PHYS-2010: General Physics I
Course Lecture Notes
Section IV

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Edition 2.6

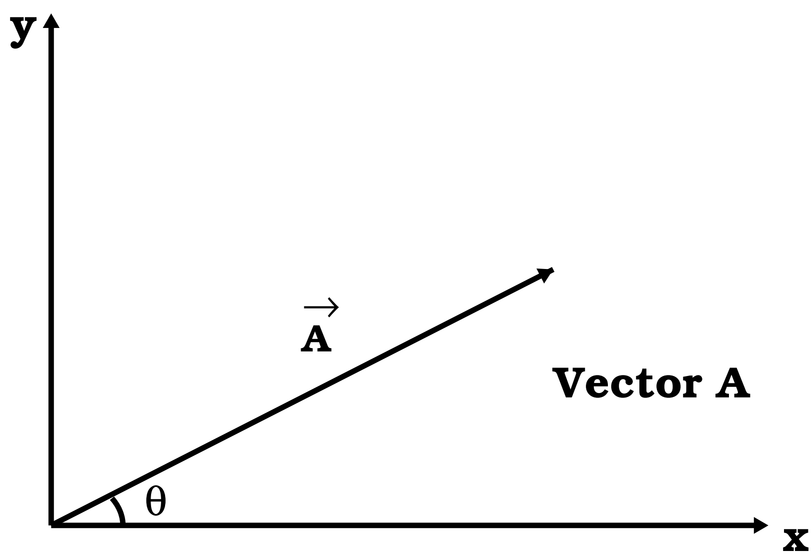
Abstract

These class notes are designed for use of the instructor and students of the course PHYS-2010: General Physics I taught by Dr. Donald Luttermoser at East Tennessee State University. These notes make reference to the *College Physics, 11th Edition* (2018) textbook by Serway and Vuille.

IV. Motion in Two Dimensions

A. Vector Arithmetic.

1. We now will be working with both the x and y axes. **Vectors** are represented on an x-y graph as an arrow with a distinct direction indicated by the arrowhead:



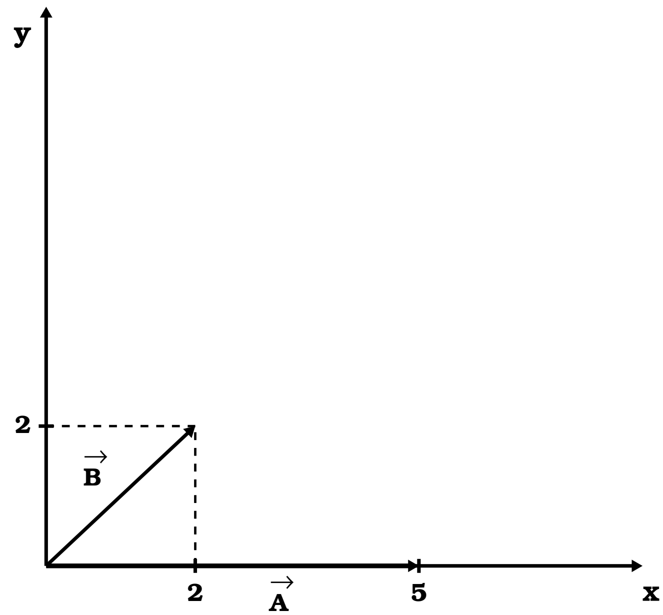
a) Note that vector $\vec{A}$ makes an angle θ with the x-axis.

b) The x-axis is always the reference line which vector direction angles are measured.

2. Vector addition.

a) Two or more vectors can be added graphically by placing the *beginning* of the *2nd* vector at the *tail* of the *1st* vector.

Example IV–1. Graphically add vectors $\vec{A}$ and $\vec{B}$ in the diagrams below.



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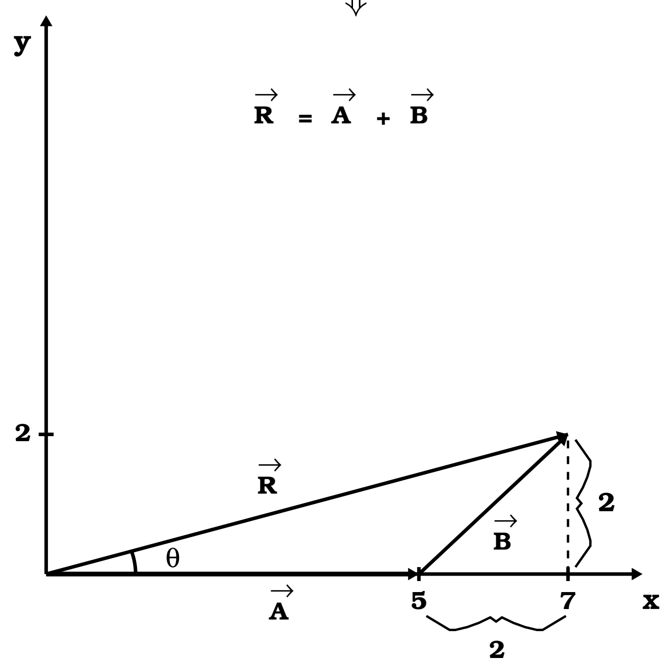

$\therefore$ $\vec{R}$ is 7 units long in the x direction and is 2 units long in the y direction.

b) Two or more vectors can be added algebraically by rewriting the vectors into 2 components, x & y, then adding the respective components.

Example IV–2. Algebraically add vectors $\vec{A}$ and $\vec{B}$ from the Example IV-1.

$$\begin{aligned} \vec{R} &= \vec{A} + \vec{B} \\ \text{where} \quad \vec{R} &= \vec{R}_x + \vec{R}_y \\ \text{and} \quad \vec{R}_x &= \vec{A}_x + \vec{B}_x \\ \vec{R}_y &= \vec{A}_y + \vec{B}_y \end{aligned}$$

$$\begin{aligned} \vec{A} &= \vec{A}_x + \vec{A}_y \\ &= 5\,\hat{x} + 0\,\hat{y} \end{aligned}$$

$$\begin{aligned} \vec{B} &= \vec{B}_x + \vec{B}_y \\ &= 2\,\hat{x} + 2\,\hat{y} \end{aligned}$$

$$\begin{aligned} \vec{R} &= \vec{A} + \vec{B} \\ &= (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} \\ &= (5+2)\,\hat{x} + (0+2)\,\hat{y} \\ &= 7\,\hat{x} + 2\,\hat{y} \end{aligned}$$

$\therefore$ $\vec{R}$ is 7 units long in the x direction and is 2 units long in the y direction.

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c) The actual length of a vector is determined from the Pythagorean theorem:

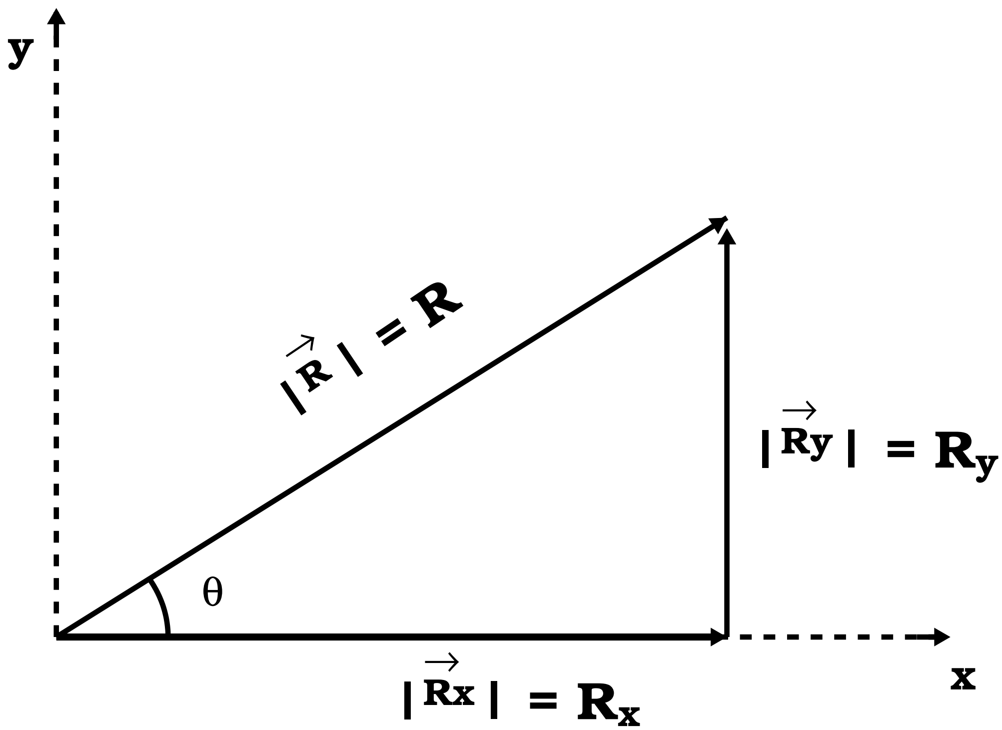
$$\boxed{R = \sqrt{R_x^2 + R_y^2}} \qquad \text{(IV-1)}$$

$\Longrightarrow$ note that the length of a vector is sometimes written as

$$\boxed{R = \left| \vec{R} \right| \ ,} \tag{IV-2}$$

and that the length is always taken to be the positive root of the square root in Equation (IV-1).

d) The angle that a vector makes with the x-axis is determined by



$$\boxed{\begin{array}{rcl} R_x & = & R \cos\theta \\ R_y & = & R \sin\theta \end{array}} \tag{IV-3}$$

$$\text{or} \quad \tan\theta = \frac{\sin\theta}{\cos\theta} \Longrightarrow \boxed{\tan\theta = \frac{R_y}{R_x} \ .} \tag{IV-4}$$

$$\text{or} \quad \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) \ .$$

Example IV–3. What are the lengths and angles of the 3 vectors, $\vec{A}, \vec{B}$, and $\vec{R}$, in Example IV-2?

$$\begin{aligned} \vec{A} = 5\,\hat{x} &\;:\; A_x = 5, \quad A_y = 0 \\ \vec{B} = 2\,\hat{x} + 2\,\hat{y} &\;:\; B_x = 2, \quad B_y = 2 \\ \vec{R} = 7\,\hat{x} + 2\,\hat{y} &\;:\; R_x = 7, \quad R_y = 2 \end{aligned}$$

$$\begin{aligned} A &= \sqrt{5^2 + 0^2} = \sqrt{25} = 5 \\ B &= \sqrt{2^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = \sqrt{4 \cdot 2} \\ &= 2\sqrt{2} \approx 2.828 \approx 3. \\ R &= \sqrt{7^2 + 2^2} = \sqrt{49+4} = \sqrt{53} \approx 7.280 \approx 7.3 \end{aligned}$$

$$\begin{aligned} \theta_A &= \tan^{-1}\left(\frac{0}{5}\right) = \tan^{-1}(0) = 0^\circ \\ \theta_B &= \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = 45^\circ \\ \theta_R &= \tan^{-1}\left(\frac{2}{7}\right) \approx \tan^{-1}(0.2857) \approx 15^\circ.9 \approx 16^\circ \;, \end{aligned}$$

though here I would also have accepted 20° as the answer for θ_R. Note that from this point forward, we will use the standard equals "=" sign even when the answer is not exact (as indicated above with the "≈" sign).

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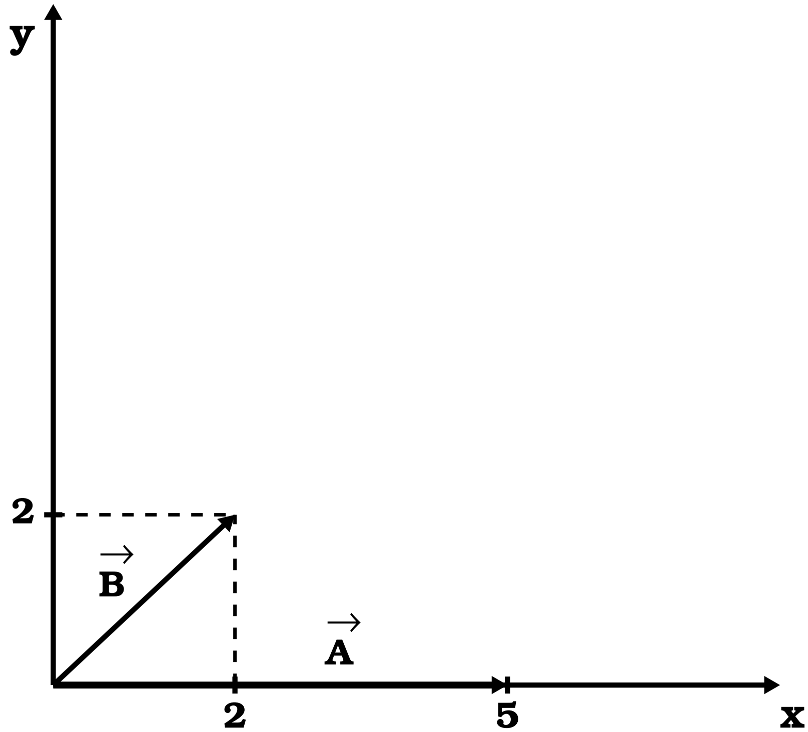
3. Vector subtraction.

a) Graphically: Flip the 2nd vector so that it is pointing in the opposite direction, then follow the rules for vector addition:

$$\begin{aligned} \vec{R} &= \vec{A} - \vec{B} \\ \vec{R} &= \vec{A} + (-\vec{B}) \end{aligned}$$

Example IV–4. Subtract $\vec{B}$ from $\vec{A}$ in Example IV-1.



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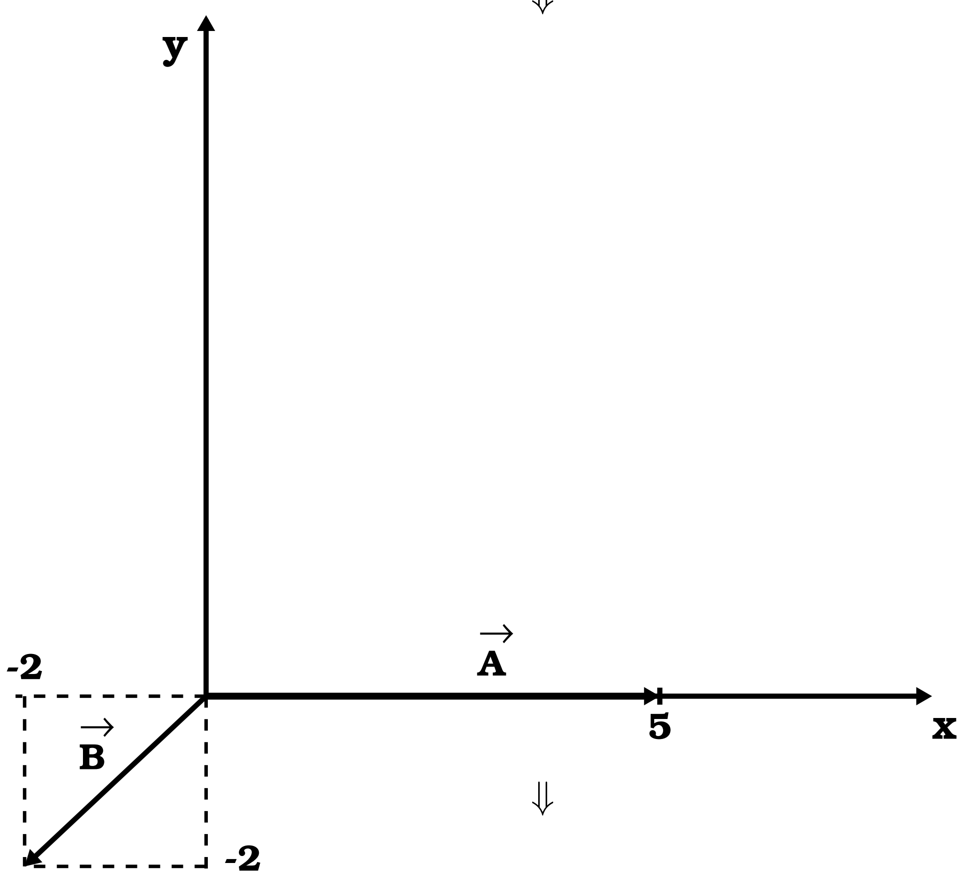

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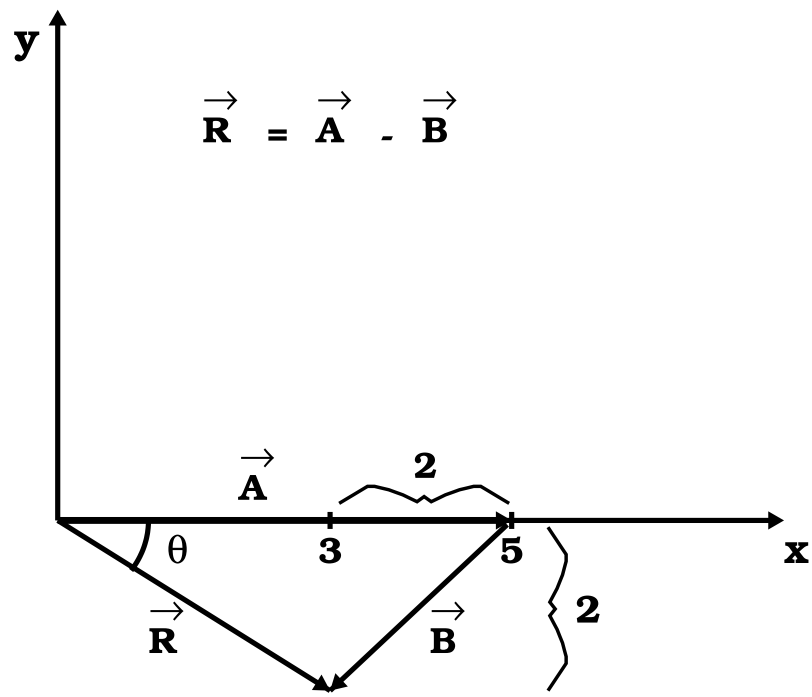

∴ $\vec{R}$ is 3 units in the x direction and is –2 units in the y direction.

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b) Subtracting one vector from another algebraically is done by subtracting the respective components:

$$\vec{R} = \vec{A} - \vec{B} = (A_x - B_x)\,\hat{x} + (A_y - B_y)\,\hat{y}\ .$$

Example IV–5. Algebraically subtract $\vec{B}$ from $\vec{A}$ in Example VI-4.

$$\begin{aligned}\vec{R} &= (5-2)\,\hat{x} + (0-2)\,\hat{y}\\ &= 3\,\hat{x} - 2\,\hat{y}\end{aligned}$$

$\Longrightarrow$ *i.e.*, 3 units over in $+x$ direction, 2 units down in $-y$ direction.

$$\begin{aligned} R &= \left|\vec{R}\right| = \sqrt{3^2 + (-2)^2} \\ &= \sqrt{9+4} = \sqrt{1\underline{3}} = 3.\underline{6}06 = 3.6 \end{aligned}$$

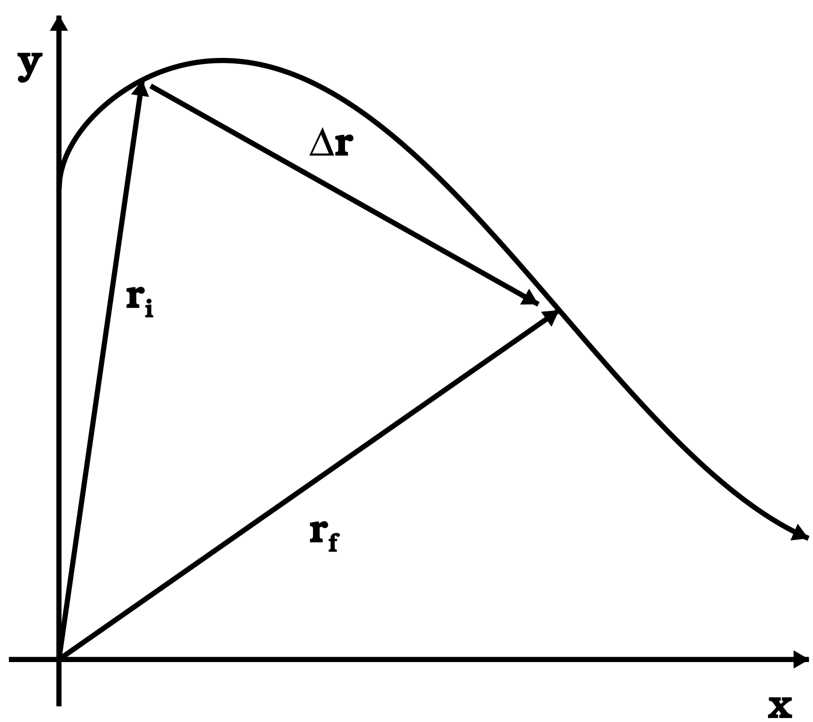
$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{-2}{3}\right) = \tan^{-1}(-0.66667) \\ &= -\underline{3}3^\circ.7 = -30^\circ \end{aligned}$$

(remember to round-off as per the significant digits of the input parameters).

B. Velocity and Acceleration in 2–D.

1. Displacement:

$$\boxed{\Delta\vec{r} = \vec{r}_{\rm f} - \vec{r}_{\rm i}\ .} \qquad \text{(IV-5)}$$



2. Velocity:

a) Average Velocity:

$$\boxed{\bar{\vec{v}} = \frac{\Delta \vec{r}}{\Delta t}\ .} \tag{IV-6}$$

b) Instantaneous Velocity $\equiv$ Velocity:

$$\boxed{\vec{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}} \equiv \frac{d\vec{r}}{dt}\ . \tag{IV-7}$$

3. Acceleration:

a) Average Acceleration:

$$\boxed{\bar{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t}\ .} \tag{IV-8}$$

b) Instantaneous Acceleration $\equiv$ Acceleration:

$$\boxed{\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}} \equiv \frac{d\vec{v}}{dt}\ . \tag{IV-9}$$

c) Note that if $\vec{a} = \bar{\vec{a}}$, then the acceleration remains constant during the time that this equation is valid.

C. Projectile Motion.

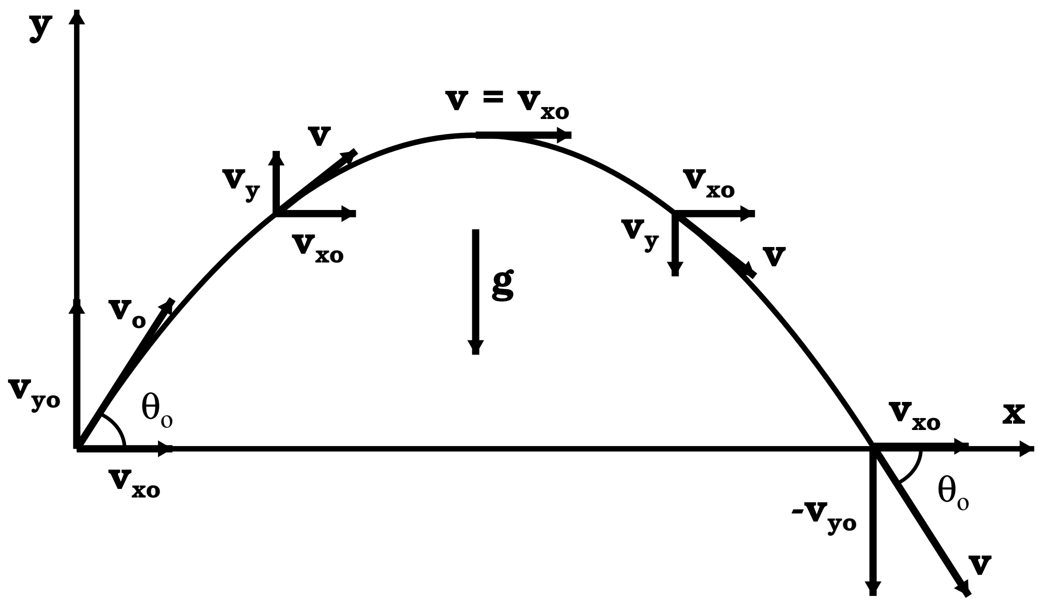
1. In this section, we will make use of the following assumptions:

a) The free-fall acceleration, $\vec{g}$, has a magnitude of 9.80 m/s^2, is constant over the range of motion, and is directed <u>downward</u>.

b) The effect of air resistance is negligible (hence the size of the object is relatively small and its surface is smooth).

c) The rotation of the Earth does not affect the motion (hence the distance traveled is small with respect to the Earth's radius).

2. For such **projectile motion**, the path is always a **parabola**.



a) The acceleration in the y direction is $-g$, just as in free fall.

b) The acceleration in the x direction is 0 (air friction is neglected).

c) $\theta_\circ$ is called the **projection angle**.

3. Breaking motion into components.

a) The vectors in our equations of motion can be broken up into x and y components:

$$v_x = v\,\cos\theta\ , \qquad v_y = v\,\sin\theta\ .$$

b) We can just use the 1–D equations of motion and set them up for each component:

x-direction (horizontal)	**y-direction (vertical)**	
$x = x_\circ + v_{\circ x}\,(t - t_\circ)$	$y = y_\circ + v_{\circ y}\,(t - t_\circ) - \frac{1}{2}\,g(t - t_\circ)^2$	(a)
$v_x = v_{\circ x}$	$v_y = v_{\circ y} - g(t - t_\circ)$	(b)
—	$v_y^2 = v_{\circ y}^2 - 2g(y - y_\circ)$	(c)

(IV-10)

c) The speed, v, of the projectile can be computed at any instant by

$$\boxed{v = \sqrt{v_x^2 + v_y^2}\ .} \tag{IV-11}$$

d) The angle with respect to the ground at which the projectile is moving can be found with

$$\boxed{\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)\ .} \tag{IV-12}$$

4. Problem-solving strategy for 2–D motion:

a) Select a coordinate system (typically Cartesian).

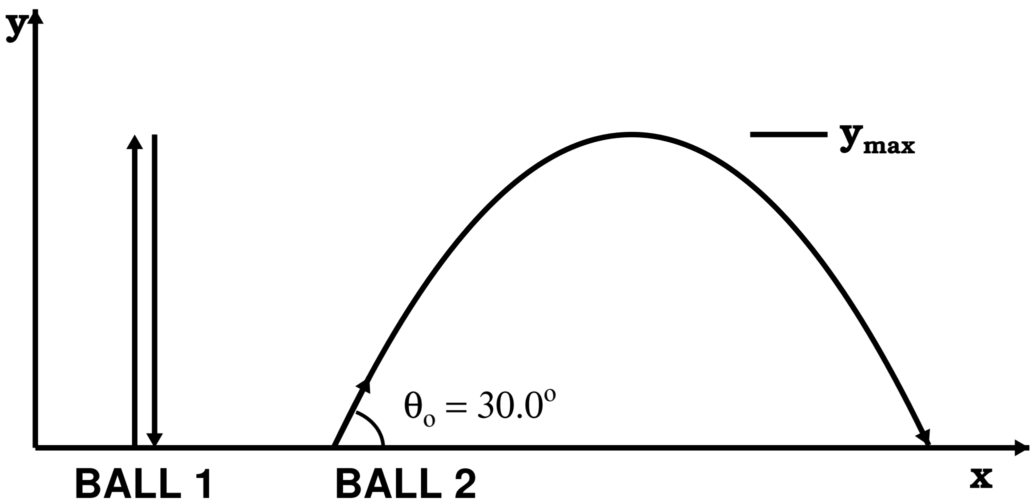
b) Resolve the initial velocity vector into x and y components.

c) Treat the horizontal motion and the vertical motion independently.

d) Horizontal motion has <u>constant velocity</u>.

e) Vertical motion has <u>constant acceleration</u>.

Example IV–6. A ball is thrown straight upward in the air and returns to the thrower's hand after 3.00 s. A second ball thrown at an angle 30.0° with respect to the horizontal direction and reaches the same maximum height as the first ball. (a) At what speed was the first ball thrown? (b) At what speed was the second ball thrown?



<u>Ball 1</u>
$t_\circ = 0$
$t_f = 3.00$ s
$\Delta x = 0$

$y = y_{max} = ?$
$y_\circ = 0$
$v_x = 0$
$v_y = 0$ (at top)

$v_\circ$ (Ball 2) = ?

Solution (a): Since the motion upward is symmetrical with the motion downward, the time it takes to get to y_{max} is half of the total time in the air:

$$t = \frac{1}{2}(3.00 \text{ s}) = 1.50 \text{ s} .$$

Since $v_y = 0$ at that position:

$$v_y = v_{\circ y} - gt$$

$$v_{\circ y} = v_y + gt = 0 + (9.80 \text{ m/s}^2 \times 1.50 \text{ s})$$

$$\boxed{v_{\circ y} = v_\circ = 14.7 \text{ m/s}}$$

Solution (b):

In order to determine the initial speed for Ball 2, we need to determine y_{max} from the data given and determined for Ball 1.

This maximum height is determined with

$$\begin{aligned}
v_y^2 &= v_{\circ y}^2 - 2gy \\
2gy &= v_{\circ y}^2 - v_y^2 \\
y &= \frac{v_{\circ y}^2 - v_y^2}{2g} \\
y_{\rm max} &= \frac{(14.7 \text{ m/s})^2 - (0 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)} \\
y_{\rm max} &= 11.0 \text{ m}
\end{aligned}$$

Now solve for $v_\circ$ for the ball thrown at 30° up to $y_{\rm max}$ (remember $v_y = 0$ at the top of the parabola):

$$\begin{aligned}
v_y^2 &= v_{\circ y}^2 - 2gy \\
v_{\circ y}^2 &= v_{y_{\rm max}}^2 + 2gy_{\rm max} \\
v_{\circ y}^2 &= 0 + 2(9.80 \text{ m/s}^2)(11.0 \text{ m}) \\
v_{\circ y}^2 &= 215.6 \text{ m}^2\,\text{s}^{-2} \\
v_{\circ y} &= 14.7 \text{ m/s},
\end{aligned}$$

which is exactly the initial velocity for the ball thrown straight up (as it should be). Then to find the total initial velocity, $v_\circ$, we just use a little trig:

$$\begin{aligned}
v_{\circ y} &= v_\circ \sin\theta \\
v_\circ &= \frac{v_{\circ y}}{\sin\theta} = \frac{14.7 \text{ m/s}}{\sin 30.0^\circ} = \frac{14.7 \text{ m/s}}{0.500}
\end{aligned}$$

$$\boxed{v_\circ = 29.4 \text{ m/s.}}$$

D. Relative Velocity.

1. Observers in different frames of reference may measure different displacements or velocities for an object in motion.

2. For measuring motion in one frame of reference with respect to a different frame of reference, we must add the motion of the other frame to our equations.

a) Displacement:

$$\boxed{\vec{r} = \vec{r}\,' + \vec{u}t} \qquad \text{(IV-13)}$$

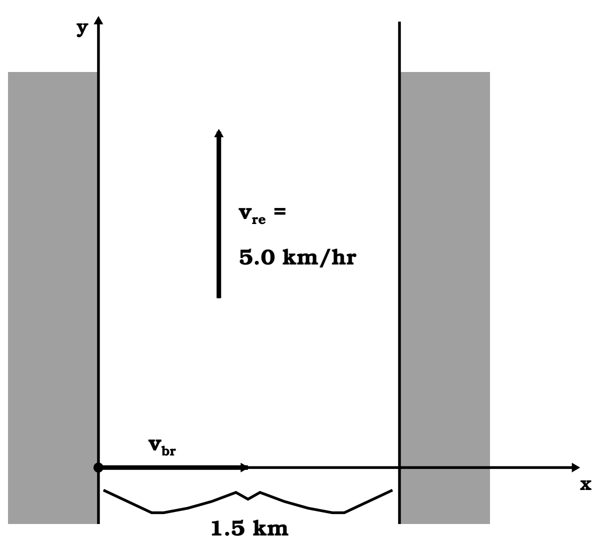
$\vec{r}$ ≡ displacement in our frame
$\vec{r}\,'$ ≡ displacement in the other frame
$\vec{u}$ ≡ velocity of the other reference frame (wrt our frame)
t ≡ time

b) Velocity:

$$\boxed{\vec{v} = \vec{v}\,' + \vec{u}} \qquad \text{(IV-14)}$$

$\vec{v}$ ≡ velocity in our frame
$\vec{v}\,'$ ≡ velocity in the other frame

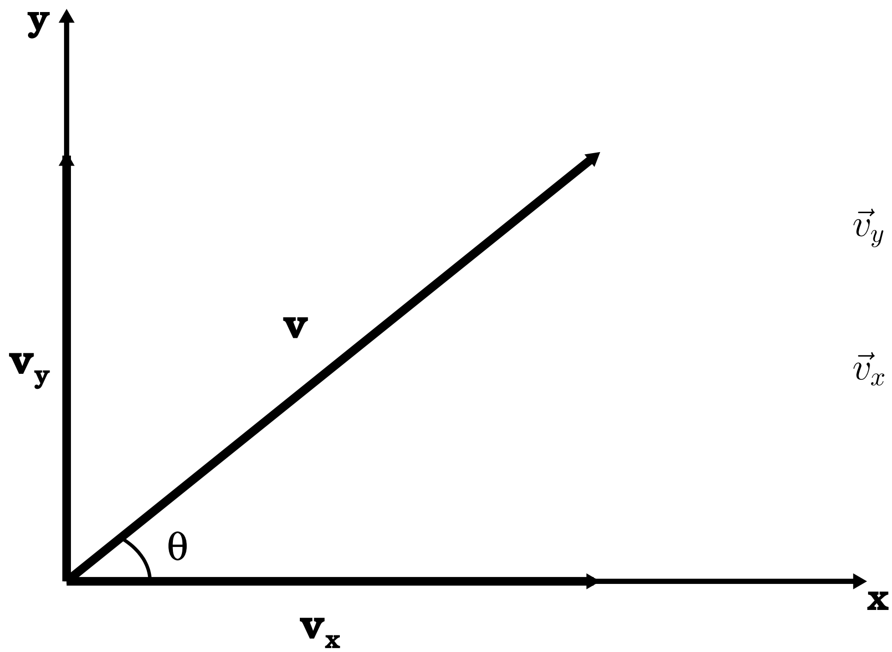
Example IV–7. A hunter wishes to cross a river that is 1.5 km wide and flows with a speed of 5.0 km/h parallel to its banks. The hunter uses a small powerboat that moves at a maximum speed of 12 km/h with respect to the water. What is the minimum time necessary for crossing?



Let $\vec{v}\,' = \vec{v}_{br}$, where "*br*" means "boat wrt river" and $\vec{u} = \vec{v}_{re}$, where "*re*" means "river wrt Earth," then

$\vec{v} = \vec{v}\,' + \vec{u}$
$\vec{v}\,' = \vec{v}_{br} = 12 \text{ km/h } \hat{x}$
$\vec{u} = \vec{v}_{re} = 5.0 \text{ km/h } \hat{y}$

In velocity space



$$\vec{v}_y = \vec{v}_{re}$$

$$\vec{v}_x = \vec{v}_{br}$$

Determine v & θ:

The actual velocity of the boat wrt the Earth is

$$\begin{aligned} v &= v_{be} = \sqrt{v_x^2 + v_y^2} = \sqrt{12^2 + 5.0^2} \text{ km/h} \\ &= \sqrt{169} \text{ km/h} = 13. \text{ km/h} \end{aligned}$$

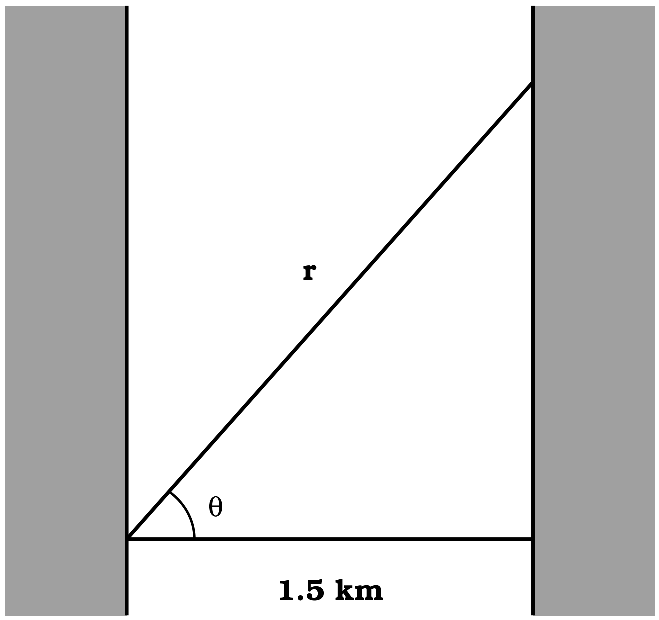
and the angle that the boat's makes wrt the x-axis is

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{5.0 \text{ km/h}}{12 \text{ km/h}}\right) \\ &= \tan^{-1}\frac{5.0}{12} = 22^\circ.6 = 23^\circ \ . \end{aligned}$$

The time it will take to cross the river is derived from

$$\begin{aligned} v &= \frac{r - r_\circ}{t - t_\circ} \qquad (\text{set } r_\circ = 0, t_\circ = 0) \\ &= \frac{r}{t} \quad \text{or} \quad t = \frac{r}{v} \end{aligned}$$

r is now the path length of our trip:



$$\cos\theta = \frac{1.5\ \text{km}}{r}$$

$$r = \frac{1.5\ \text{km}}{\cos\theta}$$

$$= \frac{1.5\ \text{km}}{\cos 22^\circ.6}$$

$$= 1.62\ \text{km}$$

$$\begin{aligned} t &= \frac{r}{v} = \frac{1.62\ \text{km}}{13.\ \text{km/h}} = 0.125\ \text{h} \\ &= 0.125\ \text{h}\ \times \frac{60\ \text{min}}{1\ \text{h}} = 7.477\ \text{min} \end{aligned}$$

$$\boxed{t = 7.5\ \text{min}}$$