ASTR-3415-001: Astrophysics
Solutions to Problem Set 4
Spring 2003

1. (10 pts) Problem 22.1, Page 977 in the Carroll and Ostlie textbook.

Solution: For this solution, let's assume that the Sun's orbit is circular and its velocity matches the local standard of rest. Then, the orbital period is just given by

$$P_\odot = \frac{2\pi}{\omega_\odot} \ ,$$

where $\omega_\odot$ is the orbital angular velocity of the Sun. From classical mechanics, we can relate the angular velocity to the tangential velocity with $v_t = \Theta_\circ = \omega_\odot R_\circ$ giving

$$P_\odot = \frac{2\pi R_\circ}{\Theta_\circ} \ ,$$

as displayed in Example 22.2 in the textbook. The galactocentric distance of the Sun is $R_\circ = 8.0 \times 10^3$ pc $\times$ $(3.0856 \times 10^{13}$ km/pc$) = 2.47 \times 10^{17}$ km (see Eq. 22.13 in the text or Table IX-2 in the notes). The circular orbital speed of the Sun is then given by Eq. (22.23) in the text and Table IX-2 in the notes as $\Theta = 220$ km/s. As such, the orbital period is

$$\begin{aligned} P_\odot &= \frac{2\pi(2.47 \times 10^{17} \text{ km})}{220 \text{ km/s}} \\ &= 7.05 \times 10^{15} \text{ s} \ \cdot \left(\frac{1 \text{ yr}}{3.1557 \times 10^7 \text{ s/yr}}\right) \\ &= 2.23 \times 10^8 \text{ yr.} \end{aligned}$$

The age of the oldest rocks in the solar system is 4.6×10^9 yr. As such, the Sun is approximately 5 billion years old. Finally the number of times the Sun has been around the galactic center since it formed is

$$\begin{aligned} N &= \frac{T_{\rm age}}{P_\odot} = \frac{5.0 \times 10^9 \text{ yr}}{2.23 \times 10^8 \text{ yr/rev}} = 22.3 \text{ rev} \\ &= \boxed{\text{22 revolutions.}} \end{aligned}$$

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2. (15 pts) Problem 22.26, Page 984 in the Carroll and Ostlie textbook.

Solution (a): (5 pts) We are given $p = 0.198'' \pm 0.006''$, $\mu = 0.66''/\text{yr}$, and $v_r = -26$ km/s. We just need to solve the parallax formula (see Eq. 3.1 in the text and Eq. IX-4 in the notes):

$$d = \frac{1}{p}\ \text{pc} \ = \frac{1}{0.198}\ \text{pc} \ = 5.05\ \text{pc}\ .$$

Since the text gives an uncertainty in the parallax angle, we should convert this to an uncertainty in the distance. On the large end,

$$d = \frac{1}{p}\ \text{pc} \ = \frac{1}{0.204}\ \text{pc} \ = 4.90\ \text{pc}\ ,$$

and on the small angle end,

$$d = \frac{1}{p}\ \text{pc} \ = \frac{1}{0.192}\ \text{pc} \ = 5.21\ \text{pc}\ ,$$

which gives an uncertainty of 0.15 pc and 0.16 pc, respectively. Therefore, the distance to Altair is

$$\boxed{d = 5.05 \pm 0.16\ \text{pc}.}$$

Solution (b): (5 pts) This angle is given in the textbook on page 962 (which uses ϕ for this angle) and by Eq. (IX-7) in the notes (which uses θ for this angle). In order to solve for this angle, we first need to determine the tangential or transverse velocity given by Eq. (1.4) in the text and (IX-5) in the notes:

$$v_t = 4.74\,\mu\,d\ \text{km/s} \ = 4.74(0.66)(5.05)\ \text{km/s} \ = 16\ \text{km/s}\ .$$

From this the angle is

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{v_t}{v_r}\right) = \tan^{-1}\left(\frac{16\ \text{km/s}}{-26\ \text{km/s}}\right) = \tan^{-1}(-0.62) \\ &= \boxed{-32^\circ\ .}\end{aligned}$$

Solution (c): (5 pts) We already had the calculate the transverse velocity above: $\boxed{16\ \text{km/s.}}$ Using this with the radial velocity, the space velocity is

$$\begin{aligned}v_\star &= \sqrt{v_r^2 + v_t^2} = \sqrt{(-26\ \text{km/s})^2 + (16\ \text{km/s})^2} \\ &= \boxed{31\ \text{km/s}\ .}\end{aligned}$$

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3. (20 pts) Problem 22.27, Page 984 in the Carroll and Ostlie textbook.

Solution: The question specifically requests that we use Newtonian gravity to solve this problem. As such, the amount of energy needed to lift $10^7 M_\odot$ out of the potential well is just the difference of the gravitational potential energy between these two points:

$$\Delta\text{PE} = -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right) ,$$

where $M = 4 \times 10^6 M_\odot$ is the mass of the supermassive black hole at the center of the Galaxy (see Example 22.4 on page 972 in your textbook), $m = 10^7 M_\odot$ is the secondary mass that was ejected, $r_1 = R_S = 0.08$ AU is the Schwarzschild radius (which defines the distance of the event horizon from the center, see Example 22.4) of the supermassive black hole, and finally $r_2 = 3$ kpc. Conversions we will need to make include 1 AU = 1.496×10^{13} cm, 1 kpc = 3.09×10^{21} cm, and 1 $M_\odot = 1.99 \times 10^{33}$ gm. This gives

$$\begin{aligned}
\Delta\text{PE} &= -(6.67 \times 10^{-8}\ \text{dyne cm}^2/\text{gm}^2)\,(4 \times 10^6 \times 1.99 \times 10^{33}\ \text{gm}) \times \\
&\quad (10^7 \times 1.99 \times 10^{33}\ \text{gm})\left(\frac{1}{(3 \times 3.09 \times 10^{21}\ \text{cm})}\right. \\
&\quad \left. - \frac{1}{(0.08 \times 1.496 \times 10^{13}\ \text{cm})}\right) \\
&= -(1.06 \times 10^{73}\ \text{dyne cm}^2)\,(-8.36 \times 10^{-13}\ \text{cm}^{-1}) = 8.86 \times 10^{60}\ \text{dyne cm} \\
&= \boxed{9 \times 10^{60}\ \text{erg}\ .}
\end{aligned}$$

Using information from page 514 in the textbook, a Type II supernova releases 10^{49} erg of energy via photons, 10^{51} erg in the kinetic energy of explosion shock wave, and 3×10^{53} ergs via neutrinos. The amount of energy it would take to move $10^7 M_\odot$ from the galactic supermassive black hole out to 3 kpc is equivalent to $\boxed{\text{900 billion times}}$ the energy release from a Type II supernova's light, $\boxed{\text{9 billion times}}$ the energy of the supernova shock wave, and $\boxed{\text{30 million times}}$ the total energy released from such a supernova.

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4. (30 pts) Problem 23.7, Page 1049 in the Carroll and Ostlie textbook.

Solution (a): (5 pts) We are given the following measured values for galaxy NGC 2639: $V_{\text{max}} = 324$ km/s and $B = 12.22$. First we are asked to determine M_B from the Tully-Fisher relation. For the information given, we will need to use Eq. (23.4) on page 1002 in the textbook for an Sa galaxy:

$$\begin{aligned}
M_B &= -9.95 \log V_{\text{max}} + 3.15 \\
&= -9.95 \log(324) + 3.15 \\
&= \boxed{-21.83\ .}
\end{aligned}$$

Solution (b): (5 pts) Since extragalactic extinction has already been subtracted, the distance from th distance modulus is

$$\begin{aligned}
B - M_B &= 5 \log \left(\frac{d}{10 \text{ pc}} \right) \\
\log \left(\frac{d}{10 \text{ pc}} \right) &= 0.2 \; (B - M_B) = 0.2 \, (12.22 + 21.83) = 6.81 \\
\frac{d}{10 \text{ pc}} &= 10^{6.81} = 6.46 \times 10^6 \\
d &= 6.46 \times 10^7 \text{ pc} = \boxed{64.6 \text{ Mpc} \, .}
\end{aligned}$$

Solution (c): (5 pts) This question is answered using Eq. (23.6) on page 1003 in the textbook:

$$\begin{aligned}
\log R_{25} &= -0.249 \, M_B - 4.00 = -0.249 \, (-21.83) - 4.00 = 1.44 \\
R_{25} &= \boxed{27.3 \text{ kpc} \, .}
\end{aligned}$$

Solution (d): (5 pts) The mass using the data supplied and calculated so far is given by Eq. (23.5) on page 1002 in the textbook where here we replace R with R_{25}:

$$\begin{aligned}
M_{25} &= \frac{V_{\text{max}}^2 R_{25}}{G} \\
&= \frac{(3.24 \times 10^7 \text{ cm/s})^2 \, (27.3 \text{ kpc } \times 3.0856 \times 10^{21} \text{ cm/kpc})}{6.673 \times 10^{-8} \text{ dyne cm}^2/\text{gm}^2} \\
&= 1.33 \times 10^{45} \text{ gm } \cdot \left(\frac{1 M_\odot}{1.99 \times 10^{33} \text{ gm}} \right) \\
&= \boxed{6.66 \times 10^{11} M_\odot = 3.3 \, M_{\text{MW}} \, .}
\end{aligned}$$

Solution (e): (5 pts) We can just use the absolute magnitude relation with luminosity (see Eq. 3.8 on page 68 of the text and Eq. II-4 in the notes, except we need to replace M_{bol} with M_B):

$$M_B - M_{B\odot} = -2.5 \log \left[\frac{L_B}{L_{B\odot}} \right] \; ,$$

where L_B is the luminosity of the galaxy in the B band and $L_{B\odot}$ is the solar B-luminosity. Using the "Hint" from Problem 23.3, we see that Appendix E page A-14 in the textbook gives the Sun's absolute visual magnitude as $M_V(\odot) = +4.83$ and its color as $(B-V)_\odot = (M_B - M_V)_\odot = +0.64$ (here I am using the color value as adopted by the text). This gives the Sun's absolute blue magnitude as $M_{B\odot} = 5.47$. Using this in the luminosity equation, we see that

$$\begin{aligned}
\log\left[\frac{L_B}{L_{B\odot}}\right] &= 0.4\,[M_{B\odot} - M_B] = 0.4\,(5.47 + 21.83) = 10.92 \\
\frac{L_B}{L_{B\odot}} &= 10^{10.92} = 8.32 \times 10^{10} \\
L_B &= \boxed{8.32 \times 10^{10}\, L_{B\odot}\ .}
\end{aligned}$$

I will accept the above answer as the final answer to this problem. However, the author's "Hint" is actually not the best way to do this since it gives the answer in an unknown quantity $L_{B\odot}$. We could calculate this solar blue luminosity using Eq. (I-12) and Table I-1 in the notes, but this is not needed since we can express the galaxy's blue luminosity in terms of the Sun's bolometric luminosity by ignoring this hint. Instead, write the luminosity equation as

$$\begin{aligned}
\log\left[\frac{L_B}{L_\odot}\right] &= 0.4\,[M_{\rm bol}(\odot) - M_B] = 0.4\,(4.76 + 21.83) = 10.64 \\
\frac{L_B}{L_\odot} &= 10^{10.64} = 4.33 \times 10^{10} \\
L_B &= \boxed{4.33 \times 10^{10}\, L_\odot\ ,}
\end{aligned}$$

or the galaxy's luminosity in the blue band is 43.3 billion solar bolometric luminosities.

Solution (f): (5 pts) Using the results from parts (d) and (e) [the first solution], we get

$$\boxed{\frac{M}{L_B} = \frac{6.66 \times 10^{11} M_\odot}{8.32 \times 10^{10}\, L_{B\odot}} = 8.00\ M_\odot/L_{B\odot}\ .}$$

If we use the second solution, we get

$$\boxed{\frac{M}{L_B} = \frac{6.66 \times 10^{11} M_\odot}{4.33 \times 10^{10}\, L_\odot} = 15.4\ M_\odot/L_\odot\ .}$$

Both of these values are on the high side of the average value for Sa galaxies given in Table 23.1 on page 1010 in the textbook.

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5. (20 pts) Problem 24.4, Page 1092 in the Carroll and Ostlie textbook.

Solution (a): (15 pts) We have the LMC's mass as $M = 2 \times 10^{10} M_\odot$ and $r = 50$ kpc. We are to assume that the MW's rotation curve is flat out to the LMC and that $C = 23$. Using the MW's rotation curve in Fig. 22.27 on page 956, we see that the orbital velocity of the LMC is approximately 240 km/s. We are to determine the time it will take for the LMC to spiral into the MW. This occurs as a result of dynamical friction (see pages 1055-1061 in the textbook). All we need to do here is follow Example 24.1 which carries out a similar calculation for a globular star cluster in orbit about M31.

However, Example 24.1 has a globular star cluster falling to the center of M31. In our question here, it is unclear as to whether the author wants us to figure out the time it will take that "back" side of the LMC to just reach the "rim" of the MW and hence have the MW completely engulf the LMC, or whether they want us to figure the time it will take to get all of the way to center. To avoid any confusion, I will present both solutions here and you will get credit for either one you chose to solve. For the "rim" calculation, we will assume the size of the LMC is negligible to the size of the MW and hence treat it like a point particle.

Spiral to the Center of the MW: All we have to use is Eq. (24.2) in Example 24.1 in the textbook. We first need to make a few unit conversions:

$$\begin{aligned} v_M &= 240 \text{ km/s } \times 10^5 \text{ cm/km } = 2.40 \times 10^7 \text{ cm/s} \\ r_i &= 50 \text{ kpc } \times 3.0856 \times 10^{21} \text{ cm/kpc } = 1.5 \times 10^{23} \text{ cm} \\ M &= 2 \times 10^{10} M_\odot \cdot (1.99 \times 10^{33} \text{ gm}/M_\odot = 4 \times 10^{43} \text{ gm} . \end{aligned}$$

The time it will take for the LMC to spiral into the MW's center is then

$$\begin{aligned} t &= \frac{2\pi v_M r_i^2}{CGM} \\ &= \frac{2\pi(2.40 \times 10^7 \text{ cm/s})(1.5 \times 10^{23} \text{ cm})^2}{23 \cdot (6.673 \times 10^{-8} \text{ dyne cm}^2/\text{gm}^2)(4 \times 10^{43} \text{ gm})} \\ &= 5.6 \times 10^{16} \text{ s } \cdot (1 \text{ yr}/3.16 \times 10^7 \text{ s}) \\ &= \boxed{1.8 \times 10^9 \text{ yr}} , \end{aligned}$$

or 1.8 billion years.

Spiral until the LMC is just inside the MW: We need to rewrite the integral on page 1059 of the textbook as

$$\int_{r_i}^{R_{\rm MW}} r\, dr = -\frac{CGM}{4\pi v_M} \int_0^t dt' ,$$

where $R_{\rm MW} = 25$ kpc ($= 7.7 \times 10^{22}$ cm). Then the time to spiral to just inside the MW is

$$t = \frac{2\pi v_M}{CGM} \left(r_i^2 - R_{\rm MW}^2 \right)$$

$$
\begin{aligned}
&= \frac{2\pi(2.40 \times 10^7 \text{ cm/s})\,[(1.5 \times 10^{23} \text{ cm})^2 - (7.7 \times 10^{22} \text{ cm})^2]}{23 \cdot (6.673 \times 10^{-8} \text{ dyne cm}^2/\text{gm}^2)(4 \times 10^{43} \text{ gm})} \\
&= 4.1 \times 10^{16} \text{ s } \cdot (1 \text{ yr}/3.16 \times 10^7 \text{ s}) \\
&= \boxed{1.3 \times 10^9 \text{ yr}} \,,
\end{aligned}
$$

or 1.3 billion years.

Solution (b): (5 pts) The textbook quotes a value of 10 billion years before the MW and the LMC merge based on detailed modeling (see page 1060). Depending upon the meaning of this merger, this is anywhere from a factor of $\boxed{5.6}$ to a factor of $\boxed{7.7}$ times longer than the approximate solutions determined above.

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6. (20 pts) Problem 25.7, Page 1151 in the Carroll and Ostlie textbook.

Solution: We simply need to make use of Kepler's 3rd law of planetary motion coupled with Newtonian mechanics to solve this problem. Using the equations and information on pages 1120 and 1121 of the textbook, since the Milky Way and M31 are on a collision course the orbital period is the sum of the Hubble time t_H plus the time it will take from present for the two galaxies to collided t_c. Assuming a constant velocity of $v =$ 119 km/s with a separation $r = 770$ kpc, this collisional time is $t_c = r/v = 6.3 \times 10^9$ yr (as given on page 1120) which converts to 2.0×10^{17} s. Making use of Eq. (XI-2) in the notes, the Hubble time for $h = 0.5$ is

$$
t_H = \frac{1}{H_\circ} = \frac{1}{100h \text{ km/s/Mpc}} = \frac{9.78 \times 10^9}{h} \text{ yr } = \frac{9.78 \times 10^9}{0.5} \text{ yrs } = 2.0 \times 10^{10} \text{ yr} \,,
$$

which converts to 6.2×10^{17} s. This gives a total orbital period of

$$
P = t_H + t_c = 2.6 \times 10^{10} \text{ yr } = 8.1 \times 10^{17} \text{ s} \,.
$$

Using this period in Eq. (25.14) in the text,

$$
v^2 - \frac{2GM}{r} + \left(\frac{2\pi GM}{P}\right)^{2/3} = 0 \,,
$$

and converting all of the known quantities into cgs units, we get

$$
1.42 \times 10^{14} \text{ cm}^2/\text{s}^2 - (5.62 \times 10^{-32} \text{ cm}^2/\text{s}^2/\text{gm})\, M + (5.18 \times 10^{-25} \text{ cm}^3/\text{s}^3/\text{gm}\, M)^{2/3} = 0 \,.
$$

Each term has a cm^2/s^2 unit in it, so we can cancel those units from the equation. If we express the mass in units of solar masses instead of grams, we get

$$
\begin{aligned}
112\, M - (1.03 \times 10^9\, M)^{2/3} &= 1.42 \times 10^{14} \\
112\, M - 1.02 \times 10^6\, M^{2/3} &= 1.42 \times 10^{14} \,.
\end{aligned}
$$

This equation has no analytic solution, so we must numerically solve for M (*i.e.*, make a guess at M, in solar masses, and see if the equation holds). The following table shows the results of the numerical iterative solution. In this table, LHS stand for left-hand side of the equation above, which is to be compared to the RHS (right-hand side) of the equation, 1.42×10^{14}. We start with the value given on page 1121 in the text.

$M/M_\odot$	LHS	Comparison	RHS
4.8×10^{12}	2.47×10^{14}	$>$	1.42×10^{14}
2.0×10^{12}	6.21×10^{13}	$<$	1.42×10^{14}
3.0×10^{12}	1.24×10^{14}	$<$	1.42×10^{14}
3.1×10^{12}	1.30×10^{14}	$<$	1.42×10^{14}
3.2×10^{12}	1.36×10^{14}	$<$	1.42×10^{14}
3.3×10^{12}	1.44×10^{14}	$>$	1.42×10^{14}
3.28×10^{12}	1.42×10^{14}	$=$	1.42×10^{14}

As such, the total mass of the Milky Way and Andromeda system is 3.28 trillion solar masses! Taking significant digits into account gives

$$\boxed{M = 3.3 \times 10^{12}\ M_\odot\ .}$$

The total luminosity of the Milky Way and M31 is given on page 1121 as $6.9 \times 10^{10} L_\odot$. Hence the mass-to-light ratio is

$$\boxed{\frac{M}{L} = \frac{3.3 \times 10^{12}\ M_\odot}{6.9 \times 10^{10}\ L_\odot} = 48\ \frac{M_\odot}{L_\odot}}$$

for $h = 0.5$, which 2/3-rds the value of 70 $M_\odot/L_\odot$ given in the text for $h = 1.0$.

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7. (20 pts) Problem 26.7, Page 1217 in the Carroll and Ostlie textbook.

Solution: The discussion of the efficiency of accretion in the vicinity of a black hole is presented on page 1182 of the text. For the non-rotating black hole, the inner-edge of an accretion disk resides at $R = 3R_s$, where R_s is the Schwarzschild radius of a non-rotating black hole given by Eq. (VIII-34) in the notes:

$$R_s = \frac{2G\,M}{c^2}\ .$$

The event horizon for a maximally rotating black hole lies at a distance of $R = 0.5R_s$. The luminosity of an accretion disk is given by Eq. (17.20):

$$L_{\rm disk} = G\,\frac{M\,\dot{M}}{2R}\ .$$

Equating this expression with the accretion luminosity as a function of efficiency η (Eq. 26.5) gives

$$L_{\rm disk} = \eta\, \dot{M}\, c^2 = G\, \frac{M\, \dot{M}}{2R}\ .$$

Solving for the efficiency gives

$$\eta = \frac{G\, M}{2\, c^2 R}\ .$$

At this point, let's introduce a parameter β which gives the scale factor of the distance from a singularity with respect to the Schwarzschild radius:

$$R = \beta\, R_s = \frac{2\beta\, G\, M}{c^2}\ .$$

Substituting this into our efficiency equation above gives

$$\eta = \frac{G\, M}{2\, c^2 \beta\, R_s} = \frac{G\, M\, c^2}{4\, c^2 \beta\, G\, M} = \frac{1}{4\, \beta}\ .$$

For the non-rotating black hole, $\beta = 3$, so $\boxed{\eta = 0.083 = 8.3\%.}$

For the rotating black hole, $\beta = 0.5$, so $\boxed{\eta = 0.50 = 50\%.}$

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8. (20 pts) Problems 27.4 & 27.5, Page 1281 in the Carroll and Ostlie textbook.

Solution (a), Problem 27.4: For this problem, we are to assume that the Universe is closed, hence $k > 0$. Eq. (27.23) is

$$R = \frac{4\pi G\, \rho_\circ}{3k\, c^2}\, [1 - \cos(x)] = \frac{1}{2}\, \frac{\Omega_\circ}{\Omega_\circ - 1}\, [1 - \cos(x)]$$

and Eq. (27.25) is

$$t = \frac{4\pi G\, \rho_\circ}{3k^{3/2} c^3}\, [x - \sin(x)] = \frac{1}{2H_\circ}\, \frac{\Omega_\circ}{(\Omega_\circ - 1)^{3/2}}\, [x - \sin(x)]\ ,$$

where x is a dummy variable used to show how R varies with t. We are to prove that these equations are the solutions to Eq. (27.10),

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G\, \rho_\circ}{3R} = -k\, c^2\ .$$

The solution follows by taking derivatives of these two equations, combining them, then inserting them into Eq. (27.10). First, let

$$\begin{aligned} \alpha &= \frac{4\pi G\, \rho_\circ}{3k\, c^2} \quad \text{and} \\ \beta &= \frac{4\pi G\, \rho_\circ}{3k^{3/2} c^3}\ , \end{aligned}$$

where the ratio of these two parameters is

$$\frac{\alpha}{\beta} = \frac{4\pi G\,\rho_\circ}{3k\,c^2} \cdot \frac{3k^{3/2}c^3}{4\pi G\,\rho_\circ} = k^{1/2}c \ ,$$

and the $R(x)$ and $t(x)$ equations take the form

$$\begin{aligned} R &= \alpha\,(1-\cos x) \\ t &= \beta\,(x-\sin x) \ . \end{aligned}$$

The derivatives of these equations are

$$\begin{aligned} \frac{dR}{dx} &= \alpha\,\sin x \\ \frac{dt}{dx} &= \beta - \beta\,\cos x = \beta\,(1-\cos x) \ . \end{aligned}$$

We can divide these equations to obtain the time derivative of R:

$$\frac{dR}{dt} = \frac{dR/dx}{dt/dx} = \frac{\alpha\,\sin x}{\beta\,(1-\cos x)} = k^{1/2}c\,\frac{\sin x}{1-\cos x} = k^{1/2}c\,\cot\frac{x}{2} \ ,$$

using one of the cotangent half-angle formulae from trigonometry. Using this in our equation of motion, we have

$$\begin{aligned} \left(\frac{dR}{dt}\right)^2 &= \frac{8\pi G\,\rho_\circ}{3R} - k\,c^2 \\ k\,c^2\cot^2\frac{x}{2} &= \frac{8\pi G\,\rho_\circ}{3R} - k\,c^2 \\ \cot^2\frac{x}{2} &= \frac{8\pi G\,\rho_\circ}{3k\,c^2 R} - 1 = \frac{2\alpha}{R} - 1 \ . \end{aligned}$$

At this point to ease the calculation, let's set $\gamma = 2\alpha/R$ and use a second cotangent trigonometric identity to write

$$\begin{aligned} \frac{1+\cos x}{1-\cos x} &= \frac{2\alpha}{R} - 1 = \gamma - 1 \\ 1+\cos x &= (\gamma-1)\,(1-\cos x) = \gamma - \gamma\,\cos x - 1 + \cos x \\ \gamma - \gamma\,\cos x &= 2 \\ \gamma\,(1-\cos x) &= 2 \\ 1-\cos x &= \frac{2}{\gamma} \ . \end{aligned}$$

At this point, we can go back to our original equation for R and solve for $(1-\cos x)$:

$$\begin{aligned} R &= \alpha\,(1-\cos x) \\ 1-\cos x &= \frac{R}{\alpha} \ . \end{aligned}$$

Using this in the preceding equation for $(1 - \cos x)$ and using our definition for γ, we immediately see that

$$\frac{R}{\alpha} = \frac{2}{\gamma} = \frac{2}{2\alpha/R} = \frac{R}{\alpha} \qquad \surd$$

hence the parameterized equations for R and t as given above are indeed solutions to the equation of motion shown above for $k > 0$.

Solution (b), Problem 27.5: For this problem, we are to assume that the Universe is open, hence $k < 0$. Eq. (27.27) is

$$R = \frac{4\pi G\,\rho_\circ}{3|k|\,c^2}\,[\cosh(x) - 1] = \frac{1}{2}\,\frac{\Omega_\circ}{1 - \Omega_\circ}\,[\cosh(x) - 1]$$

and Eq. (27.29) is

$$t = \frac{4\pi G\,\rho_\circ}{3|k|^{3/2}c^3}\,[\sinh(x) - x] = \frac{1}{2H_\circ}\,\frac{\Omega_\circ}{(1 - \Omega_\circ)^{3/2}}\,[\sinh(x) - x] \ ,$$

where x is a dummy variable used to show how R varies with t. We are to prove that these equations are the solutions to Eq. (27.10),

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G\,\rho_\circ}{3R} = -k\,c^2 \ .$$

Since k is negative here and the solutions have been presented with an absolute value sign around k, we can rewrite this equation of motion as

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G\,\rho_\circ}{3R} = |k|\,c^2 \ ,$$

where we have, from the start, taken account of the negative value of k on the right-hand side of the equation (since a negative of a negative value produces a positive value).

The solution follows by taking derivatives of these two equations, combining them, then inserting them into the modified version of Eq. (27.10) above. First, let

$$\begin{aligned} \alpha &= \frac{4\pi G\,\rho_\circ}{3|k|\,c^2} \quad \text{and} \\ \beta &= \frac{4\pi G\,\rho_\circ}{3|k|^{3/2}c^3} \ , \end{aligned}$$

where the ratio of these two parameters is

$$\frac{\alpha}{\beta} = \frac{4\pi G\,\rho_\circ}{3|k|\,c^2} \cdot \frac{3|k|^{3/2}c^3}{4\pi G\,\rho_\circ} = |k|^{1/2}c \ ,$$

and the $R(x)$ and $t(x)$ equations take the form

$$\begin{aligned} R &= \alpha\,(\cosh x - 1) \\ t &= \beta\,(\sinh x - x)\ . \end{aligned}$$

The derivatives of these equations are

$$\begin{aligned} \frac{dR}{dx} &= \alpha\ \sinh x \\ \frac{dt}{dx} &= \beta\ \cosh x - \beta = \beta\,(\cosh x - 1)\ . \end{aligned}$$

We can divide these equations to obtain the time derivative of R:

$$\frac{dR}{dt} = \frac{dR/dx}{dt/dx} = \frac{\alpha\ \sinh x}{\beta\,(\cosh x - 1)} = |k|^{1/2} c\,\frac{\sinh x}{\cosh x - 1} = |k|^{1/2} c\ \coth\frac{x}{2}\ ,$$

using one of the hyperbolic cotangent half-angle formulae from hyperbolic trigonometry. Using this in our equation of motion, we have

$$\begin{aligned} \left(\frac{dR}{dt}\right)^2 &= \frac{8\pi G\,\rho_\circ}{3R} + |k|\,c^2 \\ |k|\,c^2\coth^2\frac{x}{2} &= \frac{8\pi G\,\rho_\circ}{3R} + |k|\,c^2 \\ \coth^2\frac{x}{2} &= \frac{8\pi G\,\rho_\circ}{3|k|\,c^2 R} + 1 = \frac{2\alpha}{R} + 1\ . \end{aligned}$$

At this point to ease the calculation, let's set $\gamma = 2\alpha/R$ and use a second hyperbolic cotangent trigonometric identity to write

$$\begin{aligned} \frac{\cosh x + 1}{\cosh x - 1} &= \frac{2\alpha}{R} + 1 = \gamma + 1 \\ \cosh x + 1 &= (\gamma + 1)\,(\cosh x - 1) = \gamma\ \cosh x - \gamma + \cosh x - 1 \\ \gamma\ \cosh x - \gamma &= 2 \\ \gamma\,(\cosh x - 1) &= 2 \\ \cosh x - 1 &= \frac{2}{\gamma}\ . \end{aligned}$$

At this point, we can go back to our original equation for R and solve for $(\cosh x - 1)$:

$$\begin{aligned} R &= \alpha\,(\cosh x - 1) \\ \cosh x - 1 &= \frac{R}{\alpha}\ . \end{aligned}$$

Using this in the preceding equation for $(\cosh x - 1)$ and using our definition for γ, we immediately see that

$$\frac{R}{\alpha} = \frac{2}{\gamma} = \frac{2}{2\alpha/R} = \frac{R}{\alpha} \qquad \surd$$

hence the parameterized equations for R and t as given above are indeed solutions to the equation of motion shown above for $k < 0$.

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9. (20 pts) Problem 27.21, Page 1283 in the Carroll and Ostlie textbook.

Solution: To solve this problem, we need to write down an equation that describes the matter density history and the radiation density history. A discussion on how this is done is made on pages 1244-1246 of your textbook. Radiation density varies with the universal scale factor R as

$$\rho_r = \frac{\rho_{r\circ}}{R^4} ,$$

as given by Eq. (27.52) in the text, where the current radiation density is given by Eq. (27.51) as

$$\rho_{r\circ} = \frac{aT_\circ^4}{c^2} ,$$

and where $a = 7.5659 \times 10^{-15}$ erg/cm^3/K^4 is the radiation constant, and $c = 2.997925 \times 10^{10}$ cm/s is the speed of light. The matter density variation with the universal scale factor is given by Eq. (27.4):

$$\rho_m = \frac{\rho_\circ}{R^3} .$$

As can be seen from these history equations, radiation density dominated matter density early in the history of the universe due to its R^{-4} dependence. The cross-over from radiation domination to matter domination would have occurred when $\rho_m = \rho_r$, or

$$\begin{aligned} \frac{\rho_\circ}{R^3} &= \frac{\rho_{r\circ}}{R^4} \\ R &= \frac{\rho_{r\circ}}{\rho_\circ} = \frac{aT_\circ^4}{\rho_\circ c^2} = \frac{8\pi G a T_\circ^4}{3H_\circ^2 c^2}\frac{1}{\Omega_\circ} \\ &= \frac{2.475 \times 10^{-5}}{\Omega_\circ h^2} , \end{aligned}$$

where we have made use of Eq. (27.18) for the density parameter,

$$\Omega_\circ = \frac{\rho_\circ}{\rho_c} ,$$

Eq. (27.15) for the critical density,

$$\rho_c = \frac{3H_\circ^2}{8\pi G} = 1.88 \times 10^{-29}\, h^2 \text{ gm/cm}^3 ,$$

and Eq. (27.14),

$$H_\circ = 100\, h \text{ km/s/Mpc } = 3.24 \times 10^{-18}\, h \text{ s}^{-1} .$$

The other parameters used are $G = 6.6726 \times 10^{-8}$ dyne cm^2/gm^2, and the current temperature of the Universe as measured by COBE of $T_\circ = 2.726$ K (see Fig. 27.8).

Since we are to assume that the matter density is entirely due to the baryonic density, we can use Eq. (27.19) in the textbook,

$$\Omega_\circ = \Omega_{B,\circ} = 0.016\, h^{-2} ,$$

which will enable use to write the earlier scale factor equation as

$$R = \frac{2.475 \times 10^{-5}}{0.016\, h^{-2}\, h^2} = 1.55 \times 10^{-3} \; .$$

This would have been the scale factor of the Universe when $\rho_r = \rho_m$ for a universe composed of pure baryons as the matter component. This scale factor corresponds to a redshift (see Eq. 27.31) of

$$z = \frac{1}{R} - 1 = 645 \; .$$

Also, we can use Eq. (27.47), $T_\circ = RT =$ constant, to calculate the temperature of the Universe at that time:

$$T = \frac{T_\circ}{R} = \frac{2.726 \text{ K}}{1.55 \times 10^{-3}} = 1760 \text{ K} \; .$$

Finally using the redshift above, we can calculate the time when $\rho_r = \rho_m$ occurred. Since $z \gg 1$, we can use Eq. (27.32) to calculate this time:

$$\begin{aligned}
\frac{t}{t_{\rm H}} &= \frac{2}{3}\frac{1}{(1+z)^{3/2}\Omega_\circ^{1/2}} \\
t &= \frac{2}{3}\frac{1}{H_\circ(1+z)^{3/2}\Omega_\circ^{1/2}} \\
&= \frac{2.06 \times 10^{17}\text{ s}}{h\,(1+z)^{3/2}\Omega_\circ^{1/2}} = \frac{6.52 \times 10^{9}\text{ yr}}{h\,(1+z)^{3/2}\Omega_\circ^{1/2}} \\
&= \frac{6.52 \times 10^{9}\text{ yr}}{h\,(1+z)^{3/2}(0.016\,h^{-2})^{1/2}} = \frac{5.15 \times 10^{10}\text{ yr}}{(1+z)^{3/2}} \\
&= \boxed{3.1 \times 10^{6}\text{ yr} \; .}
\end{aligned}$$

Next, we use the Saha equation to figure out the temperature when recombination occurs. This is typically set when $N_{\rm H\,I} = N_{\rm H\,II}$. Assuming the electron density scales as Eq. (27.58), the authors have already determined an equation for recombination time (see Eq. 27.60):

$$t_{\rm rec} = \frac{1.8 \times 10^{5}\text{ yr}}{\Omega_\circ^{1/2} h} = \frac{1.8 \times 10^{5}\text{ yr}}{(0.016\,h^{-2})^{1/2} h} = \boxed{1.4 \times 10^{6}\text{ yr} \; .}$$

The ratio of our cross-over time to the recombination time gives

$$\boxed{\frac{t}{t_{\rm rec}} = 2.2 \; .}$$

Since these times are within a factor of 2, this is considered to be approximately the same time on a cosmic timescale.

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10. (25 pts) Problem 28.5, Page 1324 in the Carroll and Ostlie textbook.

Solution: To figure out the total baryon mass in an observable universe, we will assume that the density of these baryons are uniform and use the definition of mass density of $\rho = M/V$. Then for a spherically symmetric volume, the total mass is

$$M_B = V\,\rho_B = \frac{4}{3}\pi d^3 \rho_B \ ,$$

where d is the path length that a photon travels since the beginning time: $t_{\rm init} = 0$. Since velocity ($v = c$ for light) is just distance traveled over a time interval, we then have

$$d = c\,t \ ,$$

where t is the time since the beginning. For this problem, we are interested in the cross-over time when matter density starts to dominate radiation (*i.e.*, the time when $\rho_B = \rho_r$, where we are labeling the matter density as the baryon density ρ_B). This time is given by Eq. (27.53) in the textbook as

$$t = \frac{2}{3}\,\frac{1}{H_\circ(1+z)^{3/2}\Omega_\circ^{1/2}} = 2.6 \times 10^{10}\,\Omega^{-2}\,h^{-4}\ \text{s} \ .$$

In this problem, we are told to assume that $\Omega_\circ = 0$ and $h = 1$, giving

$$t = 2.6 \times 10^{10}\ \text{s} \ = 820\ \text{yr} \ .$$

Using this time in the size of the observable universe equation, we get

$$d = ct = (3.0 \times 10^{10}\ \text{cm/s})\,(2.6 \times 10^{10}\ \text{s}) = 7.8 \times 10^{20}\ \text{cm} \ = 820\ \text{ly} \ .$$

From an equation given on page 1245 in the textbook, we can also calculate the temperature of the Universe at that time as

$$T = \frac{T_\circ}{R} = 1.1 \times 10^5\,\Omega_\circ h^2\ \text{K} = 1.1 \times 10^5\ \text{K} \ .$$

With the use of this temperature, we can calculate the baryon density since it is equal to the radiation density at that point:

$$\begin{aligned} \rho_B &= \rho_r = \frac{aT^4}{c^2} = \frac{(7.566 \times 10^{-15}\ \text{erg/cm}^3/\text{K}^4)(1.1 \times 10^5\ \text{K})^4}{(3.0 \times 10^{10}\ \text{cm/s})^2} \\ &= 1.2 \times 10^{-15}\ \text{gm/cm}^3 \end{aligned}$$

Using this in our original mass-density equation, we get

$$\begin{aligned} M_B &= \frac{4}{3}\pi d^3 \rho_B = \frac{4}{3}\pi d^3 \frac{aT^4}{c^2} = 2.4 \times 10^{48}\ \text{gm} \\ &= \boxed{1.2 \times 10^{15} M_\odot \ .} \end{aligned}$$

Finally, if we compare this baryon mass to the Jeans' mass as a function of temperature plot of Fig. 28.3 in the textbook, we can compare how our calculated baryon mass corresponds to the Jeans' mass. The calculated temperature for the epoch when $\rho_B = \rho_r$ is 1.1×10^5 K or $\log T = 5.04$. As such, we will be interested to see how M_B varies with M_J for temperatures in this regime and cooler temperatures. The above baryon mass gives $\log(M_B/M_\odot) = 15.1$. We see that $M_B \leq M_J$ in this portion of the radiation era near the cross-over temperature. As such, no gravitational instabilities will arise, hence no density fluctuations will result in collapse. However, in the matter dominated universe just after recombination has taken place, $M_B \gg M_J$. Hence as soon as recombination occurs, gravitational instabilities will quickly arise causing growing density fluctuations (*i.e.*, fragmentation) and gravitational collapse.

We also can do some direct calculations of the Jeans' mass during the radiation era and the matter era by using the equations supplied in the notes. During the radiation era, the baryonic Jeans' mass is given by Eq. (XI-95):

$$M^r_{B-J} = 5.4 \times 10^{18}\, \Omega_B h^2 T_{\rm eV}^{-3}\ M_\odot\ ,$$

where $T_{\rm eV}$ is the temperature of the radiation field expressed in units of eV (1 eV = 1.1605×10^4 K). If we use the temperature calculated for the cross-over time and use a temperature slightly higher than this (say 1.50×10^5 K = 12.9 eV) for the temperature of the radiation field just prior to cross-over, setting $\Omega_B = \Omega_\circ$ gives a Jeans' mass of

$$M^r_{B-J} = \frac{5.4 \times 10^{18}\ M_\odot}{(12.9)^3} = 2.5 \times 10^{15}\ M_\odot\ .$$

As can be seen, $M_B < M^r_{B-J}$, so no gravitational instabilities will grow and any density fluctuations that exist will not result in a collapse.

As we have seen in Problem 9, the radiation era to matter era cross-over occurs very near the the recombination time (within a factor of 2). Hence, recombination occurs slightly after cross-over, at a z of 1100. As such, we can use Eq. (XI-97),

$$M^m_{B-J} = 1.3 \times 10^5\ \left(\Omega_B h^2\right)^{-1/2} \left(\frac{z}{1100}\right)\ M_\odot\ .$$

Since $z < 1100$ past the recombination time, $M^m_{B-J} < 1.3 \times 10^5\ M_\odot$ for the remainder of the matter era. As such, $M_B \gg M^m_{B-J} \rightarrow$ the mass of baryons will become gravitationally unstable which will cause density fluctuations to grow and collapse to occur. Note that due to this large inequality, the collapse occurs very rapidly.