PHYS-4007/5007: Computational Physics
Course Lecture Notes
Section IX

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Abstract

These class notes are designed for use of the instructor and students of the course PHYS-4007/5007: Computational Physics taught by Dr. Donald Luttermoser at East Tennessee State University.

IX. Computing Trajectories

A. The Physics of Trajectories

1. This section concerns itself with the numerical solution to Newton's Second Law of Motion in a gravitational potential field near the surface of a large gravitating body.

2. As we saw with Eq. (VIII-1), Newton's Second Law of Motion can be written as:

$$F_{\rm tot} = \sum_{i=1}^{N} F\left[t, r(t), v(t)\right] = ma = m\frac{dv(t)}{dt} = m\frac{d^2r(t)}{dt^2} \;, \qquad \text{(IX-1)}$$

where here we are using the standard notation of $r \equiv$ displacement, $v \equiv$ linear velocity, $a \equiv$ linear acceleration, $t \equiv$ time, $m \equiv$ mass of the object, and N is the total number of independent forces that are acting on the body.

   a) The position $r(t)$ of a particle of mass m is acted upon by the net force $F_{\rm tot}$, which may be a function of time t, position $r(t)$, and the velocity $v(t) = dr(t)/dt$.

   b) The motion of the object can then be described completely by specifying $F_{\rm tot}$ and setting initial conditions at $t_\circ$ to r and v.

   c) Note that derivatives with respect to time are often written with the 'dot' notation:

$$\begin{aligned} v &= \frac{dr(t)}{dt} = \dot{r} \qquad \text{(for velocity)} \\ a &= \frac{dv(t)}{dt} = \frac{d^2r(t)}{dt^2} = \ddot{r} \qquad \text{(for acceleration)} \end{aligned}$$

   so that Newton's 2nd law can be written as $F = m\ddot{r}$.

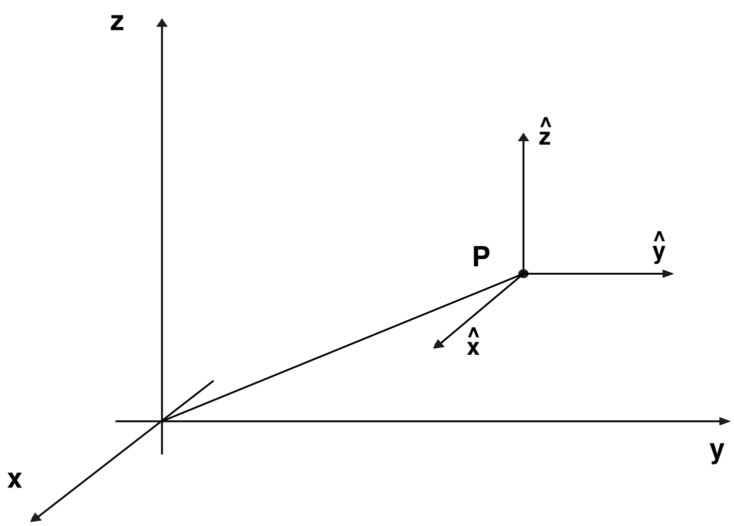

Figure IX–1: The orthogonal (Cartesian) coordinate system.

3. In classical mechanics, the *static* potential field is related to a **conservative** force by the equation

$$\vec{F} = -\vec{\nabla} U \ , \tag{IX-2}$$

where U is the potential energy of the field (a scalar), and the "del" operator acting on a scalar is referred to as taking the **gradient** of the scalar $\Longrightarrow$ it converts a scalar to a vector. The "del" operator takes first derivatives on each coordinate of the vector space. As such, it has a different form depending on the coordinate system:

a) Orthogonal (Cartesian) Coordinates (x, y, z):

$$\vec{\nabla} = \hat{x}\,\frac{\partial}{\partial x} + \hat{y}\,\frac{\partial}{\partial y} + \hat{z}\,\frac{\partial}{\partial z} \ , \tag{IX-3}$$

such that

$$\vec{\nabla} f(x, y, z) = \hat{x}\,\frac{\partial f}{\partial x} + \hat{y}\,\frac{\partial f}{\partial y} + \hat{z}\,\frac{\partial f}{\partial z} \ .$$

b) Spherical-Polar (Spherical) Coordinates (r, θ, ϕ):

$$\vec{\nabla} = \hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial}{\partial \theta} + \hat{\phi}\,\frac{1}{r\,\sin\theta}\,\frac{\partial}{\partial \phi} \ . \tag{IX-4}$$

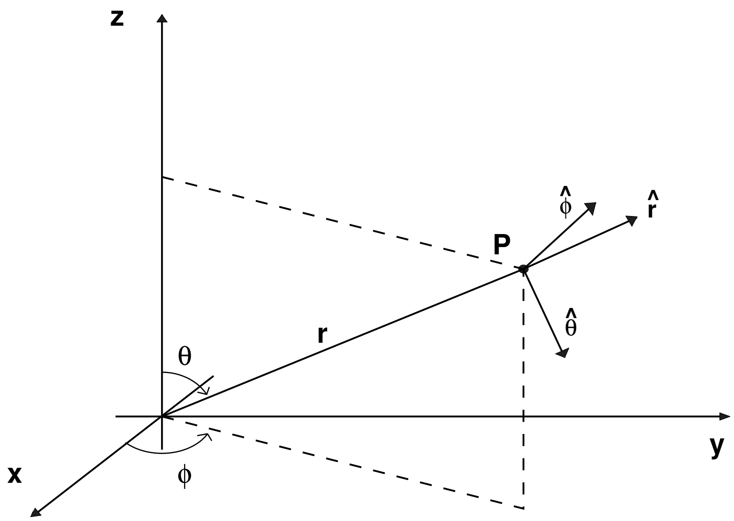

Figure IX–2: The spherical-polar coordinate system.

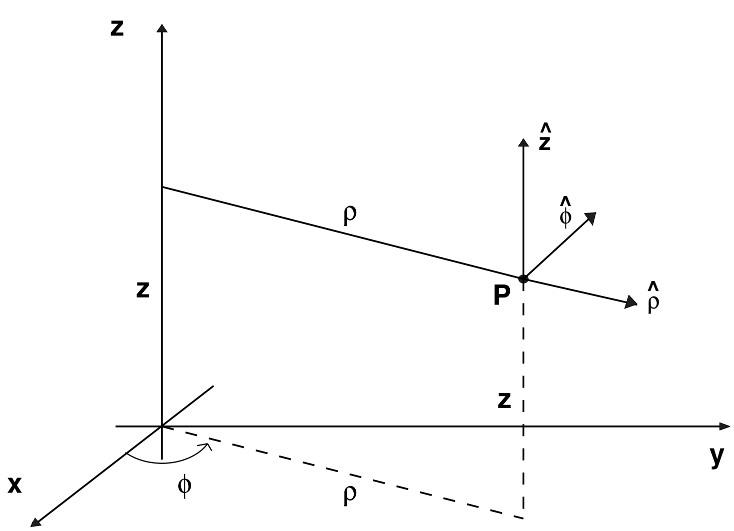

Figure IX–3: The circular-cylindrical coordinate system.

c) Circular-Cylindrical (Cylindrical) Coordinates (ρ, ϕ, z):

$$\vec{\nabla} = \hat{\rho}\,\frac{\partial}{\partial \rho} + \hat{\phi}\,\frac{1}{\rho}\,\frac{\partial}{\partial \phi} + \hat{z}\,\frac{\partial}{\partial z}\ . \qquad \text{(IX-5)}$$

4. In this section, we will be using the gravitational potential field to describe the potential energy, then from Newton's Universal Law of Gravity,

$$\vec{F}_g(r) = -\frac{G\,m_1\,m_2}{r^2}\,\hat{r}\ , \qquad \text{(IX-6)}$$

we see that this force only depends upon r and is negative since it points in the opposite direction of $\hat{r}$ due to its attractive nature. In this equation the m's represent the masses of bodies 1 and 2 which are separated by a distance r. $G = 6.673 \times 10^{-11}$ N m^2/kg^2 $= 6.673 \times 10^{-8}$ dyne cm^2/g^2 is the Universal Gravitation Constant.

a) Using Eq. (IX-6) in conjunction with Eq. (IX-4), we immediately see that $U = U(r)$ since $F = F(r)$.

i) To solve the gradient equation for the potential energy (Eq. IX-2), we can take the dot product (*i.e.*, inner product) of both sides of the equation of Eq. (IX-2) with the differential displacement $d\vec{r}$ and integrate from the initial (1) to the final position (2).

ii) Since the force here is conservative, the work done by this force is independent of the path or trajectory taken. Hence, we can use the standard definite integral instead of a path integral to solve for U:

$$\begin{aligned}
\int_1^2 \vec{F}\cdot d\vec{r} &= -\int_1^2 \vec{\nabla}U\cdot d\vec{r} && \text{(IX-7)}\\
-\int_1^2 \left(\frac{G\,m_1\,m_2}{r^2}\,\hat{r}\right)\cdot(dr\,\hat{r}) &= -\int_1^2 \left(\frac{\partial U}{\partial r}\,\hat{r}\right)\cdot(dr\,\hat{r}) &&\\
\int_1^2 \frac{G\,m_1\,m_2}{r^2}\,dr &= \int_1^2 \frac{dU}{dr}\,dr && \text{(IX-8)}
\end{aligned}$$

where the partials become full derivatives since $U = U(r)$.

iii) Integrating Eq. (IX-8), we get

$$-\frac{G\,m_1\,m_2}{r}\bigg|_1^2 = \int_1^2 dU = U_2 - U_1 \ ,$$

or

$$U_1 - U_2 = G\,m_1\,m_2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \ . \qquad \text{(IX-9)}$$

b) From Eq. (IX-9), we can immediately see that the potential energy for a gravitational field takes on the form

$$U = -\frac{G\,m_1\,m_2}{r} \ . \qquad \text{(IX-10)}$$

i) From this equation we see that $U \rightarrow 0$ as $r \rightarrow \infty$.

ii) Also we can see that gravitational potential energy is a negative energy.

5. Besides potential energy, any body in motion has a *kinetic* energy associated with it.

a) The left-hand side of Eq. (IX-7) is the definition of the work done on a particle by a force field:

$$W \equiv \int_1^2 \vec{F} \cdot d\vec{r} = U_1 - U_2 \ . \qquad \text{(IX-11)}$$

b) Using Newton's Second Law of Motion (Eq. IX-1) and rewriting $d\vec{r}$ as $(d\vec{r}/dt)dt$, we can write

$$\vec{F} \cdot d\vec{r} = \left(m\frac{d\vec{v}}{dt}\right) \cdot \left(\frac{d\vec{r}}{dt}\,dt\right) = m\frac{d\vec{v}}{dt} \cdot \vec{v}\,dt \ .$$

Note that

$$\frac{d}{dt}(\vec{v} \cdot \vec{v}) = \vec{v} \cdot \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} \cdot \vec{v} = 2\frac{d\vec{v}}{dt} \cdot \vec{v} \ ,$$

so

$$\frac{d\vec{v}}{dt} \cdot \vec{v} = \frac{1}{2}\frac{d}{dt}(\vec{v} \cdot \vec{v})$$

and hence,

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= \frac{1}{2} m \frac{d}{dt}(\vec{v} \cdot \vec{v})\, dt = \frac{1}{2} m \frac{d}{dt}\left(v^2\right)\, dt \\ &= d\left(\frac{1}{2} m v^2\right) . \qquad \text{(IX-12)} \end{aligned}$$

c) Now using Eq. (IX-12) in the first part of Eq. (IX-11), we get

$$\begin{aligned} W \equiv \int_1^2 \vec{F} \cdot d\vec{r} &= \int_1^2 d\left(\frac{1}{2} m v^2\right) = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \\ &= T_2 - T_1 , \qquad \text{(IX-13)} \end{aligned}$$

where 'T' represents kinetic energy.

d) Inserting this value for the work in Eq. (IX-11), we see

$$W = T_2 - T_1 = U_1 - U_2$$

or

$$\begin{aligned} T_1 + U_1 &= T_2 + U_2 \qquad \text{(IX-14)} \\ E_1 &= E_2 , \end{aligned}$$

where E represents the total mechanical energy of the system.

e) Eq. (IX-14) is called the **conservation of mechanical energy**, which is valid only for a conservative force $\rightarrow$ forces that do not depend on time, nor the work done by such a force depending upon the path taken in a trajectory.

6. Whereas the force is equal to the negative gradient of the potential energy (see Eq. IX-2), the acceleration due to a conservative force can be determined from the **potential** Φ of the force ('potential' is the potential energy per unit mass). For gravity

$$\vec{g} \equiv -\vec{\nabla}\Phi \ , \tag{IX-15}$$

thus

$$\Phi = \frac{U}{m} = -\frac{GM}{r} \ . \tag{IX-16}$$

Note this could also be proven by following the technique shown in Eqs. (IX-7) through (IX-10). To determine the gravitational potential, one must know the manner in which mass is distributed throughout a body.

a) The potential due to a continuous distribution of matter is

$$\Phi = -G \int_V \frac{\rho(\vec{r'})}{r'} \, dv' \ , \tag{IX-17}$$

(*i.e.*, a volume integral) where $\rho(\vec{r'})$ is the mass density over volume element dv' at distance $\vec{r'}$ from the origin integrated over the entire volume V.

b) If mass is distributed over a thin shell (*i.e.*, a surface distribution), then

$$\Phi = -G \int_S \frac{\rho_s}{r'} \, da' \ , \tag{IX-18}$$

(*i.e.*, a surface integral) where ρ_s is the surface density of mass (areal mass density), da' is an area differential, and the integral is taken over the surface S.

c) Finally, if there is a line source with linear mass density ρ_ℓ of length L,

$$\Phi = -G \int_L \frac{\rho_\ell}{r'} \, ds' \ , \tag{IX-19}$$

(*i.e.*, a line integral) where ds' is a length differential on the line source of gravity.

7. The work done by a potential field is

$$\begin{aligned} dW &= -\vec{g} \cdot d\vec{r} = \vec{\nabla}\Phi \cdot d\vec{r} \\ &= \sum_{i=1}^{N} \frac{\partial \Phi}{\partial x_i}\, dx_i = d\Phi \ . \end{aligned} \tag{IX-20}$$

8. When dealing with motion in gravitational fields, there are two regimes that are typically encountered: (1) *trajectories* (near a surface of a large mass, *e.g.*, Earth) and (2) *orbits* (where 2 masses can be considered as point-like).

a) For orbits, we use the general form of the gravitational potential as described in Eqs. (IX-10,16):

$$U_g = -\frac{GMm}{r} \ , \tag{IX-21}$$

where we are now using M (the larger mass) for m_1 and m (the smaller mass) for m_2 and

$$\Phi = -\frac{GM}{r} \ . \tag{IX-22}$$

Orbits will be covered in §X of these notes.

b) For trajectories, typically the maximum height ($y_{\max} = h$) reached is small with respect to $R_\oplus$ and hence $g \approx$ constant. As such, we can write Eqs. (IX-9 and IX-11) as

$$W = U_1 - U_2 = GM_\oplus m \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \ .$$

i) If we take point '2' to be the Earth's surface and '1' to be the position of the projectile, then

$$\begin{aligned} \Delta U &= G\,M_\oplus\,m \left(\frac{1}{R_\oplus} - \frac{1}{R_\oplus + h}\right) \\ &= G\,M_\oplus\,m \left(\frac{R_\oplus + h}{R_\oplus\,(R_\oplus + h)} - \frac{R_\oplus}{R_\oplus\,(R_\oplus + h)}\right) \\ &= G\,M_\oplus\,m \left(\frac{R_\oplus + h - R_\oplus}{R_\oplus\,(R_\oplus + h)}\right) \end{aligned}$$

$$= G\,M_\oplus\,m \left(\frac{h}{R_\oplus\,(R_\oplus + h)} \right) .$$

ii) If $R_\oplus$ is much greater than h (which it will be for experiments near the Earth's surface), $h \ll R_\oplus$. As such, $R_\oplus + h \approx R_\oplus$ and the equation above becomes

$$\begin{aligned} \Delta U &= G\,M_\oplus\,m \left(\frac{h}{R_\oplus\,(R_\oplus)} \right) = G\,M_\oplus\,m \left(\frac{h}{R_\oplus^2} \right) \\ &= \frac{G\,M_\oplus\,m\,h}{R_\oplus^2} = m\,\frac{G\,M_\oplus}{R_\oplus^2}\,h \ . \end{aligned} \qquad \text{(IX-23)}$$

iii) Using Eqs. (IX-15 and IX-16), we see that

$$\begin{aligned} \vec{g} &= -\vec{\nabla}\Phi = -\vec{\nabla} \left(\frac{G\,M_\oplus}{r} \right) \\ &= -\frac{d}{dr} \left(\frac{G\,M_\oplus}{r} \right) \hat{r} = \frac{G\,M_\oplus}{r^2}\,\hat{r} \end{aligned} \qquad \text{(IX-24)}$$

and at the Earth's surface,

$$g = \frac{G\,M_\oplus}{R_\oplus^2} \ , \qquad \text{(IX-25)}$$

where 'g' is referred to as the Earth's **surface gravity**.

iv) Using Eq. (IX-25) in Eq. (IX-23), we finally get

$$\Delta U = mgh = mg\Delta y \ ,$$

where $\Delta y = y - y_\circ = h$ is just the change in height from our initial position $y_\circ$ (typically the ground) and y is an arbitrary position in the trajectory above $y_\circ$.

v) If we arbitrarily set $y_\circ = 0$, then the potential at that position is zero, and y represents the position above the ground ($y_\circ$). As such, the potential energy becomes

$$U = mgy \ . \tag{IX-26}$$

9. Trajectory calculations can be difficult due to non-gravitational forces that enter the calculations.

a) A **drag force** due to air friction which usually takes the form

$$\vec{F}_r = -mkv^n \frac{\vec{v}}{v} \ , \tag{IX-27}$$

where $\vec{F}_r$ represents the retarding (*i.e.*, drag) force, v is the magnitude of the velocity, $\vec{v}$ is the velocity vector (hence $\vec{F}_r$ is in the opposite direction of $\vec{v}$ due to the negative sign, note that the ratio $\vec{v}/v$ is essentially just a unit vector in the direction of $\vec{v}$), and k is the drag coefficient. As such, the total force on the object now becomes

$$\vec{F} = \vec{F}_g + \vec{F}_r \ . \tag{IX-28}$$

b) If the downrange distance of the projectile is large enough such that the Earth's surface can no longer be represented as a flat plane, the Earth's rotation has to be taken into account. To an observer in the rotating coordinate system, the effective force (ignoring air friction) is

$$\vec{F}_{\rm eff} = m\vec{a}_f - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r \ . \tag{IX-29}$$

i) $\vec{F}_f = m\vec{a}_f$ is the force in the *fixed* coordinate system (which is just Newton's 2nd law). This force is said to be an *inertial* force since it applies only to a static coordinate system.

ii) $\vec{F}_{\rm cf} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is the **centrifugal force**, which results from trying to write an inertial force law for a noninertial (*i.e.*, *accelerating*) reference frame (note that $\vec{\omega}$ is called the angular velocity). The minus sign in this term implies that this *pseudo*-force (see below) is directed outwards from the center of rotation (perpendicular to the axis of rotation).

iii) $\vec{F}_{\rm cor} = -2m\vec{\omega} \times \vec{v}_r$ is the **Coriolis force**, which results from the Earth's rotation ($\vec{v}_r \equiv$ Earth's rotational velocity) $\Longrightarrow$ this "force" arises when an attempt is made to describe motion relative to a rotating body (*i.e.*, the ground moves out from under you when the projectile is in the air).

iv) Note that the centrifugal and Coriolis forces are not *forces* in the usual sense of the word. They are only introduced so that the *inertial (non-accelerating) frame* equation

$$\vec{F} = m\vec{a}_f$$

(*i.e.*, Newton's 2nd law) can have a *noninertial (accelerating) frame* analogous equation:

$$\vec{F}_{\rm eff} = m\vec{a}_r \ ,$$

so that

$$\vec{F}_{\rm eff} = m\vec{a}_f + (\text{noninertial terms}).$$

B. Numerical Solutions for Trajectories.

1. Trajectories with $h \ll R_\oplus$ and $x \ll R_\oplus$.

a) Combining Eqs. (IX-1, 27, & 28), Newton's 2nd law becomes:

$$\sum \vec{F} = \vec{F}_g + \vec{F}_r = m\frac{d^2\vec{r}}{dt^2}$$

or

$$m\frac{d^2\vec{r}}{dt^2} = -mg\,\hat{y} - mkv^n\,\frac{\vec{v}}{v}\ , \tag{IX-30}$$

where we have defined $+\hat{y}$ in the upward direction.

b) Since the atmospheric drag force is a function of $\vec{v}$, it is more convenient to write Eq. (IX-30) as

$$\begin{aligned} m\frac{d\vec{v}}{dt} &= -mg\,\hat{y} - mkv^n\,\frac{\vec{v}}{v} \\ \frac{d\vec{v}}{dt} &= -g\,\hat{y} - kv^n\,\frac{\vec{v}}{v} \end{aligned} \tag{IX-31}$$

i) At low velocities, $n \approx 1$ and the magnitude of the drag force follows

$$F_r \approx -B_1\,v \tag{IX-32}$$

$\Longrightarrow$ this is known as **Stoke's law**.

ii) As v increases, $n \rightarrow 2$ and the drag force follows

$$F_r \approx -B_2\,v^2\ . \tag{IX-33}$$

iii) As such, we can write a general form of the drag force as

$$F_r \approx -B_1\,v - B_2\,v^2 \tag{IX-34}$$

or

$$F_r = -\sum_{i=1}^{N} B_i\,v^i \tag{IX-35}$$

$\Longrightarrow$ a simple power series in v (note that $B_i \rightarrow 0$ faster than $v^i \rightarrow \infty$ as $i \rightarrow \infty$).

iv) The B_i's are related to the drag coefficient k in Eqs. (IX-30 & 31).

c) Since we are dealing with projectiles here, the drag force will simply be described by Eq. (IX-33). As such, we need to solve each component of Eq. (IX-31) — hence, we need to break the drag force into its component forces:

$$F_r = -mkv^2 \ .$$

i) Since

$$v^2 = v_x^2 + v_y^2 \ ,$$

we can write

$$F_{r,x} = F_r \cos\theta = F_r \frac{v_x}{v} \ , \qquad \text{(IX-36)}$$

where θ is the angle between $\vec{v}_x$ and $\vec{v}$.

ii) Likewise

$$F_{r,y} = F_r \sin\theta = F_r \frac{v_y}{v} \ . \qquad \text{(IX-37)}$$

iii) Hence, the drag force has components of

$$\begin{aligned} F_{r,x} &= -mkvv_x \\ F_{r,y} &= -mkvv_y \ . \end{aligned} \qquad \text{(IX-38)}$$

d) The x-component of Newton's 2nd law gives

$$\frac{dv_x}{dt} = -kvv_x \qquad \text{(IX-39)}$$

with

$$\frac{dx}{dt} = v_x \ . \qquad \text{(IX-40)}$$

e) The y-component gives

$$\frac{dv_y}{dt} = -g - kvv_y \qquad \text{(IX-41)}$$

with

$$\frac{dy}{dt} = v_y \ . \qquad \text{(IX-42)}$$

f) The solution to these first-order DEs can be found with simple forward-difference equations:

$$\begin{aligned} x_{i+1} &= x_i + v_{x,i}\,\Delta t \\ v_{x,i+1} &= v_{x,i} - kvv_{x,i}\,\Delta t \\ y_{i+1} &= y_i + v_{y,i}\,\Delta t \\ v_{y,i+1} &= v_{y,i} - g\,\Delta t - kvv_{y,i}\,\Delta t \;. \end{aligned} \qquad \text{(IX-43)}$$

g) Eqs. (IX-43) can be then solved as an initial value problem as described in §VI with $x_\circ$, $y_\circ$, $v_\circ$, and $\theta_\circ$ supplied by the user. From $v_\circ$ and $\theta_\circ$, we first calculate the initial velocity vector components: $v_{x\circ} = v_\circ \cos\theta_\circ$, and $v_{y\circ} = v_\circ \sin\theta_\circ$.

i) The Δt steps are chosen to give the errors that follow Eq. (VI-15).

ii) Calculations are carried out until a certain τ = *total time* is reached or some condition of x_i, y_i, $v_{x,i}$, or $v_{y,i}$ is satisfied.

h) But what is the drag coefficient k?

i) Since the projectile is trying to push air of mass $dm_{\rm air}$ out of the way, where

$$dm_{\rm air} \approx \rho\,A\,v\,dt \;, \qquad \text{(IX-44)}$$

where ρ is the density of the air and A is the frontal area, we can guess that k is a function of ρ [and possibly t if the object is rotating $\rightarrow$ then $A = A(t)$].

ii) $k(y = 0) = k_\circ$ is usually given (based on air tunnel measurements), and the following expression is used for $k(y)$:

$$k(y) = \frac{\rho(y)}{\rho_\circ}\,k_\circ \;, \qquad \text{(IX-45)}$$

where $\rho_\circ$ is the density of air when the $k_\circ$ measurement was made.

i) Now we need a description of $\rho(y)$!

i) One could supply a data table of ρ as a function of height (see Appendix B.1 in *Fundamentals of Atmospheric Modeling* by Mark Jacobson, 1999, Cambridge University Press).

ii) One could solve the following set of differential equations to determine $\rho(y)$:

$$P = N k_B T \quad \text{(ideal gas law)} \tag{IX-46}$$

$$\rho = \sum N_i \, m_i \quad \text{(mass conservation)} \tag{IX-47}$$

$$\frac{dP}{dy} = -\rho \, g \quad \text{(hydrostatic equilibrium)} \tag{IX-48}$$

$$\frac{dT}{T} = \left(\frac{R}{c_p}\right) \frac{dP}{P} \quad \text{(Poisson's equation)} \tag{IX-49}$$

$$dQ = c_p \, dT - \alpha \, dP \quad \text{(energy conserv.)} \tag{IX-50}$$

where P = pressure, T = temperature, N = total particle density, ρ total mass density, N_i = number density of species i, m_i = mass of species i, k_B = Boltzmann's constant, g = surface gravity, R = universal gas constant, c_p = specific heat of air at constant pressure, Q = total heat (determined from solar radiation incident on the Earth's atmosphere), and α = specific volume of air.

iii) Note that the solution to these equation would still only be an approximation since we have left out condensation, evaporation, sublimation, chemical reactions, and wind from these equations.

2. Trajectories with $h \sim R_\oplus$ and $x \sim R_\oplus$.

a) Rotating Coordinate Systems.

i) Let's consider 2 sets of coordinate axes:
$\Longrightarrow$ one 'fixed' = **inertial frame** (the *primed* [$'$] coordinates),
$\Longrightarrow$ one 'rotating' with respect to the fixed system and possibly in linear motion with respect to the fixed frame = **noninertial frame**.

ii) Let point 'P' be a point in space that can both be measured from either frames, then

$$\vec{r}\,' = \vec{R} + \vec{r}\,, \tag{IX-51}$$

where $\vec{r}\,'$ is the radius vector if P in the fixed system, $\vec{r}$ is the radius vector of P in the moving system, and $\vec{R}$ locates the origin of the moving system with respect to the fixed system as shown in Figure IX-4.

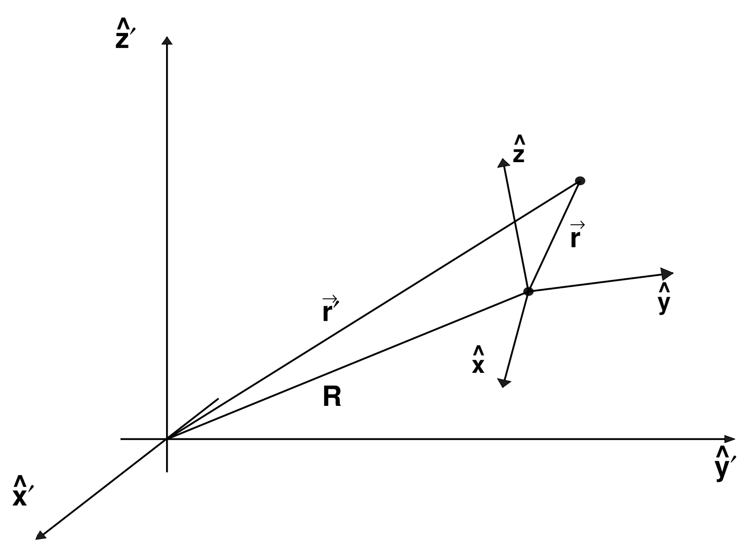

Figure IX–4: Locating a point in an inertial frame and a noninertial frame.

iii) The radius vector differential of the moving frame as measured in the fixed frame is related to the rotation angle differential of the noninertial frame:

$$(d\vec{r})_{\rm fixed} = d\vec{\theta} \times \vec{r} \; , \qquad \text{(IX-52)}$$

where here, the LHS of the equation is measured in the fixed frame and the RHS is measured in the rotating frame.

iv) The time rate of change of $\vec{r}$ as measured in the fixed frame is

$$\left(\frac{d\vec{r}}{dt}\right)_{\rm fixed} = \frac{d\vec{\theta}}{dt} \times \vec{r} = \vec{\omega} \times \vec{r} \; , \qquad \text{(IX-53)}$$

since the angular velocity $\vec{\omega}$ is defined by

$$\vec{\omega} \equiv \frac{d\vec{\theta}}{dt} \; . \qquad \text{(IX-54)}$$

v) If point P has a velocity $(d\vec{r}/dt)_{\rm rot}$ with respect to the rotating system, this velocity must be added to $\vec{\omega} \times \vec{r}$ to obtain the time rate of change of $\vec{r}$ in the fixed system:

$$\left(\frac{d\vec{r}}{dt}\right)_{\rm fixed} = \left(\frac{d\vec{r}}{dt}\right)_{\rm rot} + \vec{\omega} \times \vec{r} \; . \qquad \text{(IX-55)}$$

vi) This expression is not just limited to the displacement vector $\vec{r}$, in fact, for any arbitrary vector $\vec{Q}$, we have

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{\rm fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{\rm rot} + \vec{\omega} \times \vec{Q}} \; . \qquad \text{(IX-56)}$$

vii) Note that the angular acceleration $\dot{\vec{\omega}}$ is the same in both the fixed and rotating systems:

$$\begin{aligned} \left(\frac{d\vec{\omega}}{dt}\right)_{\rm fixed} &= \left(\frac{d\vec{\omega}}{dt}\right)_{\rm rot} + \vec{\omega}\times\vec{\omega} \\ \left(\frac{d\vec{\omega}}{dt}\right)_{\rm fixed} &= \left(\frac{d\vec{\omega}}{dt}\right)_{\rm rot} \equiv \dot{\vec{\omega}} \; . \end{aligned} \tag{IX-57}$$

viii) As such, the velocity of point P as measured in the fixed coordinate system is

$$\begin{aligned} \left(\frac{d\vec{r}\,'}{dt}\right)_{\rm fixed} &= \left(\frac{d\vec{R}}{dt}\right)_{\rm fixed} + \left(\frac{d\vec{r}}{dt}\right)_{\rm fixed} \\ \left(\frac{d\vec{r}\,'}{dt}\right)_{\rm fixed} &= \left(\frac{d\vec{R}}{dt}\right)_{\rm fixed} + \left(\frac{d\vec{r}}{dt}\right)_{\rm rot} + \vec{\omega}\times\vec{r} \; . \end{aligned}$$

If we define

$$\vec{v}_f \equiv \dot{\vec{r}}_f \equiv \left(\frac{d\vec{r}\,'}{dt}\right)_{\rm fixed} \tag{IX-58}$$

$$\vec{V} \equiv \dot{\vec{R}}_f \equiv \left(\frac{d\vec{R}}{dt}\right)_{\rm fixed} \tag{IX-59}$$

$$\vec{v}_r \equiv \dot{\vec{r}}_r \equiv \left(\frac{d\vec{r}}{dt}\right)_{\rm rot} \tag{IX-60}$$

we may write

$$\boxed{\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega}\times\vec{r}} \; , \tag{IX-61}$$

where

$$\begin{aligned} \vec{v}_f &= \text{velocity relative to the fixed axes} \\ \vec{V} &= \text{linear velocity of the moving origin} \\ \vec{v}_r &= \text{velocity relative to the rotating axes} \\ \vec{\omega} &= \text{angular velocity of the rotating axes} \\ \vec{\omega}\times\vec{r} &= \text{velocity due to the rotation of the moving axes.} \end{aligned}$$

b) The Coriolis Force.

i) Newton's 2nd law $\vec{F} = m\vec{a}$ is only valid in an inertial frame, therefore

$$\vec{F} = m\vec{a}_f = m\left(\frac{d\vec{v}_f}{dt}\right)_{\rm fixed} , \qquad \text{(IX-62)}$$

where the differentiation must be carried out with respect to the fixed system.

ii) If we limit ourselves to cases of *constant angular acceleration* ($\dot{\omega} = 0$), using Eq. (IX-61) we can write

$$\vec{F} = m\ddot{\vec{R}}_f + m\left(\frac{d\vec{v}_r}{dt}\right)_{\rm fixed} + m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\rm fixed} . \qquad \text{(IX-63)}$$

iii) The second term can be evaluated by substituting $\vec{v}_r$ for $\vec{Q}$ in Eq. (IX-56):

$$\begin{aligned}\left(\frac{d\vec{v}_r}{dt}\right)_{\rm fixed} &= \left(\frac{d\vec{v}_r}{dt}\right)_{\rm rot} + \vec{\omega}\times\vec{v}_r \\ &= \vec{a}_r + \vec{\omega}\times\vec{v}_r , \end{aligned} \qquad \text{(IX-64)}$$

where $\vec{a}_r$ is the acceleration in the rotating coordinate system.

iv) The last term in Eq. (IX-63) can be obtained directly from Eq. (IX-55):

$$\begin{aligned}\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\rm fixed} &= \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\rm rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) \\ &= \vec{\omega}\times\vec{v}_r + \vec{\omega}\times(\vec{\omega}\times\vec{r}) . \end{aligned} \qquad \text{(IX-65)}$$

v) Combining Eqs. (IX-63)-(IX-65), we obtain

$$\vec{F} = m\ddot{\vec{R}}_f + m\vec{a}_r + 2m\vec{\omega}\times\vec{v}_r + m\vec{\omega}\times(\vec{\omega}\times\vec{r}) , \quad \dot{\vec{\omega}} = 0 \qquad \text{(IX-66)}$$

where $\ddot{\vec{R}}_f$ is the acceleration of the origin of the moving coordinate system relative to the fixed system.

vi) In the case of trajectories on the Earth's surface, the origin of the rotating coordinate system is stationary (in the $\vec{r}$ direction) with respect to the fixed coordinate system. As such, $\ddot{\vec{R}}_f = 0$ we can finally write

$$\vec{F} = m\vec{a}_f = m\vec{a}_r + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2m\vec{\omega} \times \vec{v}_r \ . \tag{IX-67}$$

vii) From this equation, the effective force on a particle measured by an observer in the rotating frame is then

$$\boxed{\vec{F}_{\rm eff} = m\vec{a}_r = m\vec{a}_f - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r} \ . \tag{IX-68}$$

which is what we wrote down in §IX.A of this section of the notes in Eq. (IX-29).

c) Coding Problems with $h \sim R_\oplus$ and $x \sim R_\oplus$ — Part 1: The Fixed Coordinate System.

i) Unlike the preceding case where $h \ll R_\oplus$ and $x \ll R_\oplus$, for our current problem we must use a three-dimensional coordinate system.

ii) In reality, the Earth is constantly being subjected to a variety of motions:
$\Longrightarrow$ The Earth's own rotation.
$\Longrightarrow$ The Earth's orbital velocity around the Sun.

$\Longrightarrow$ The Sun's orbital velocity about the center of the Milky Way Galaxy.

$\Longrightarrow$ The Milky Way's and Andromeda (M31) galaxy's motion towards each other.

$\Longrightarrow$ The Local Group of galaxy's motion in the Virgo Supercluster of galaxies.

$\Longrightarrow$ The Hubble flow of all of the galaxies, clusters, and superclusters in the Universe.

As can be seen, quite a lot of velocities to worry about. However, for our problem here, the Earth's rotational velocity will dominate these other velocities.

iii) Choose the fixed coordinate system's origin to lie at the Earth's center and its $\hat{z}'$ direction to lie from a line from the center to the North Pole (following the right-hand rule since the Earth is rotating counterclockwise as viewed from the North Pole) $\Longrightarrow \vec{\omega} = \omega\,\hat{z}' = (7.272205 \times 10^{-5}\ \mathrm{rad/s})\ \hat{z}'$.

iv) We will use spherical coordinates to describe our fixed coordinate system, however, instead of θ' which is measured from the z' axis, we will use the **latitude** angle $= \lambda' = 90^\circ - \theta'$ which is measured from the x'-y' plane $\Longrightarrow$ the **equatorial plane**. Hence, $\sin\theta' = \cos\lambda'$ and $\cos\theta' = \sin\lambda'$.

v) The $\vec{r}\,'$ vector is measured with respect to the Earth's center ($= R_\oplus + z$, where z will be the height in the rotating frame with respect to sea level — see below). Since the Earth is an oblate spheroid, an accurate radius for the Earth is obtained with the equation

$$R_\oplus = \frac{R_{eq}(\oplus)}{e\,|\sin\lambda| + 1} \ , \qquad \text{(IX-69)}$$

where $R_{eq}(\oplus) = 6.37853 \times 10^6$ m is the equatorial radius of the Earth (at sea level), λ is the latitude angle, and $e = 0.003393$ is the obliquity of the Earth.

vi) To define a reference point from which ϕ' is measured (*i.e.*, the x' axis — see Figure IX-2), the background stars are used as the reference and the direction of the x' axis is from the center of the Earth to the **vernal equinox** on the celestial sphere. The vernal equinox is the intersection point of the Sun's path on the sky (*i.e.*, the **ecliptic**), which is the Earth's orbit projected onto the sky, and the **celestial equator**, which is the Earth's equatorial plane projected on the sky.

$\Longrightarrow$ ϕ' completes one revolution in one **sidereal day** (= 24 *sidereal* hours exactly).

vii) Whereas the Earth's coordinate system is based on **latitude** (lines parallel to the equator) and **longitude** (lines $\perp$ to latitude that run from the North to South Poles), both measured in degrees, the sky's coordinates are **declination** (DEC, like latitude and measured in degrees) and **right ascension** (RA, like longitude, but measured in units of time).

viii) *Local* sidereal time is determined by the current RA directly on the local celestial meridian $\Longrightarrow$ hence $0^{\rm hr}$ sidereal time occurs when the vernal equinox is on the local meridian.

ix) The time used in the changing ϕ' calculation is based on **Universal Time** $\Longrightarrow$ $0^{\rm hr}$ UT occurs when the vernal equinox is on the **celestial meridian** of the **Prime Meridian** in Greenwich, England which marks 0° longitude. The celestial meridian is an imaginary line on the sky that connects the north point on the horizon, the zenith (point directly overhead), and the south point on the horizon – it separates the east side of the sky from the west side of the sky. As such, Universal Time is the local sidereal time at the Prime Meridian.

x) The Earth is subjected to extremely small, but unpredictable, variations in its rotation rate (mainly due to gravitational perturbations from other bodies in the Solar System and a non-smooth slowing down due to the tides raised from the Sun and Moon). To precisely predict the positions of bodies in the Solar System, we require a steady time standard.

$\Longrightarrow$ Universal Time is then replaced by **Ephemeris Time** (E.T.) in celestial mechanics.

$\Longrightarrow$ At the beginning of 1900 A.D., an **ephemeris second** was defined as 1/31,556,925.97474 the length of the tropical year 1900 and both U.T. and E.T. were in agreement.

$\Longrightarrow$ Today, these times differ by about 56 seconds.

xi) However for this course, we will just use U.T. To calculate ϕ', use the following formula:

$$\phi' = 2\pi \, \frac{t_{\rm U.T.}}{24 \text{ hr}} \,, \tag{IX-70}$$

where $t_{\rm U.T.}$ is the current Universal Time in decimal sidereal hours.

xii) As such, your code should have the ability for the user to input:

$\Longrightarrow$ **A time of launch (in U.T. sidereal hours).**

$\Longrightarrow$ **The latitude of launch (in decimal degrees).**

$\Longrightarrow$ **The altitude of the ground with respect to sea level.**

d) Coding Problems with $h \sim R_\oplus$ and $x \sim R_\oplus$ — Part 2: The Rotating Coordinate System.

i) We now have all of the fixed (*primed*) frame coordinates (*i.e.*, r', λ', and ϕ'), we now need the rotating frame coordinates.

$\Longrightarrow$ x is defined in the **eastern** direction.

$\Longrightarrow$ y is defined in the **northern** direction.

$\Longrightarrow$ z is the altitude (which follows from the right-hand rule).

ii) Rotational coordinate frame transformations are then made with

$$r \;=\; \left(x^2 + y^2 + z^2\right)^{1/2} \tag{IX-71}$$

$$\theta = \cos^{-1}\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right) \quad \text{(IX-72)}$$

$$\phi = \tan^{-1}\frac{y}{x}\,, \quad \text{(IX-73)}$$

where r is measured from the launch point (the rotating origin), θ is the angle from the z (*i.e.*, altitude) axis (note that this makes θ different from what it was in the $h \ll R_\oplus$ case when it was measured from the ground), and ϕ is the angle subtended from the east point on the horizon moving towards north.

iii) Now your code should allow the user to input:

> $\Longrightarrow$ **A launch velocity (typically in m/s) $= \dot{r}$.**
>
> $\Longrightarrow$ **The projection angle of launch (in decimal degrees) $= \gamma$.**
>
> $\Longrightarrow$ **The direction angle with respect to the east direction rotating towards north $= \phi$.**

iv) Then from these in input, r will be determined in the next step and $\theta = 90° - \gamma$.

e) Coding Problems with $h \sim R_\oplus$ and $x \sim R_\oplus$ — Part 3: Solving the System of Equations.

i) Calculate the Coriolis and centrifugal cross products in Eq. (IX-68).

ii) Solve for v_r in Eq. (IX-61) by using the forward-difference technique shown in Eqs. (IX-43). However, now, replace y with z and the two x equations become four — 2 for x and 2 for y which will now have the same functional form. Also, instead of g, now use a_r from Eq. (IX-68) in the z direction where $a_f = g$ in this equation.

iii) Solve for the unknowns: downrange distance, maximum height, impact velocity, etc.