kappa}$ is the mean mass absorption coefficient (cm^2/gm), which I gave as k in the notes, and not opacity (cm^{-1}) as the authors call it.

   Solution: In the notation that I have introduced in the notes (which by the way is the <u>correct</u> notation), we are trying to prove

$$\frac{dP}{d\tau} = \frac{g}{\overline{k}} ,$$

   where $\overline{k}$ is the mean mass-absorption coefficient (measured in cm^2/gm) and is related to the opacity (measured in cm^{-1}) by

$$\overline{k} = \frac{\overline{\kappa}}{\rho} .$$

   Note that we did not include the 'ν' subscripts since these are mean values averaged over all frequencies. If we use the definition of optical depth from the notes (see Eq. I-18):

$$d\tau = -\overline{\kappa}\, ds ,$$

   where once again we have dropped the 'ν' subscripts since this ia a mean opacity. Since we are working only in the radial direction with the HSE equation, the length differential in the optical depth equation is just $ds = dr$. As such, we can rewrite the optical depth equation as

$$d\tau = -\overline{k}\rho\ dr$$

   or

$$dr = -\frac{d\tau}{\overline{k}\rho} .$$

   The HSE equation is given by Eq. (10.7) in the textbook and Eq. (IV-14) in the course notes as

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} .$$

   Now plugging in the equations developed above into the HSE equation we get

$$-\frac{\overline{k}\rho\, dP}{d\tau} = -\frac{GM_r\rho}{r^2}$$

$$\frac{dP}{d\tau} = \frac{GM_r}{\overline{k}r^2} .$$

The acceleration due to gravity is

$$g = \frac{GM_r}{r^2} \ ,$$

and plugging this into the HSE equation, we finally get

$$\boxed{\frac{dP}{d\tau} = \frac{g}{\bar{k}} \ .}$$

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2. (20 pts) Problem 10.2, Page 375 in the Carroll and Ostlie textbook.

Solution: There are several ways to prove this statement. One way is to just write Newton's Universal Law of Gravitation:

$$\vec{F}_g = -\frac{GM_r m}{r^2} \hat{r} \ ,$$

where m is the mass of the test particle at distance r from the center of mass (which will be at the center of M_r since $m \ll M_r$ for a test mass point), and M_r is the mass of the body of total mass M that lies within point r. Since there is no mass inside of point r for the test mass inside a mass shell, $M_r = 0$, hence $F_g = 0$. $\surd$

Though I won't do the solution here (since you will have probably seen [or will see] this in your mechanics course), we could have also proved this with a differential equation approach where after showing that the gravitational potential Φ within the sphere is constant, the force $\vec{F}_g = -m\vec{\nabla}\Phi = 0$ (see Eq. IV-5).

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3. (20 pts) Problem 10.3, Page 375 in the Carroll and Ostlie textbook.

Solution: The energy released per particle for chemical energy is $E_p = 10$ eV/particle. Therefore, the total energy released over the lifetime of the Sun is this energy times the total number of particles N in the Sun: $E_{\rm tot} = NE_p$. Since we are told to assume that the Sun is composed entirely of H, then $M_\odot = Nm_{\rm H}$, or

$$N = \frac{M_\odot}{m_{\rm H}} = \frac{1.99 \times 10^{33} \text{ gm}}{1.67 \times 10^{-24} \text{ gm/particle}} = 1.19 \times 10^{57} \text{ particles,}$$

so the total energy released through these chemical reactions over the lifetime of the Sun is

$$\begin{aligned} E_{\rm tot} = NE_p &= (1.19 \times 10^{57} \text{ particles})(10 \text{ eV/particle } \cdot 1.60 \times 10^{-12} \text{ erg/eV}) \\ &= 1.9 \times 10^{46} \text{ erg.} \end{aligned}$$

The luminosity of any shining object is just the energy released per unit time interval. Assuming a constant luminosity over the entire "chemical" lifetime of the star t, we have $L_\odot = E_{\text{tot}}/t$, or solving for t we get

$$t = \frac{E_{\text{tot}}}{L_\odot} = \frac{1.9 \times 10^{46} \text{ erg}}{3.83 \times 10^{33} \text{ erg/s}} = 5.0 \times 10^{12} \text{ s}$$

$$= \boxed{1.6 \times 10^5 \text{ yr}} \, ,$$

or only 160,000 years. Since the oldest Earth rocks have been dated at ages over 3.5 billion years old, and the Moon rocks and meteorites are dated at 4.6 billion (4.6×10^9) years, and these rocks could not have existed before the Sun existed, chemical reactions are overwhelmingly insufficient to explain the energy output of the Sun.

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4. (20 pts) Problem 10.11, Page 376 in the Carroll and Ostlie textbook. Calculate this with the equations in the textbook, just don't use the numbers I supplied to you in the course notes.

Solution: Besides the mass values given to us, other useful information about masses are given on page 332 in the textbook. Here are a list of the givens:

$$m(^1\text{H}) = 1.00782u \quad m(^2\text{H}) = 2.0141u \quad m_{e^+} = m_e = 5.48617 \times 10^{-3}u$$
$$m(^3\text{He}) = 3.0160u \quad m(^4\text{He}) = 4.002603u \quad m_\nu \approx 1 \text{ eV} = 1.0706 \times 10^{-9}u$$

where u = 1 atomic mass unit (amu) = 1.660540×10^{-24} gm. The energy released is given by $Q = (M_{\text{i}} - M_{\text{f}})c^2$, where '$i$' is the 'initial' stage of a given reaction and 'f' is the final stage after the reaction has occurred. We will not have to include the mass of the neutrino in these calculations since its mass is negligible in comparison to both the masses given and the differences of the baryon masses calculated (see below). Also, we will not include the mass loss from the positron here (even though its mass is not negligible to the baryon mass differences), since the energy of this mass loss is immediately resupplied to the gas through an electron-positron annihilation. PP I has three steps in it reaction chain, the third reaction happens once for every 2 times the first two occur. The energy released for each stage is

$$\frac{Q_a}{c^2} = m(^1\text{H}) + m(^1\text{H}) - m(^2\text{H}) = 0.00154\,u$$
$$\frac{Q_b}{c^2} = m(^2\text{H}) + m(^1\text{H}) - m(^3\text{He}) = 0.00592\,u$$
$$\frac{Q_c}{c^2} = 2m(^3\text{He}) - m(^4\text{He}) - 2m(^1\text{H}) = 0.013757\,u$$

$$\begin{aligned}
Q_{\text{tot}} &= 2\,(Q_a + Q_b) + Q_c \\
&= [2\,(0.00154\,u + 0.00592\,u) + 0.013757\,u]c^2 \\
&= 0.028677\,u\,c^2 = 0.028677\,(1.660540 \times 10^{-24} \text{ gm})(2.997925 \times 10^{10} \text{ cm/s})^2 \\
&= 4.2798 \times 10^{-5} \text{ erg}
\end{aligned}$$

or converting to MeV = 10^6 eV, where 1 eV = 1.60219×10^{-12} erg, we get

$$\boxed{Q_{\rm tot} = 26.712 \text{ MeV} .}$$

As a check to see if we have none this correctly, note that in the PP chain, 4 hydrogen nuclei come together to make one helium nucleus, or

$$\begin{aligned}
\frac{Q_{\rm tot}}{c^2} &= 4m(^3\text{He}) - m(^4\text{He}) \\
&= 4(1.00782\,u) - 4.002603\,u = 0.028677\,u \\
Q_{\rm tot} &= 0.028677\,u\,c^2 = 0.028677\,(1.660540 \times 10^{-24} \text{ gm})(2.997925 \times 10^{10} \text{ cm/s})^2 \\
&= 4.2798 \times 10^{-5} \text{ erg } = 26.712 \text{ MeV} \quad \surd
\end{aligned}$$

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5. (10 pts) Problem 10.13, Page 377 in the Carroll and Ostlie textbook.

Solution (a): To balance these reactions, we need to invoke the conservation of baryon number, lepton number, and charge (and perhaps energy if needed).

$$^{27}\text{Si} \rightarrow \, ^{?}\text{Al} \, + e^+ + \, ?$$

For charge Z, baryon number B, and lepton number L conservation we have

$$\begin{aligned}
Z: \quad 14 &= \; 13 + 1 + q \\
B: \quad 27 &= \; (n + 13) + 0 + 0 \\
L: \quad 0 &= \; 0 + (-1) + l
\end{aligned}$$

or $q = 0$, hence the last particle has no charge; $n = 14$, hence the nucleon number for aluminum is 27; and $l = 1$, hence the third particle is a matter lepton and judging from q must be an electron neutrino (to balance the positron). So

$$\boxed{^{27}\text{Si} \rightarrow \, ^{27}\text{Al} \, + e^+ + \, \nu_e \, .}$$

Solution (b): Following the same technique as we had in (a):

$$^{?}\text{Al} \, + \, ^{1}\text{H} \, \rightarrow \, ^{24}\text{Mg} \, + \, ^{4}?$$

Before proceeding, the last unknown has to be helium since we are given its charge of 2 and its total nucleon number of 4. Continuing our conservation laws for the remaining unknown:

$$\begin{aligned}
Z: \quad 13 + 1 &= \; 12 + 2 \quad \surd \\
B: \quad (n + 13) + 1 &= \; 24 + 4 \\
L: \quad 0 + 0 &= \; 0 + 0 \quad \surd
\end{aligned}$$

hence $n = 14$ to give aluminum a nucleon number of 27 or

$$\boxed{{}^{27}\text{Al} \ + \ {}^{1}\text{H} \ \rightarrow \ {}^{24}\text{Mg} \ + \ {}^{4}\text{He} \ .}$$

Solution (c): Once again, following the same technique as we had in (a):

$${}^{35}\text{Cl} \ + \ {}^{1}\text{H} \ \rightarrow \ {}^{36}\text{Ar} \ + \ ?$$

Here we have

$$\begin{aligned} Z: \ & 17 + 1 = & 18 + q \\ B: \ & 35 + 1 = & 36 + n \\ L: \ & 0 + 0 = & 0 + l \end{aligned}$$

or $q = 0$, $n = 0$, and $l = 0$, hence, the extra particle must be a photon, or

$$\boxed{{}^{35}\text{Cl} \ + \ {}^{1}\text{H} \ \rightarrow \ {}^{36}\text{Ar} \ + \gamma \ .}$$

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6. (20 pts) Problem 10.15, Page 377 in the Carroll and Ostlie textbook. Use the equations in the textbook and not from your notes here.

Solution: There are two ways to answer this problem, an approximate solution, and an accurate solution. I'll accept either in this case. First note that for the low mass star, $\log T_e = 3.438$ gives $T_e = 2742$ K, and $\log L/L_\odot = -3.297$ gives $L = 5.047 \times 10^{-4} L_\odot$, and for the high mass star, $\log T_e = 4.722$ gives $T_e = 52{,}720$ K, and $\log L/L_\odot = 6.045$ gives $L = 1.109 \times 10^6 L_\odot$.

The Approximate Solution: Just follow Example 10.4 on page 332 in the textbook:

$$t_{\text{nuc}} = \frac{f \, E_{\text{nuc}}}{L_\star} \ ,$$

where E_{nuc} is the energy released via H fusion in the core over the main sequence (MS) lifetime of the star and f is the fraction of the stellar mass that lies in the thermonuclear zone at some point during the MS life:

$$E_{\text{nuc}} = \eta M_\star c^2 \ ,$$

where $\eta = 0.0071$ is the mass difference fraction of 4 hydrogen nuclei to one helium nuclei. So for our lower mass star where $f = 1$ due to the star being completely convective, we have

$$
\begin{aligned}
t_{\rm nuc}(\text{lo}) &= \frac{1.0 \cdot 0.0071 \cdot 0.085 M_\odot \cdot (3.00 \times 10^{10}\ \text{cm/s})^2}{5.047 \times 10^{-4} L_\odot} = 1.06 \times 10^{21}\, \frac{M_\odot}{L_\odot} \\
&= 1.1 \times 10^{21}\, \frac{1.99 \times 10^{33}\ \text{gm}}{3.83 \times 10^{33}\ \text{erg/s}} = 5.5 \times 10^{20}\ \text{s} \\
&= \boxed{1.7 \times 10^{13}\ \text{yr}\ .}
\end{aligned}
$$

For the upper limit on the MS, $f = 0.1$ (actually 0.3 is a more accurate value, but we will follow the book here):

$$
\begin{aligned}
t_{\rm nuc}(\text{hi}) &= \frac{0.1 \cdot 0.0071 \cdot 90 M_\odot \cdot (3.00 \times 10^{10}\ \text{cm/s})^2}{1.109 \times 10^{6} L_\odot} = 5.11 \times 10^{13}\, \frac{M_\odot}{L_\odot} \\
&= 5.1 \times 10^{13}\, \frac{1.99 \times 10^{33}\ \text{gm}}{3.83 \times 10^{33}\ \text{erg/s}} = 2.7 \times 10^{13}\ \text{s} \\
&= \boxed{8.4 \times 10^{5}\ \text{yr}\ .}
\end{aligned}
$$

The Accurate Solution: In the above solution, we assumed that all of the mass in the core gets converted into helium, hence the core is pure hydrogen. If we had no information about the luminosity of the stars, we would have to use Equation (10.50) to get the energy production rate per unit mass for the proton-proton chain for the low mass star and Equation (10.54) for the CNO energy production rate. These two equations would require us to figure out the central temperature. We could do this, following the central temperature calculation we did for the Sun using the ideal gas law on page IV-13 in the notes and HSE (to get the central pressure) on page IV-7 in the notes. But we don't need to do this since luminosity is already given to us. As such, we can get a more accurate main sequence lifetime by using Eq. (IV-1) [Note that all I was looking for here was the approximate solution as listed above. I show the more accurate solution here using the notes, even though I told you not to use this equation. I'm showing you this just to be complete.]

Here we have an additional factor $X_{\rm init}$ which is the initial mass fraction of hydrogen. Note that $X_{\rm init} = 0.70$ (page IV-13 of the notes) for Population I stars. Then for the low mass star

$$
t_{\rm nuc} = \frac{f\, \eta\, X_{\rm init}\, M_\star}{L_\star}
$$

$$
\begin{aligned}
t_{\rm nuc}(\text{lo}) &= \frac{1.0 \cdot 0.0071 \cdot 0.70 \cdot 0.085 M_\odot \cdot (3.00 \times 10^{10}\ \text{cm/s})^2}{5.047 \times 10^{-4} L_\odot} = 7.42 \times 10^{20}\, \frac{M_\odot}{L_\odot} \\
&= 7.4 \times 10^{20}\, \frac{1.99 \times 10^{33}\ \text{gm}}{3.83 \times 10^{33}\ \text{erg/s}} = 3.9 \times 10^{20}\ \text{s} \\
&= \boxed{1.2 \times 10^{13}\ \text{yr}\ .}
\end{aligned}
$$

and for the upper limit on the main sequence

$$
\begin{aligned}
t_{\rm nuc}({\rm hi}) &= \frac{0.1 \cdot 0.0071 \cdot 0.70 \cdot 90 M_\odot \cdot (3.00 \times 10^{10}\ {\rm cm/s})^2}{1.109 \times 10^6 L_\odot} = 3.58 \times 10^{13}\,\frac{M_\odot}{L_\odot} \\
&= 3.6 \times 10^{13}\,\frac{1.99 \times 10^{33}\ {\rm gm}}{3.83 \times 10^{33}\ {\rm erg/s}} = 1.9 \times 10^{13}\ {\rm s} \\
&= \boxed{5.9 \times 10^5\ {\rm yr}\ .}
\end{aligned}
$$

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7. (10 pts) Problem 10.16, Page 377 in the Carroll and Ostlie textbook.

Solution: Here, we just need to use the blackbody relation given by Eq. (V-9) on page V-19 in the notes:

$$
\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4 ,
$$

where $T_\odot = 5770$ K is the effective temperature of the Sun. Solving this for R for the low mass star gives

$$
\begin{aligned}
\frac{R}{R_\odot} &= \sqrt{\frac{L}{L_\odot}} \left(\frac{5770\ {\rm K}}{T}\right)^2 \\
&= \sqrt{5.047 \times 10^{-4}} \left(\frac{5770\ {\rm K}}{2742\ {\rm K}}\right)^2 \\
&= 9.95 \times 10^{-2}
\end{aligned}
$$

or

$$
\boxed{R_{\rm lo} = 0.0992\, R_\odot = 6.90 \times 10^9\ {\rm cm}\ .}
$$

Now solving for the high mass star,

$$
\begin{aligned}
\frac{R}{R_\odot} &= \sqrt{\frac{L}{L_\odot}} \left(\frac{5770\ {\rm K}}{T}\right)^2 \\
&= \sqrt{1.109 \times 10^6} \left(\frac{5770\ {\rm K}}{52,720\ {\rm K}}\right)^2 \\
&= 1.26 \times 10^1
\end{aligned}
$$

or

$$
\boxed{R_{\rm hi} = 12.6\, R_\odot = 8.78 \times 10^{11}\ {\rm cm}\ .}
$$

The ratio of their radii is

$$
\boxed{\frac{R_{\rm lo}}{R_{\rm hi}} = \frac{0.0992\, R_\odot}{12.6\, R_\odot} = 7.87 \times 10^{-3}\ .}
$$

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8. (20 pts) Problem 12.4, Page 479 in the Carroll and Ostlie textbook.

Solution: We are given the following:

$$\tau_{\rm H} = 0.5 \qquad T = 100 \text{ K}$$
$$\Delta v = 10 \text{ km/s} \qquad n_{\rm H} = 10 \text{ cm}^{-3}$$

In Eq. (12.4),

$$\tau_{\rm H} = 5.2 \times 10^{-19} \frac{N_{\rm H}}{T\,\Delta v} ,$$

where $\tau_{\rm H}$ is the optical depth (unitless), T is the ISM gas temperature (K), Δv is the full-width-at-half-maximum of the line measured in Doppler velocity units (km/s), and $N_{\rm H}$ is the column density (cm^{-2}). The numerical constant term in front includes various natural constants and conversion scale factors to allow the parameters to be measured in the units given. Here, we are to solve for the column density:

$$\begin{aligned} N_{\rm H} &= \frac{\tau_{\rm H}\, T\, \Delta v}{5.2 \times 10^{-19}} \\ &= \frac{0.5 \cdot 100 \cdot 10}{5.2 \times 10^{-19}} \text{ cm}^{-2} = 9.6 \times 10^{20} \text{ cm}^{-2} . \end{aligned}$$

This number represents the number of hydrogen atoms in a cylindrical column of unit area. Note that for a cylinder, $V = A \cdot \ell$, where ℓ is the length of the cylinder or the depth (*i.e.*, thickness) that we see into the column. Since $V = n_{\rm H}^{-1}$ and $A = N_{\rm H}^{-1}$, we can solve for the thickness ℓ:

$$\begin{aligned} \ell &= \frac{V}{A} = \frac{n_{\rm H}^{-1}}{N_{\rm H}^{-1}} = \frac{N_{\rm H}}{n_{\rm H}} \\ &= \frac{9.6 \times 10^{20} \text{ cm}^{-2}}{10 \text{ cm}^{-3}} = 9.6 \times 10^{19} \text{ cm} . \end{aligned}$$

Since there are 3.0856×10^{18} cm in one parsec, the thickness is

$$\boxed{31 \text{ pc} .}$$

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9. (20 pts) Problem 13.5, Page 537 in the Carroll and Ostlie textbook.

Solution (a): Eq. (13.23) is Reimer's law of mass loss which states,

$$\dot{M} = -4 \times 10^{-13} \eta \frac{L}{g\,R} M_\odot \text{ yr}^{-1} ,$$

where R is measured in solar radii, L in solar radii, and g in surface gravities in units of the Sun's surface gravity of $g_\odot = 2.74 \times 10^4 \text{ cm/s}^2$. Making use of this info, we can write (using Eq. (IV-10) from the notes),

$$\frac{g}{g_\odot} = \frac{G\,M/R^2}{G\,M_\odot/R_\odot^2} = \frac{M/M_\odot}{(R/R_\odot)^2} .$$

Note that we can rewrite this equation as

$$g = \frac{M}{R^2} \ ,$$

with the understanding that g, M, and R are being measured in solar units. Making use of this fact in the above equation, we can write

$$\dot{M} = -4 \times 10^{-13} \eta \frac{L\, R^2}{M\, R} M_\odot \ \mathrm{yr}^{-1}$$

or

$$\boxed{\dot{M} = -4 \times 10^{-13} \eta \frac{L\, R}{M} M_\odot \ \mathrm{yr}^{-1} \ .}$$

Solution (b): Rewrite the mass loss as a derivative and set $C = -4 \times 10^{-13} M_\odot/\mathrm{yr}$, then

$$\dot{M} = \frac{dM}{dt} = C\eta \frac{L\, R}{M} \ ,$$

or

$$M\, dM = C\eta LR\, dt \ .$$

Set the initial time to be 0 when $M = M_\circ$ (*i.e.*, the initial mass), then set up the following integral equation

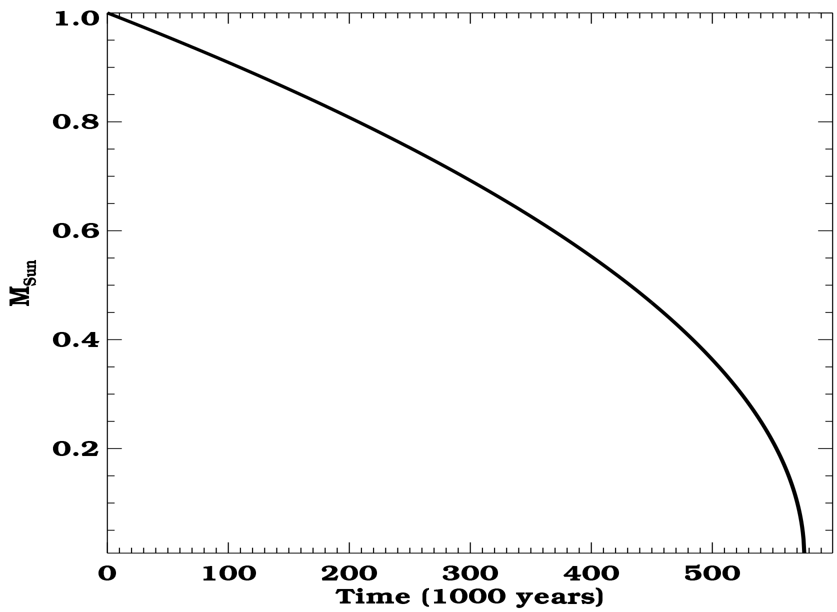
$$\begin{aligned} \int_{M_\circ}^{M} M\, dM &= \int_0^t C\eta LR\, dt = C\eta LR \int_0^t dt \\ \frac{1}{2}(M^2 - M_\circ^2) &= C\eta LRt \ , \end{aligned}$$

or finally,

$$\boxed{M = \sqrt{M_\circ^2 - 8 \times 10^{-13}\, \eta\, L\, R\, t} \ ,}$$

where t is measured in years and the M's, L, and R are in solar units.

Solution (c): I have attached a copy of an IDL procedure that makes a plot of the equation written above in part (b). I calculated a thousand different times from 0 to 600 thousand years in this plot as shown on the next page.



Solution (d): We just need to take the equation in part (b), set $M_\circ = 1.0$, $M = 0.6$, and solve for t. Note that the mass in this part ($1 M_\odot$) is the same as that in part (b), as such, take $R = 310$, $\eta = 1$, and $L = 7000$ in this equation, then

$$\begin{aligned} t &= \frac{M_\circ^2 - M^2}{8 \times 10^{-13}\, \eta\, L\, R}\ \text{yr} \\ &= \frac{1.0^2 - 0.6^2}{8 \times 10^{-13} \cdot 1 \cdot 7000 \cdot 310}\ \text{yr} \\ &= \frac{0.64}{8 \times 10^{-13} \cdot 1 \cdot 7000 \cdot 310}\ \text{yr} \end{aligned}$$

or

$$\boxed{t = 3.7 \times 10^5\ \text{yr}\ ,}$$

which is consistent with the graph shown in part (c).

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10. (10 pts) Problem 13.15, Page 539 in the Carroll and Ostlie textbook.

Solution: Using Eq. (V-11) from the notes (12.20b) in the textbook), the Eddington limit is

$$\frac{L_{\text{Ed}}}{L_\odot} \cong 3.8 \times 10^4\, \frac{M}{M_\odot}\ .$$

Information about η Car is given on page 528 of the textbook: $M = 150 M_\odot$, $L = 6 \times 10^6 L_\odot$, $\dot{M} = 3 \times 10^{-4} M_\odot/\mathrm{yr}$, and $v_{\mathrm{wind}} = 450$ km/s. Plugging this mass into the Eddington limit equation gives

$$\frac{L_{\mathrm{Ed}}}{L_\odot} \cong 3.8 \times 10^4 \, \frac{150 M_\odot}{M_\odot} = 5.7 \times 10^6 \ ,$$

or

$$\boxed{L_{\mathrm{Ed}} \cong 5.7 \times 10^6 \, L_\odot \ .}$$

The luminosity of the star is

$$L = 6 \times 10^6 \, L_\odot \cong 1.05 \, L_{\mathrm{Ed}} \ ,$$

which exceeds the Eddington limit. This accounts for this star's large mass loss and high speed stellar wind.

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11. (20 pts) Suppose an electron with a kinetic energy of 10^9 eV is moving through a magnetic field in a supernova remnant. If most of its radiated light is emitted at 100 MHz, what is the strength of the magnetic field? At what velocity is this electron moving? What is the mass of this electron?

Solution: We just use Eq. (VII-33) from the notes:

$$\nu_{\mathrm{max}} = \frac{e\,H}{4\pi m_e\, c} \left(\frac{E}{m_e\, c^2} \right)^2 \ ,$$

where ν_{max} is the frequency where the radiation reaches its greatest intensity = 100 MHz = 10^8 Hz = 10^8 s^{-1}, $e = 4.80 \times 10^{-10}$ esu, H is the magnetic field strength in gauss, $m_e = 9.11 \times 10^{-28}$ gm, $c = 3.00 \times 10^{10}$ cm/s (hence $m_e c^2$ is the rest mass energy of the electron), and E is the energy of the electron in ergs. Solving for H we get

$$\begin{aligned} H &= \frac{4\pi m_e\, c\, \nu_{\mathrm{max}}}{e} \left(\frac{m_e\, c^2}{E} \right)^2 \\ &= \frac{3.43 \times 10^{-8} \text{ gm cm/s}^2}{4.80 \times 10^{-10} \text{ esu}} \left(\frac{8.20 \times 10^{-7} \text{ erg}}{10^9 \text{ eV} \ \cdot 1.60 \times 10^{-12} \text{ erg/eV}} \right)^2 \\ &= 71.5 \text{ gauss} \left(\frac{8.20 \times 10^{-7} \text{ erg}}{1.60 \times 10^{-3} \text{ erg}} \right)^2 \\ &= \boxed{1.88 \times 10^{-5} \text{ gauss} \ = 18.8 \ \mu\text{-gauss} \ .} \end{aligned}$$

As can be seen, the electron is relativistic since $m_e c^2 \ll E$. As such, we can't use the Newtonian form of the kinetic energy equation (since this would give us a velocity greater than the speed of light!). Instead, we need to make use of the total relativistic energy equation given by Eq. (4.6) in the textbook and solve for v:

$$\begin{aligned}
E &= \frac{m_e\,c^2}{\sqrt{1-v^2/c^2}} \\
1-\frac{v^2}{c^2} &= \left(\frac{m_e\,c^2}{E}\right)^2 \\
\frac{v^2}{c^2} &= 1-\left(\frac{m_e\,c^2}{E}\right)^2 \\
&= 1-\left(\frac{8.20\times10^{-7}\text{ erg}}{1.60\times10^{-3}\text{ erg}}\right)^2 \\
&= 1-2.63\times10^{-7} = 0.999999738 \\
\frac{v}{c} &= 0.999999869
\end{aligned}$$

or

$$\boxed{v = 0.999999869\,c\ ,}$$

which is very close to (<u>but less than</u>) the speed of light.

The mass of the electron can be determined from the relativistic momentum equation (see Eq. 4.44) in the textbook:

$$\begin{aligned}
p = m\,v &= \frac{m_\circ\,v}{\sqrt{1-v^2/c^2}} \\
m &= \frac{m_\circ}{\sqrt{1-v^2/c^2}}\ ,
\end{aligned}$$

where $m_\circ = m_e = 9.11\times10^{-28}$ gm is the rest mass of the electron and m is the mass of the electron traveling at velocity v. Using the results from above we get

$$\begin{aligned}
m &= \frac{m_e}{\sqrt{1-(0.999999869\,c)^2/c^2}} \\
&= \frac{m_e}{\sqrt{1-0.999999738}} = \frac{m_\circ}{\sqrt{2.62\times10^{-7}}} \\
&= 1950\,m_e\ ,
\end{aligned}$$

or

$$\boxed{m = 1.78\times10^{-24}\text{ gm}}$$

$\Longrightarrow$ nearly 4 million times the rest mass of the electron!

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12. (20 pts) Using the conservation of angular momentum ($L = m\omega r$), calculate how fast the Sun would spin if it were to collapse down to a neutron star. (Note that $P_\odot = 25$ days.)

Solution: CAM requires

$$\begin{aligned} L_{\text{before}} &= L_{\text{after}} \\ m_{\text{b}}\,\omega_{\text{b}}\,r_{\text{b}} &= m_{\text{a}}\,\omega_{\text{a}}\,r_{\text{a}}\ . \end{aligned}$$

During the collapse, we will assume that the Sun loses no mass, hence $m_{\text{b}} = m_{\text{a}} = M_\odot$ and these variables cancel out. Here, we are trying to solve for ω_{a} with $\omega_{\text{b}} = 1$ rotation/25 days. Finally, $r_{\text{b}} = R_\odot = 6.96 \times 10^5$ km and $r_{\text{a}} = R_{\text{NS}} = 10$ km (see page 598 of the textbook which gives the radius of a 1.4 $M_\odot$ neutron star as 10 km — since $R_{\text{NS}} \propto M^{-1/3}$ as shown by Eq. 15.22, the radius of a 1 $M_\odot$ will be very similar to a 1.4 $M_\odot$ NS). Solving for ω_{a} gives

$$\begin{aligned} \omega_{\text{a}} &= \frac{R_\odot}{R_{\text{NS}}}\,\omega_{\text{b}} \\ \omega_{\text{a}} &= \frac{6.96 \times 10^5\ \text{km}}{10\ \text{km}} \cdot \frac{1\ \text{rot}}{25\ \text{day}} \\ &= 2800\ \text{rot/day}\ . \end{aligned}$$

Since the period is just the reciprocal of this angular speed, we get

$$P_{\text{NS}} = \boxed{3.6 \times 10^{-4}\ \text{day}\ = 31\ \text{s}\ .}$$