PHYS-2010: General Physics I
Course Lecture Notes
Section IX

Dr. Donald G. Luttermoser
East Tennessee State University

Edition 2.6

Abstract

These class notes are designed for use of the instructor and students of the course PHYS-2010: General Physics I taught by Dr. Donald Luttermoser at East Tennessee State University. These notes make reference to the *College Physics, 11th Edition* (2018) textbook by Serway and Vuille.

IX. Gravitation

A. Newton's Law of Gravity.

1. Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distances between them. The was first realized by Sir Isaac Newton, and hence, this is referred to as **Newton's Law of Gravity** and can be expressed mathematically as

$$\boxed{\vec{F}_g = G\,\frac{m_1\,m_2}{r^2}\,\hat{r}\ .} \qquad \text{(IX-1)}$$

a) m_1 and $m_2 \equiv$ masses of objects #1 and #2 (measured in kg).

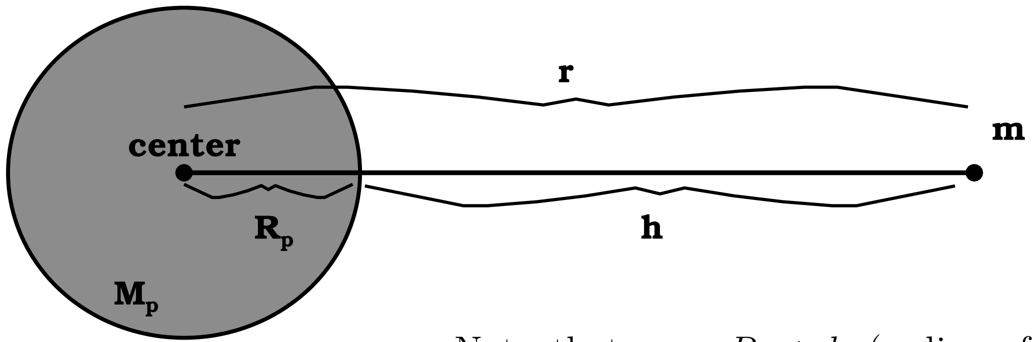
b) $r \equiv$ distance between the masses (measured in m).

c) $G \equiv$ constant of universal gravitation $\Longrightarrow$ this is the proportionality constant between the force and the dependent parameters.

$$\boxed{G = 6.673 \times 10^{-11}\ \frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}\ .} \qquad \text{(IX-2)}$$

d) Newton's law of gravity is thus an **inverse-square law**.

2. The gravitational force exerted by a spherical mass on a particle outside the sphere is the same as if the entire mass of the sphere were concentrated at its center.



Note that $r = R_p + h$ (radius of sphere, *i.e.*, planet, plus the height above the sphere).

Example IX–1. Two objects attract each other with a gravitational force of magnitude 1.00×10^{-8} N when separated by 20.0 cm. If the total mass of the two objects is 5.00 kg, what is the mass of each?

Solution:

Let the individual masses be represented by m_1 and m_2 and the total mass of the two objects be represented as $M = m_1 + m_2$ (= 5.00 kg). Then we can express the second mass as $m_2 = M - m_1$. The distance between the two masses is $r = 20.0$ cm $= 0.200$ m, the gravitational force between them is $F = 1.00 \times 10^{-8}$ N, and the gravitational constant is $G = 6.67 \times 10^{-11}$ N m^2/kg^2.

Now make use of Newton's Law of Gravitation:

$$\begin{aligned} F &= \frac{Gm_1 m_2}{r^2} = \frac{Gm_1(M - m_1)}{r^2} \\ Fr^2 &= Gm_1(M - m_1) = GMm_1 - Gm_1^2 \\ 0 &= Gm_1^2 - GMm_1 + Fr^2 \end{aligned}$$

Algebra has a well know solution to the quadratic equation of the form

$$ax^2 + bx + c = 0 \ ,$$

as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ .$$

Here, $x = m_1$, $a = G$, $b = -GM$, and $c = Fr^2$, so

$$\begin{aligned} m_1 &= \frac{GM \pm \sqrt{G^2M^2 - 4GFr^2}}{2G} \\ &= \frac{GM \pm G\sqrt{M^2 - 4Fr^2/G}}{2G} \\ &= \frac{1}{2}\left(M \pm \sqrt{M^2 - \frac{4Fr^2}{G}}\right) \\ &= \frac{1}{2}\left[5.00 \text{ kg } \pm \sqrt{(5.00 \text{ kg})^2 - \frac{4(1.00\times10^{-8} \text{ N})(0.200 \text{ m})^2}{6.67\times10^{-11} \text{ N m}^2/\text{kg}^2}}\right] \\ &= \frac{1}{2}\left[5.00 \text{ kg } \pm \sqrt{25.0 \text{ kg}^2 - 24.0 \text{ kg}^2}\right] \\ &= \frac{1}{2}\left[5.00 \text{ kg } \pm \sqrt{1.0 \text{ kg}^2}\right] . \end{aligned}$$

As such, m_1 either equals (4.0 kg)/2 = 2.0 kg or (6.0 kg)/2 = 3.0 kg (either answer is correct). If we chose $m_1 = 3.0$ kg, then $m_2 = M - m_1 = 5.00 \text{ kg} - 3.0 \text{ kg} = 2.0$ kg. Hence the solution is

$$\boxed{m_1 = 3.0 \text{ kg and } m_2 = 2.0 \text{ kg} .}$$

---

3. Why do objects fall independent of their mass on the Earth's (or any planetary body) surface?

a) Gravity: $F_g = G\,\dfrac{M_\oplus\, m}{R_\oplus^2}$.

b) Motion: $F = m\,a = m\,g$.

c) Set $F_g = F$, then

$$G\,\frac{M_\oplus\, m}{R_\oplus^2} = m\,g$$

or

$$\boxed{g = \frac{G\,M_\oplus}{R_\oplus^2} .} \qquad \text{(IX-3)}$$

d) Plugging in values:

$$g = \frac{(6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2)(5.9763 \times 10^{24} \text{ kg})}{(6.378 \times 10^6 \text{ m})^2}$$
$$= \frac{3.98845 \times 10^{14} \text{ N m}^2/\text{kg}}{4.0679 \times 10^{13} \text{ m}^2} = 9.8036 \times 10^0 \text{ N/kg} ,$$

but, since 1 N = 1 kg m/s^2, and paying attention to our input significant digits, we get

$$g = 9.80 \, \frac{\text{kg m/s}^2}{\text{kg}} = 9.80 \text{ m/s}^2 \; !$$

e) Since g can be measured from the ground (*e.g.*, motion experiments) and $R_\oplus$ can be measured astronomically (*e.g.*, simple shadow length measurements at two different latitudes — Eratosthenes did this in 200 BC!), we can then determine the Earth's mass with rewriting Eq. (IX-3):

$$M_\oplus = \frac{g\, R_\oplus^2}{G} \; .$$

4. The gravitational constant G.

a) Measuring G in the laboratory accurately is a difficult task!

b) Cavendish was the first to measure it while he was trying to determine the density of the Earth in 1798 (see the textbook for details).

c) More sophisticated experiments have been carried out since that time $\Longrightarrow$ G's accuracy is known to only 5 significant digits:

$$G = 6.6730(41) \times 10^{-11} \text{ N m}^2/\text{kg}^2 \; ,$$

where the '(41)' digits are uncertain.

$\Longrightarrow$ in comparison, the speed of light's accuracy is known to 9 significant digits!

$$c = 2.99792458(1) \times 10^8 \text{ m/s}.$$

d) Future space-based experiments (in free-fall and a vacuum) should increase the accuracy in the measurement of G!

5. The farther we get from the Earth's center, the smaller the acceleration due to gravity:

$$\boxed{g = \frac{G\, M_\oplus}{r^2}\ ,} \tag{IX-4}$$

where $r \equiv$ distance from Earth's center.

a) When g is measured on the Earth's surface (or some other planetary surface), it is called the **surface gravity**.

b) When g is measured elsewhere (*i.e.*, not on a planetary surface), it is called the **acceleration due to gravity**, and if the object is in free-fall, it also is often called the **free-fall acceleration**.

Example IX–2. Mt. Everest is at a height of 29,003 ft (8840 m) above sea level. The greatest depth in the sea is 34,219 ft (10,430 m). Compare the Earth's surface gravity at these two points.

Solution:

Let $h_{\rm e}$ be the height of Mount Everest and $h_{\rm s}$ be the greatest depth of the sea. Using Eq. (IX-4) and the fact that the radius of the Earth at sea level at the Earth's equator is $R_\oplus = 6.378077 \times 10^6$ m, we get the distances for the highest and lowest points of the surface from the center of the Earth as

$$r_{\rm h} \;=\; R_\oplus + h_{\rm e} = 6.378077 \times 10^6 \text{ m } + 8.840 \times 10^3 \text{ m}$$

$$
\begin{aligned}
&= 6.386917 \times 10^6 \text{ m} \\
r_{\rm l} &= R_\oplus - h_{\rm s} = 6.378077 \times 10^6 \text{ m} - 1.0430 \times 10^4 \text{ m} \\
&= 6.367647 \times 10^6 \text{ m}
\end{aligned}
$$

Which gives the surface gravities of

$$
\begin{aligned}
g_{\rm h} &= \frac{G\,M_\oplus}{r_{\rm h}^2} = \frac{(6.6730 \times 10^{-11} \text{ N m}^2/\text{kg}^2)(5.9763 \times 10^{24} \text{ kg})}{(6.386917 \times 10^6 \text{ m})^2} \\
&= \boxed{9.7762 \text{ m/s}^2} , \\
g_{\rm l} &= \frac{G\,M_\oplus}{r_{\rm l}^2} = \frac{(6.6730 \times 10^{-11} \text{ N m}^2/\text{kg}^2)(5.9763 \times 10^{24} \text{ kg})}{(6.367647 \times 10^6 \text{ m})^2} \\
&= \boxed{9.8355 \text{ m/s}^2} ,
\end{aligned}
$$

and $\dfrac{g_{\rm l} - g_{\rm h}}{g} \times 100\% = \dfrac{9.8355 - 9.7762}{9.8036} \times 100\% = 0.605\%$,

the surface gravity of the lowest point on the Earth's surface is a little more than half of a percent larger than at the highest point.

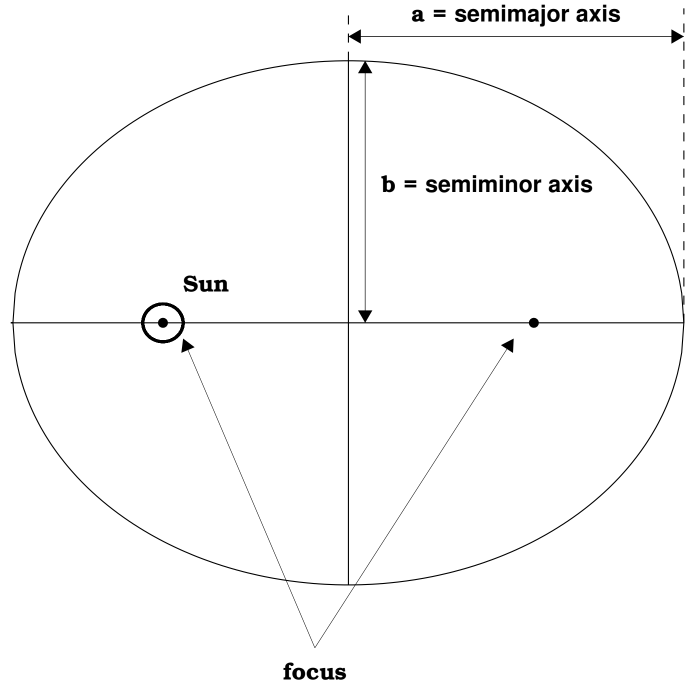
B. Kepler's Laws of Planetary Motion.

1. Johannes Kepler (1571 – 1630) was a German mathematician and astronomer who used Tycho Brahe's observations of Mars to derive the 3 laws of planetary motion. The data showed that the Copernican model of heliocentric (Sun-centered) solar system was correct, except that the planets move in elliptical and not circular paths around the Sun as Copernicus had assumed.

2. The laws:

a) **Law 1:** The orbit of a planet about the Sun is an ellipse with the Sun at one focus. The so-called **elliptical orbit**. The equation for the ellipse in Cartesian coordinates (when the foci are on the x axis) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; . \qquad \text{(IX-5)}$$



i) Semimajor axis (a): Half of the longest axis of an ellipse.

ii) Semiminor axis (b): Half of the shortest axis of an ellipse.

iii) 1 Astronomical Unit (A.U.) is the length of the Earth's semimajor axis,

$$1 \text{ A.U.} = 1.4960 \times 10^{11} \text{ m.}$$

iv) The relative flatness of an ellipse is measured by the **eccentricity** e:

$$e = \frac{\sqrt{a^2 - b^2}}{a} \; . \qquad \text{(IX-6)}$$

v) The distance from the center to either focus is given by $\sqrt{a^2 - b^2}$.

vi) The ellipse is just one type of **conic section**. If $a = b$, Eq. (IX-6) gives $e = 0$ and we have a **circular orbit** $\Longrightarrow$ a second type of conic section (*i.e.*, a circle).

vii) If we let 'a' get bigger and bigger such that $a \gg b$, then Eq. (IX-6) gives $e \approx \sqrt{a^2}/a = a/a = 1$ as $a \rightarrow \infty$. When this happens, we have a **parabolic orbit**. Such an orbit is said to be **open** (both circular and elliptical orbits are **closed**) and never return. A parabolic orbit is achieved when the velocity of a satellite just equals the **escape velocity**, $v_{\rm esc}$.

viii) There also are orbits that are "more open" than parabolic orbits $\Longrightarrow$ the so-called **hyperbolic orbits**. These orbits have $e > 1$ and can be achieved if $v > v_{\rm esc}$.

b) **Law 2:** A line joining a planet and the Sun sweeps out equal areas in equal amounts of time (*law of equal areas*).

i) This means that planets move faster when closer to the Sun in its orbit than when it is farther away.

ii) Objects in very elliptical orbits don't stay near the Sun for a very long time $\Longrightarrow$ comets.

iii) **Perihelion:** Point on an orbit when a planet is closest to the Sun (r_p = perihelion distance).

iv) Aphelion: Point on an orbit when a planet is farthest from the Sun (r_a = aphelion distance).

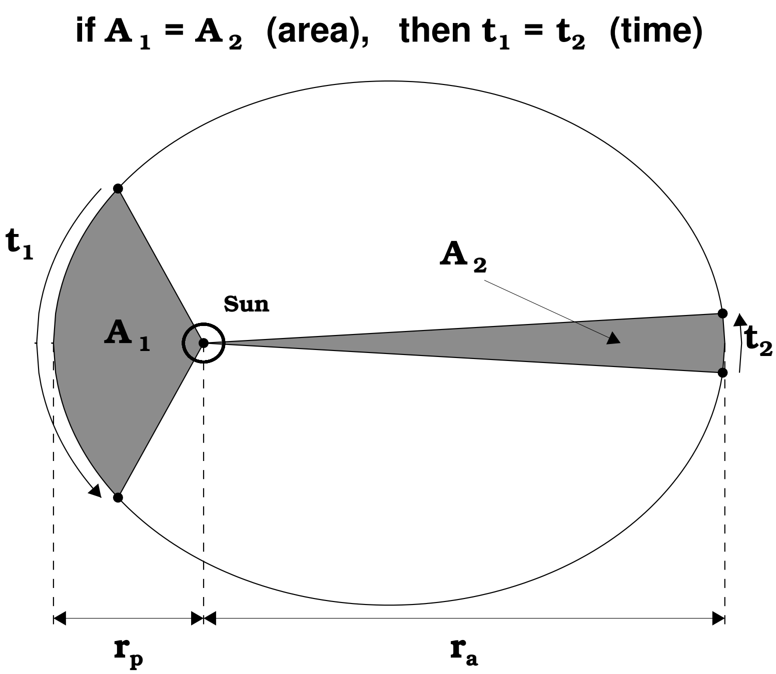
v) The perihelion and aphelion of a solar orbit can be determined from the semimajor axis and the eccentricity with

$$r_p = a(1-e) \tag{IX-7}$$

$$r_a = a(1+e) \ , \tag{IX-8}$$

also note that

$$r_p + r_a = 2a \ . \tag{IX-9}$$



c) Law 3: The square of the orbital period (T) of any planet is proportional to the cube of the semimajor axis (a) of a planet's orbit about the Sun (*harmonic law*):

$$T^2 \propto a^3 \ . \tag{IX-10}$$

3. Kepler's laws were empirically deduced. Nearly 100 years after these planetary laws were developed, Newton came along and showed theoretically why they are valid.

a) Gravity (for a planet in orbit about the Sun [$\equiv \odot$]):

$$F_g = \frac{G\,M_\odot\,M_p}{r^2} \ .$$

b) Centripetal force: $$F_c = \frac{M_p\,v^2}{r} \ .$$

c) Set these two forces equal:

$$\begin{aligned} \frac{G\,M_\odot\,M_p}{r^2} &= \frac{M_p\,v^2}{r} \ , \\ v^2 &= \frac{G\,M_\odot}{r} \ . \end{aligned}$$

d) Note, however, that the orbital velocity of a planet can be expressed as the length of the circumference C of the orbit divided by its orbital period. For demonstration purposes, we will assume that the orbit is circular, but the same result is obtained for elliptical orbits, then

$$v = \frac{C}{T} = \frac{2\pi r}{T} \ .$$

e) Plugging this equation into the force equation above gives

$$\begin{aligned} v^2 = \left(\frac{2\pi r}{T}\right)^2 &= \frac{G\,M_\odot}{r} \\ \frac{4\pi^2 r^2}{T^2} &= \frac{G\,M_\odot}{r} \\ \frac{4\pi^2 r^3}{G\,M_\odot} &= T^2 \ . \end{aligned}$$

Since we have assumed a circular orbit here, $r = a$ (the semimajor axis), and

$$\boxed{T^2 = \left(\frac{4\pi^2}{G\,M_\odot}\right) a^3 = K_\odot\,a^3 \ .} \qquad \text{(IX-11)}$$

i) $K_\odot = \dfrac{4\pi^2}{G\,M_\odot} = 2.97 \times 10^{-19}$ s^2/m^3.

ii) $K_\odot$ is independent of the planet's mass $\Longrightarrow K$ only depends upon the larger, central object's mass.

iii) $T^2 \propto a^3$ is valid whether the orbit is circular (where $a = r$, the orbital radius) or elliptical (where 'a' is used intact), though the elliptical orbit solution has a slightly different form for the $K_\odot$ constant. The proof for elliptical orbits requires advanced calculus (hence we will not show it here).

4. We can base other planets in solar system with respect to the Earth:

$$\frac{T_p^2}{T_\oplus^2} = \frac{K_\odot\, a_p^3}{K_\odot\, a_\oplus^3} = \frac{a_p^3}{a_\oplus^3}\ . \tag{IX-12}$$

Since $T_\oplus = 1$ yr and $a_\oplus = 1$ A.U., Eq. (IX-12) becomes

$$\frac{T_p^2}{1\ \mathrm{yr}^2} = \frac{a_p^3}{1\ \mathrm{A.U.}^3}\ ,$$

or

$$\boxed{T_{\mathrm{yr}}^2 = a_{\mathrm{AU}}^3\ .} \tag{IX-13}$$

Example IX–3. The *Voyager 1* spacecraft has just passed the 140 A.U. mark in its distance from the Sun. At this time, the Death Star from an evil galactic empire intercepts *Voyager* and figures out which planet sent it based on the gold record that was included on the spacecraft. The Death Star alters its course and starts to orbit the Sun from that point and sets its heading such that it will reach perihelion at the Earth's location. Calculate the following about its orbit: (a) the semimajor axis (in A.U.); (b) the eccentricity, and (c) the length of time (in years) it will take to get to Earth.

Solution (a):

We are given that $r_p = 1.00$ A.U. and since it starts its free fall orbit from the position of *Voyager 1*, that position marks it aphelion position, $r_a = 140$ A.U. (note that we do not have to convert to SI units for this problem). Using Eq. (IX-9) we can easily calculate the semimajor axis of the Death Star's orbit:

$$a = \frac{r_p + r_a}{2} = \frac{1.00 \text{ A.U.} + 140 \text{ A.U.}}{2} = \boxed{70.5 \text{ A.U.}}$$

Solution (b):

Using Eq. (IX-7) and solving for e, we get,

$$e = 1 - \frac{r_p}{a} = 1 - \frac{1.00 \text{ A.U.}}{70.5 \text{ A.U.}} = 1 - 0.0142 = \boxed{0.986\ ,}$$

nearly a parabolic orbit!

Solution (c):

First we need to calculate the period of a complete orbit from Kepler's 3rd law, then the amount of time it will take to go from aphelion to perihelion is one-half that period. As such,

$$\begin{aligned} \left(\frac{T}{T_\oplus}\right)^2 &= \left(\frac{a}{a_\oplus}\right)^3 = \left(\frac{70.5 \text{ A.U.}}{1.00 \text{ A.U.}}\right)^3 = (70.5)^3 = 3.50 \times 10^5 \\ \frac{T}{1.00 \text{ yr}} &= \sqrt{3.50 \times 10^5} \\ T &= 592 \text{ yr} \end{aligned}$$

Since this is the full period of the Death Star's orbit, the time to get to Earth will be

$$t = \frac{1}{2} T = \boxed{296 \text{ yr}\ .}$$

C. Conservation of Energy in a Gravitational Field.

1. Up until this point, we handled conservation of mechanical energy for gravity for a constant surface gravity (*i.e.*, a constant acceleration due to gravity). Here, we will relax that requirement and let g vary with r as shown in Eq. (IX-4).

a) For a constant acceleration due to gravity, we have seen from Eq. (VI-11) that

$$\text{PE} = mgh \ , \tag{IX-14}$$

where m is the mass of the object, g is the surface gravity (*i.e.*, acceleration), and h is the height above the ground.

b) However, this is not the most general form of the potential energy of a gravitational field — it is an *approximation* to the general form. In higher-level physics, potential energy is related to a force field by the equation

$$\vec{F} = -\vec{\nabla}(\text{PE}) = -\frac{d(\text{PE})}{dr}\,\hat{r} \ , \tag{IX-15}$$

where $\vec{F} = \vec{F}_g$ as described by Eq. (IX-1) for a gravitational field and the "del" symbol $\vec{\nabla}$ is the spatial derivative in three dimensions — for Eq. (IX-15) to be valid, we only use the component of $\vec{\nabla}$ in the $\hat{r}$ direction (*i.e.*, d/dr) since $\vec{F}_g$ only points in the r-direction.

i) Since this is an algebra-based course, we won't solve that differential equation with calculus. However, by realizing that the derivative symbol 'd' just means *infinitesimally* small 'delta' ($\Delta \equiv$ change of) and we can approximate Eq. (IX-15) with

$$\begin{aligned} \Delta(\text{PE}) &= -\left(\frac{G\,M\,m}{r^2}\right)\Delta r \\ \text{PE} - \text{PE}_\circ &= -\left(\frac{G\,M\,m}{r^2}\right)(r - r_\circ) \ , \end{aligned} \tag{IX-16}$$

where the '$\circ$' terms are the initial values.

ii) Now if we define the initial $\text{PE}_\circ$ as 0 at the center of the gravitating body (so $r_\circ = 0$), Eq. (IX-16) becomes

$$\begin{aligned} \text{PE} &= -\left(\frac{G\,M\,m}{r^2}\right) r \\ &= -\frac{G\,M\,m}{r} \,. \end{aligned}$$

iii) The calculus solution to the differential equation above (Eq. IX-15) would have given exactly the same answer.

iv) As such, the most general form of the potential energy of a mass in a gravitating field is

$$\boxed{\text{PE} = -\frac{G\,M\,m}{r}} \,. \qquad \text{(IX-17)}$$

c) With this general form of the gravitational potential energy, we can see how the potential energy equation near the Earth's surface (Eq. IX-14) arises.

i) Let's say we have a projectile that we launch from the ground. While on the ground, $r = R_\oplus$ (the radius of the Earth), which gives a potential energy of

$$\text{PE} = -\frac{G\,M_\oplus\,m}{R_\oplus} \,. \qquad \text{(IX-18)}$$

ii) Now, when the projectile reaches its highest point above the ground, h, it is a distance of $r = R_\oplus + h$ from the center of the Earth. At this point, it has a potential energy of

$$\text{PE} = -\frac{G\,M_\oplus\,m}{R_\oplus + h} \,. \qquad \text{(IX-19)}$$

iii) The change in potential energy between these two points is

$$\begin{aligned}\Delta\text{PE} &= \frac{-G\,M_\oplus\,m}{R_\oplus + h} - \frac{-G\,M_\oplus\,m}{R_\oplus} \\ &= G\,M_\oplus\,m\left(\frac{1}{R_\oplus} - \frac{1}{R_\oplus + h}\right) \\ &= G\,M_\oplus\,m\left(\frac{R_\oplus + h}{R_\oplus\,(R_\oplus + h)} - \frac{R_\oplus}{R_\oplus\,(R_\oplus + h)}\right) \\ &= G\,M_\oplus\,m\left(\frac{R_\oplus + h - R_\oplus}{R_\oplus\,(R_\oplus + h)}\right) \\ &= G\,M_\oplus\,m\left(\frac{h}{R_\oplus\,(R_\oplus + h)}\right)\ .\end{aligned}$$

iv) If $R_\oplus$ is much greater than h (which it will be for experiments near the Earth's surface), $h \ll R_\oplus$. As such, $R_\oplus + h \approx R_\oplus$ and the equation above becomes

$$\begin{aligned}\Delta\text{PE} &\approx G\,M_\oplus\,m\left(\frac{h}{R_\oplus\,(R_\oplus)}\right) = G\,M_\oplus\,m\left(\frac{h}{R_\oplus^2}\right) \\ &= \frac{G\,M_\oplus\,m\,h}{R_\oplus^2}\ .\end{aligned}$$

v) Now, remembering our defining equation for surface gravity (*e.g.*, Eq. IX-3):

$$g = \frac{G\,M_\oplus}{R_\oplus^2}\ ,$$

we use this in the potential equation we just wrote:

$$\Delta\text{PE}\ = m\,\frac{G\,M_\oplus}{R_\oplus^2}\,h = m\,g\,h\ ,$$

hence we have proven Eq. (IX-14) from first principles.

2. We can now use this general potential energy equation to figure out high trajectory, orbital, and space trajectory problems. First, let's develop a relationship between the initial velocity, $v_\circ$, of a projectile and the maximum height, h, it will reach.

a) If we were limiting ourselves to vertically directed trajectories near the Earth's surface, we would have

$$\begin{aligned} \mathrm{KE}_i + \mathrm{PE}_i &= \mathrm{KE}_f + \mathrm{PE}_f \\ \frac{1}{2} m v_\circ^2 + mgy_\circ &= \frac{1}{2} m v^2 + mgy \ . \end{aligned}$$

$y_\circ = 0$, since it represents the ground and the projectile will reach its maximum height ($y = h$) when the velocity goes to zero ($v = 0$). From this, we can solve for the initial velocity and get

$$v_\circ = \sqrt{2gh} \ .$$

b) Using the more general form of the potential, our initial position will be on the surface of the Earth ($r = R_\oplus$) and we will reach a distance from the center of the Earth of $r = R_\oplus + h$, which we will simply write as r. The conservation of mechanical energy then gives

$$\begin{aligned} \mathrm{KE}(R_\oplus) + \mathrm{PE}(R_\oplus) &= \mathrm{KE}(r) + \mathrm{PE}(r) \\ \frac{1}{2} m v_\circ^2 - \frac{GM_\oplus m}{R_\oplus} &= \frac{1}{2} m v^2 - \frac{GM_\oplus m}{r} \ . \end{aligned}$$

c) Once again, $v = 0$ at the top of the trajectory, and as such, the initial velocity is

$$\begin{aligned} \frac{1}{2} m v_\circ^2 - \frac{GM_\oplus m}{R_\oplus} &= 0 - \frac{GM_\oplus m}{r} \\ \frac{1}{2} m v_\circ^2 &= \frac{GM_\oplus m}{R_\oplus} - \frac{GM_\oplus m}{r} \\ \frac{1}{2} m v_\circ^2 &= GM_\oplus m \left(\frac{1}{R_\oplus} - \frac{1}{r} \right) \\ v_\circ^2 &= \frac{2GM_\oplus m}{m} \left(\frac{1}{R_\oplus} - \frac{1}{r} \right) \end{aligned}$$

$$
\begin{aligned}
v_\circ^2 &= 2GM_\oplus \left(\frac{1}{R_\oplus} - \frac{1}{r} \right) \\
v_\circ &= \sqrt{2GM_\oplus \left(\frac{1}{R_\oplus} - \frac{1}{r} \right)} \ . \qquad \text{(IX-20)}
\end{aligned}
$$

d) Now if we once again define g to be the acceleration due to gravity at the Earth's surface (*i.e.*, surface gravity), we can rewrite Eq. (IX-3) to read

$$
G\,M_\oplus = g\,R_\oplus^2 \ .
$$

e) Plugging this into Eq. (IX-20) gives

$$
v_\circ = \sqrt{2gR_\oplus^2 \left(\frac{1}{R_\oplus} - \frac{1}{r} \right)} \ , \qquad \text{(IX-21)}
$$

and replacing r by $R_\oplus + h$, we get

$$
\begin{aligned}
v_\circ &= \sqrt{2gR_\oplus^2 \left(\frac{1}{R_\oplus} - \frac{1}{R_\oplus + h} \right)} \\
&= \sqrt{2gR_\oplus^2 \left(\frac{R_\oplus + h}{R_\oplus\,(R_\oplus + h)} - \frac{R_\oplus}{R_\oplus\,(R_\oplus + h)} \right)} \\
&= \sqrt{2gR_\oplus^2 \left(\frac{R_\oplus + h - R_\oplus}{R_\oplus\,(R_\oplus + h)} \right)} \\
&= \sqrt{2gR_\oplus^2 \left(\frac{h}{R_\oplus\,(R_\oplus + h)} \right)} \\
&= \sqrt{2gR_\oplus \left(\frac{h}{R_\oplus + h} \right)} \\
v_\circ &= \sqrt{\frac{2ghR_\oplus}{R_\oplus + h}} \ . \qquad \text{(IX-22)}
\end{aligned}
$$

f) Finally, one can immediately see that if $h \ll R_\oplus$, then $R_\oplus + h \approx R_\oplus$ and Eq. (IX-22) becomes

$$
v_\circ \approx \sqrt{\frac{2ghR_\oplus}{R_\oplus}} = \sqrt{2gh} \ .
$$

As can be seen, this equation (as written above) is just an approximation to a more general equation (*i.e.*, Eq. IX-22).

Example IX–4. A satellite of mass 200 kg is launched from a site on the Equator into an orbit at 200 km above the Earth's surface. (a) If the orbit is circular, what is the orbital period of this satellite? (b) What is the satellite's speed in orbit? (c) What is the minimum energy necessary to place this satellite in orbit, assuming no air friction?

Solution (a):

The radius of the satellite's orbit is

$$r = R_\oplus + h = 6.38 \times 10^6 \text{ m } + 200 \times 10^3 \text{ m } = 6.58 \times 10^6 \text{ m },$$

$m = 200$ kg is the satellite's mass, and $M_\oplus = 5.98 \times 10^{24}$ kg is the Earth's mass. The orbital velocity will just be the tangential velocity of the circular orbit. Since the gravitational force provides the centripetal acceleration, we have

$$\begin{aligned}
F_{\rm c} &= F_{\rm g} \\
m\left(\frac{v_{\rm orb}^2}{r}\right) &= \frac{GM_\oplus m}{r^2} \\
v_{\rm orb}^2 &= \frac{GM_\oplus}{r} \\
v_{\rm orb} &= \sqrt{\frac{GM_\oplus}{r}} \\
&= \sqrt{\frac{(6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2)(5.98 \times 10^{24} \text{ kg})}{6.58 \times 10^6 \text{ m}}} \\
&= 7.79 \times 10^3 \text{ m/s} .
\end{aligned}$$

For a circular orbit, the orbital period is

$$T = \frac{2\pi r}{v_{\rm orb}} = \frac{2\pi(6.58 \times 10^6 \text{ m})}{7.79 \times 10^3 \text{ m/s}}$$

$$= \; 5.31 \times 10^3 \text{ s} \; = \boxed{\; 1.48 \text{ hr} \; .}$$

Solution (b):

The orbital speed was computed in part (a):

$$\boxed{v_{\rm orb} = 7.79 \times 10^3 \text{ m/s} \; .}$$

Solution (c):

The minimum energy to reach orbit can be determined by calculating the difference between the total mechanical energy of the satellite in orbit and total mechanical energy of the satellite prior to launch. The satellite itself prior to launch is not moving, but the surface of the Earth is moving due to the Earth's rotation (we will ignore the Earth's orbital velocity here since we are not leaving the Earth's gravitational influence). Hence, minimum launch energy = $E_{\rm min}$ = (KE + PE)$_{\rm orb}$ − (KE + PE)$_{\rm rot}$. The Earth's rotational period is the definition of a 'day', so $T_{\rm rot}$ = 1.000 day = 86,400 s. From this, we get the Earth's rotation velocity of

$$v_{\rm rot} = \frac{2\pi R_\oplus}{T_{\rm rot}} = \frac{2\pi(6.38 \times 10^6 \text{ m})}{8.6400 \times 10^4 \text{ s}} = 464 \text{ m/s} \; .$$

Hence, the minimum launch energy is

$$\begin{aligned}
E_{\rm min} &= \left(\frac{1}{2}mv_{\rm orb}^2 - \frac{GmM_\oplus}{r}\right) - \left(\frac{1}{2}mv_{\rm rot}^2 - \frac{GmM_\oplus}{R_\oplus}\right) \\
&= m \left[\left(\frac{1}{2}v_{\rm orb}^2 - \frac{GM_\oplus}{r}\right) - \left(\frac{1}{2}v_{\rm rot}^2 - \frac{GM_\oplus}{R_\oplus}\right)\right] \\
&= m \left[\frac{v_{\rm orb}^2 - v_{\rm rot}^2}{2} + GM_\oplus \left(\frac{1}{R_\oplus} - \frac{1}{r}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
E_{\rm min} &= (200 \text{ kg}) \left[\frac{(7.79 \times 10^3 \text{ m/s})^2 - (464 \text{ m/s})^2}{2} + \right. \\
&\quad \left(6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}\right) (5.98 \times 10^{24} \text{ kg}) \times \\
&\quad \left. \left(\frac{1}{6.38 \times 10^6 \text{ m}} - \frac{1}{6.58 \times 10^6 \text{ m}} \right) \right] \\
&= (200 \text{ kg}) \left(3.02 \times 10^7 \text{ m}^2/\text{s}^2 + 1.90 \times 10^6 \text{ m}^2/\text{s}^2 \right) \\
&= \boxed{6.43 \times 10^9 \text{ J}} .
\end{aligned}
$$

---

3. The absolute magnitude of the potential energy due to a large gravitating body is often many times larger than the kinetic energy of the object in motion.

a) We can define the total mechanical energy $E_{\rm tot}$ of body in motion as the sum of the kinetic and potential energies:

$$E_{\rm tot} = \text{KE} + \text{PE} . \qquad \text{(IX-23)}$$

b) Since PE is negative in a gravitational field, $E_{\rm tot} < 0$ (*i.e.*, negative) for projectile trajectories $\Longrightarrow$ a *bound* state.

c) In higher-level physics, **bound states** have $E_{\rm tot} < 0$, whereas **free states** have $E_{\rm tot} > 0$.

4. For an object to just overcome any gravitating body's (like the Earth's) potential field, an object has to be launched with **zero** total energy, that is, $|\text{KE}| = |\text{PE}|$ to escape the primary body's (*e.g.*, Earth's) gravitational field $\Longrightarrow$ the **escape velocity**.

a) Hence, to calculate the escape velocity from the surface of a large body of mass M and radius R, we just have to set the initial kinetic and potential energy sum to zero and solve for the velocity:

$$E_{\rm tot} = \frac{1}{2} m v_{\rm esc}^2 - \frac{GMm}{R} = 0 \ . \qquad \text{(IX-24)}$$

b) This gives the equation for the **escape velocity** (or escape speed):

$$\boxed{v_{\rm esc} = \sqrt{\frac{2GM}{R}}} \ . \qquad \text{(IX-25)}$$

c) As can be seen from Eq. (IX-25), the escape velocity does not depend upon the mass of the rocket. Using the values for Earth in Eq. (IX-25), we get

$$v_{\rm esc} = \sqrt{\frac{2GM_\oplus}{R_\oplus}} \ , \qquad \text{(IX-26)}$$

or $v_{\rm esc} = 11.2$ km/s to leave the Earth's gravitational field.

d) For an object to escape the gravitational field of a primary body, it must achieve a velocity greater than or equal to the escape velocity: $v \geq v_{\rm esc}$.

Example IX–5. The Sun has a mass of 1.9892×10^{30} kg and a radius of 6.9598×10^{8} m. Calculate the escape velocity from the solar surface (in km/s) and compare it to the escape velocity from the surface of the Earth.

Solution:

From Eq. (IX-26), the escape velocity from the solar surface is:

$$v_{\rm esc}(\odot) = \sqrt{\frac{2GM_\odot}{R_\odot}}$$

$$= \sqrt{\frac{2(6.6730 \times 10^{-11}\ {\rm N\ m^2/kg^2})(1.9892 \times 10^{30}\ {\rm kg})}{6.9598 \times 10^{8}\ {\rm m}}}$$

$$= \; 6.1761 \times 10^5 \text{ m/s} \left(\frac{1 \text{ km}}{1000 \text{ m}}\right) = \boxed{617.61 \text{ km/s}}\;.$$

Since $v_{\rm esc}(\oplus) = 11.2$ km/s, $v_{\rm esc}(\odot) = (617.61$ km/s / 11.2 km/s$)$ $v_{\rm esc}(\oplus) = 55.1\ v_{\rm esc}(\oplus)$, or 55.1 times bigger than the escape velocity from the Earth's surface!