PHYS-4007/5007: Computational Physics Course Lecture Notes Section X

Dr. Donald G. Luttermoser
East Tennessee State University

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Abstract

These class notes are designed for use of the instructor and students of the course PHYS-4007/5007: Computational Physics taught by Dr. Donald Luttermoser at East Tennessee State University.

X. Computing Orbits

A. The Physics of Orbits

1. Reduced Mass

a) For two masses m_1 and m_2, Newton's Law of Gravity is described as a force between two particles separated by a distance $\vec{r}$. However, if the origin does not lie along that line, then $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$.

b) The Lagrangian for such a system may be written as

$$\mathcal{L} = \frac{1}{2}m_1|\dot{r}_1|^2 + \frac{1}{2}m_2|\dot{r}_2|^2 - U(r) \ . \qquad \text{(X-1)}$$

c) For two bodies in orbit about each other, it is convenient to select the origin such that it lies on the **center or mass** along $\vec{r}$. The center of mass is defined by the equation:

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \ . \qquad \text{(X-2)}$$

d) This equation, combined with $\vec{r} = \vec{r}_1 - \vec{r}_2$, yields

$$\left.\begin{array}{rcl} \vec{r}_1 & = & \frac{m_2}{m_1+m_2}\vec{r} \\ \vec{r}_2 & = & -\frac{m_1}{m_1+m_2}\vec{r} \end{array}\right\} \ . \qquad \text{(X-3)}$$

e) Substituting Eq. (X-3) into Eq. (X-1) gives

$$\boxed{\mathcal{L} = \frac{1}{2}\mu|\dot{\vec{r}}|^2 - U(r)} \ , \qquad \text{(X-4)}$$

where μ is the **reduced mass**,

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M} \ , \qquad \text{(X-5)}$$

and $M = m_1 + m_2$ is the total mass of the system.

f) By introducing the reduced mass, we have reduced the motion of two bodies to an *equivalent one-body problem* in which we must determine only the motion of a "particle" of mass μ in the central potential field described by $U(r)$.

2. Conservation Theorems — First Integrals of the Motion

a) In a spherically symmetric force field, angular momentum is conserved:

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}} = \text{ constant}, \tag{X-6}$$

where $\vec{p}$ is the linear momentum.

b) From this conservation law, we see that the radius vector and the linear velocity are coplanar and $\perp$ to the constant angular momentum vector.

i) This means that the $\dot{\phi}$ term is zero since $\vec{L}$ lies in the direction of $\hat{z}$ from the angular momentum equation above.

ii) As such, we can write $\dot{\vec{r}}$ in Eq. (X-4) using only plane polar coordinates:

$$\boxed{\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)}\,. \tag{X-7}$$

c) Since the Lagrangian is cyclic in θ, the angular momentum conjugate to the coordinate θ is conserved:

$$\dot{p}_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \tag{X-8}$$

or

$$p_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{ constant}. \tag{X-9}$$

d) The quantity p_θ is a *first integral* of the motion. From this point forward, we will denote its constant value by the symbol ℓ:

$$\boxed{\ell \equiv \mu r^2 \dot{\theta} = \text{ constant.}} \qquad \text{(X-10)}$$

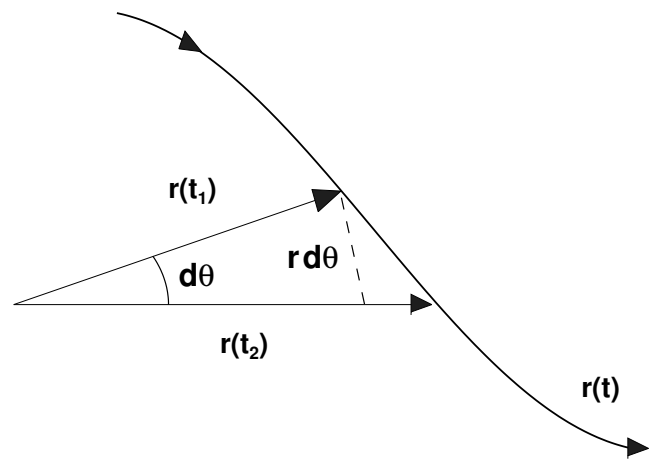


Figure X–1: Geometry setup of Kepler's 2nd Law of Motion.

e) Referring to Fig. (X-1), we see that describing the path $\vec{r}(t)$, the radius vector sweeps out an area $\frac{1}{2}r^2\,d\theta$ in a time interval dt:

$$dA = \frac{1}{2}r^2\,d\theta \ , \qquad \text{(X-11)}$$

and dividing by the time interval, the *areal velocity* is

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}r^2\,\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} \\ &= \frac{\ell}{2\mu} = \text{ constant,} \end{aligned} \qquad \text{(X-12)}$$

which shows that the areal velocity is constant in time $\Longrightarrow$ **Kepler's Second Law of Planetary Motion**.

f) The conservation of energy gives us

$$E = T + U = \text{ constant} \qquad \text{(X-13)}$$

$$= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) \ , \qquad \text{(X-14)}$$

or

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)} \ . \qquad \text{(X-15)}$$

3. Equation of Motion.

a) Using the total energy equation (Eq. X-15), we can solve for the velocity:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U) - \frac{\ell^2}{\mu^2 r^2}} \ . \qquad \text{(X-16)}$$

b) Note that

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}\,dr = \frac{\dot{\theta}}{\dot{r}}\,dr = \frac{\ell/\mu r^2}{\dot{r}}\,dr \ , \qquad \text{(X-17)}$$

or

$$\dot{r} = \frac{\ell}{\mu r^2}\frac{dr}{d\theta} \ , \qquad \text{(X-18)}$$

hence

$$\frac{\ell}{\mu r^2}\frac{dr}{d\theta} = \sqrt{\frac{2}{\mu}(E - U) - \frac{\ell^2}{\mu^2 r^2}} \ . \qquad \text{(X-19)}$$

c) This leads to an equation of $\theta(r)$:

$$\theta(r) = \int \frac{(\ell/r^2)\,dr}{\sqrt{2\mu(E - U - \ell^2/2\mu r^2)}} + \text{ constant.} \qquad \text{(X-20)}$$

d) If $F(r) \propto r^n$, Eq. (X-20) becomes an *elliptical integral.*

i) For $n = 1, -2$, and $-3 \rightarrow$ solutions are circular functions.

ii) $n = 1$ gives harmonic motion.

iii) $n = -2$ gives a central-force law (*e.g.*, gravity, see later).

iv) $n = -3$ gives an equation that is not important in physics.

e) We can also solve this problem with Lagrangians:

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \ , \qquad \text{(X-21)}$$

where the Lagrangian is given by Eq. (X-7).

i) Term one of Eq. (X-21) is

$$\frac{\partial \mathcal{L}}{\partial r} = \mu r \dot{\theta}^2 - \frac{\partial U}{\partial r} \ .$$

ii) Term two of Eq. (X-21) is

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \dot{r} \ , \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \ddot{r} \ .$$

iii) Putting these two equations together gives

$$\mu r \dot{\theta}^2 - \mu \ddot{r} - \frac{\partial U}{\partial r} = 0$$

or

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r) \ . \qquad \text{(X-22)}$$

f) Now let

$$u \equiv \frac{1}{r} \ , \qquad \text{(X-23)}$$

then

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} \ . \qquad \text{(X-24)}$$

g) Since $\dot{\theta} = \ell/\mu r^2$, we get

$$\frac{du}{d\theta} = -\frac{\mu}{\ell}\dot{r} \ . \qquad \text{(X-25)}$$

h) Next,

$$\begin{aligned}\frac{d}{d\theta}\frac{du}{d\theta} &= \frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\mu}{\ell}\,\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{\ell}\,\dot{r}\right) \\ &= -\frac{\mu}{\ell\dot{\theta}}\,\ddot{r} = -\frac{\mu^2}{\ell^2}\,r^2\,\ddot{r}\ . \end{aligned} \tag{X-26}$$

i) Therefore since $u = 1/r$,

$$\ddot{r} = -\frac{\ell^2}{\mu^2}\,u^2\,\frac{d^2u}{d\theta^2} \tag{X-27}$$

$$r\dot{\theta}^2 = \frac{\ell^2}{\mu^2}\,u^3\ . \tag{X-28}$$

j) Using this in Eq. (X-22) gives

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\,\frac{1}{u^2}\,F(u)\ , \tag{X-29}$$

or

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}\,F(r)} \tag{X-30}$$

$\Longrightarrow$ this equation is useful if we wish to find $r = r(\theta)$.

4. Orbits in a Central Field.

a) From Eq. (X-16), the velocity vanishes when

$$E - U(r) - \frac{\ell^2}{2\mu r^2} = 0\ . \tag{X-31}$$

i) These are *turning points* in the motion.

ii) Two roots are obtained from Eq. (X-31): $r_{\rm min}$ and $r_{\rm max}$ such that $r_{\rm min} \le r \le r_{\rm max}$.

b) Certain combinations of $U(r)$, E, and ℓ will produce a single root for Eq. (X-31), then $\dot{r} = 0$ for all times, hence $r =$ constant $\rightarrow$ the orbit is *circular*.

c) If the motion of the particle in $U(r)$ is periodic $\rightarrow$ orbit is **closed**.

d) If the orbit does not close on itself (*i.e.*, does not repeat itself) $\rightarrow$ orbit is **open**.

e) Orbital motion is symmetric in time between $r_{\rm min}$ and $r_{\rm max}$, hence $\Delta\theta = 2\,\theta(r)$ or

$$\Delta\theta(r) = 2\int_{r_{\rm min}}^{r_{\rm max}} \frac{(\ell/r^2)\,dr}{\sqrt{2\mu(E - U - \ell^2/2\mu r^2)}}. \qquad \text{(X-32)}$$

i) The path will be closed only if $\Delta\theta$ is a rational function of 2π ($\Delta\theta = 2\pi \cdot \frac{a}{b}$, where '$a$' and '$b$' are integers).

ii) After 'b' periods, r will have made 'a' complete revolutions and return to its original position.

iii) If $U(r) \propto r^{n+1}$, then closed non-circular paths can result only if $n = -2$ (inverse-square law) or $+1$ (harmonic oscillator).

5. Centrifugal Energy and Effective Potential.

a) Note that $\left[\frac{\ell^2}{2\mu r^2}\right] = [E]$ (see Eq. X-32) and

$$\frac{\ell^2}{2\mu r^2} = \frac{1}{2}\mu r^2 \dot{\theta}^2 \ . \qquad \text{(X-33)}$$

Now let

$$U_c \equiv \frac{\ell^2}{2\mu r^2} \qquad \text{(X-34)}$$

and

$$F_c = -\frac{\partial U_c}{\partial r} = \frac{\ell^2}{\mu r^3} = \mu r \dot{\theta}^2 = \begin{array}{c}\text{centrifugal}\\ \text{force}\end{array} \qquad \text{(X-35)}$$

(although not a 'force' in the ordinary sense of the word).

b) Hence, let $U_c \equiv$ **centrifugal potential energy**.

c) Then, the **effective potential energy** is

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2} \ . \tag{X-36}$$

d) For gravity,

$$F(r) = -\frac{Gm_1m_2}{r^2} \ , \tag{X-37}$$

so

$$U(r) = -\int F(r)\,dr = -\frac{Gm_1m_2}{r} \tag{X-38}$$

and

$$V(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2} \ , \tag{X-39}$$

where $V \to 0$ as $r \to \infty$. See Fig. (X-2) on the next page for plots of these potentials.

i) For the bottom plot in Fig. (X-2), if $E = E_1 > 0$, the motion is unbounded.

ii) If $0 > E(= E_2) > E_3$, the motion has turning points (*i.e.*, bounded between r_2 and $r_4 \to$ the *apsidal* distances).

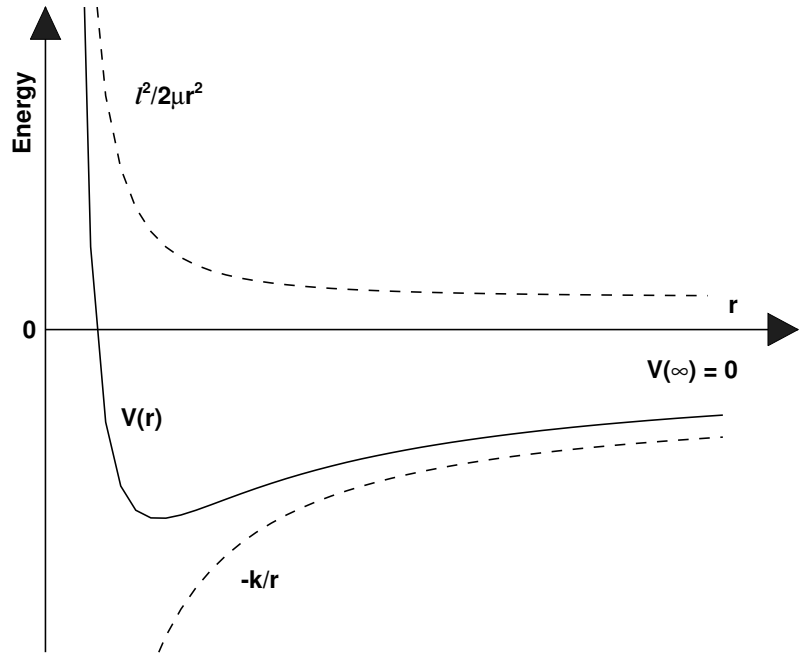
iii) If $E = E_3$, only one solution exists $\to$ motion is circular.

iv) Note that $E < E_3$ is not possible $\Longrightarrow \dot{r}$ would be imaginary!

6. Planetary Motion — Kepler's Problem.

a) Let us re-examine Eq. (X-20).

i) For the potential energy given by Eq. (X-38) let's define $k = Gm_1m_2$ (typically we will let m_1 be the



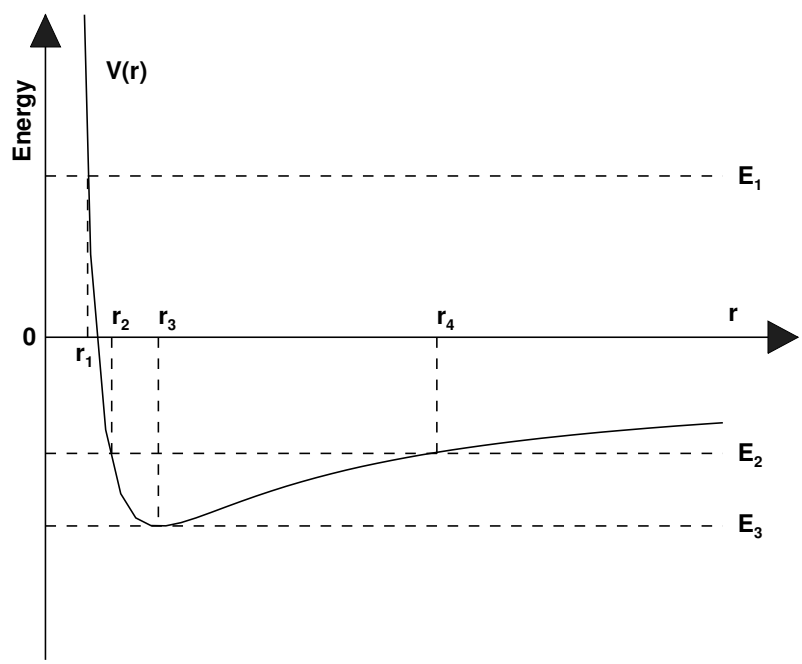


Figure X–2: The various potentials in a gravitational field (*top*) and the various energy regimes in the gravitational effective potential field (*bottom*).

bigger mass such that $m_1 \gg m_2$) giving the potential energy as

$$U(r) = -\frac{Gm_1m_2}{r} = -\frac{k}{r} \ . \tag{X-40}$$

ii) If we choose the origin of θ so that the integration constant in Eq. (X-20) is zero, we can show (with a little algebra) that

$$\cos\theta = \frac{\frac{\ell^2}{\mu k}\cdot\frac{1}{r} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}} \ . \tag{X-41}$$

iii) Let us now define the following constants:

$$\alpha \equiv \frac{\ell^2}{\mu k} \tag{X-42}$$

$$\varepsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu k^2}} \ . \tag{X-43}$$

b) Then we can rewrite Eq. (X-41) as

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta \ .} \tag{X-44}$$

i) This is the equation of a conic section with one focus at the origin.

ii) The quantity ε is called the **eccentricity** of the orbit.

iii) The quantity 2α is termed the **latus rectum** of the orbit.

c) The minimum value for r occurs when $\cos\theta$ is a maximum (*i.e.*, for $\theta = 0$). Thus the choice of the integration constant in Eq. (X-32) be zero corresponds to measuring θ from $r_{\min}$.

i) For an arbitrary orbit, $r_{\rm min}$ is called the **pericenter**.

ii) For solar orbits it is called the **perihelion**.

iii) For objects in orbit about the Earth, it is called the **perigee**.

d) The maximum value for r occurs when $\cos\theta$ is a minimum (*i.e.*, for $\theta = 180°$).

i) For an arbitrary orbit, $r_{\rm max}$ is called the **apocenter**.

ii) For solar orbits it is called the **aphelion**.

iii) For objects in orbit about the Earth, it is called the **apogee**.

e) Various values of the eccentricity (and, hence, of the energy) classify the orbits according to the different conic sections (see Figure X-3):

$\varepsilon > 1$	$E > 0$	(hyperbola)
$\varepsilon = 1$	$E = 0$	(parabola)
$0 < \varepsilon < 1$	$U_{\rm min} < E < 0$	(ellipse)
$\varepsilon = 0$	$E = U_{\rm min}$	(circle)
$\varepsilon < 0$	$E < U_{\rm min}$	(not allowed)

f) For the case of planetary motion, the orbits are ellipses as per **Kepler's First Law of Planetary Motion** (see Figure X-4).

i) The **semimajor axis** of an elliptical orbit is given

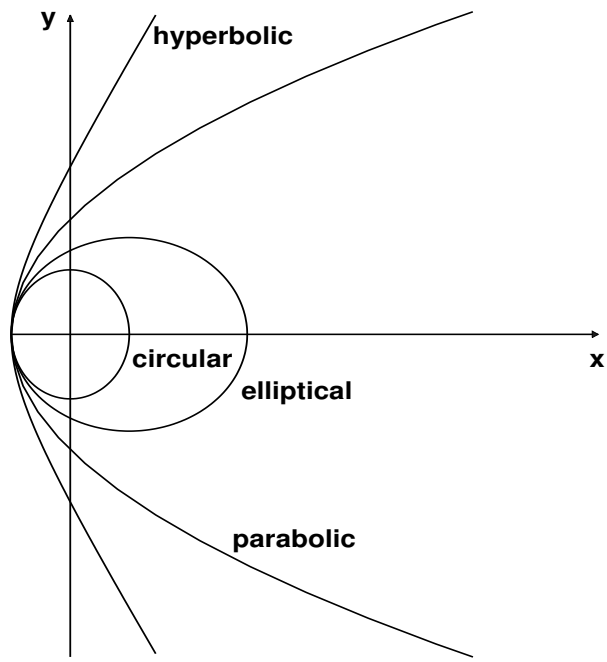


Figure X–3: Possible orbits in a $1/r$ potential follow the conic sections shown here.

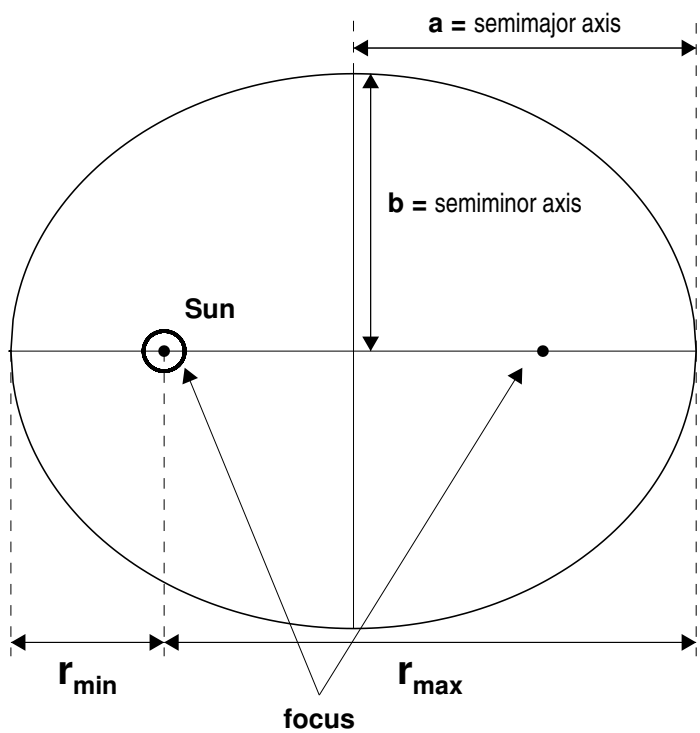


Figure X–4: Details of the elliptical orbit.

by

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|} = \frac{G m_1 m_2}{2|E|} \ , \qquad \text{(X-45)}$$

as such, the semimajor axis depends only on the energy of the particle, or vise versa,

$$|E| = \frac{k}{2a} = \frac{G m_1 m_2}{2a} \ . \qquad \text{(X-46)}$$

ii) The **semiminor axis** of an elliptical orbit is given by

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{\ell}{2\mu|E|} \ , \qquad \text{(X-47)}$$

as such, the semiminor axis depends both on E and ℓ. Comparing Eq. (X-45) with Eq. (X-47), we can write the semiminor axis in terms of the semimajor as

$$b = a\sqrt{1-\varepsilon^2} = \sqrt{\alpha a} \ . \qquad \text{(X-48)}$$

g) The extrema (*i.e.*, turning points) of the orbit is given by

$$r_{\rm min} = a(1-\varepsilon) = \frac{\alpha}{1+\varepsilon} \qquad \text{(X-49)}$$

$$r_{\rm max} = a(1+\varepsilon) = \frac{\alpha}{1-\varepsilon} \qquad \text{(X-50)}$$

h) To find the period of elliptical motion, take the areal velocity equation (Eq. X-12) and integrate:

$$\begin{aligned} dt &= \frac{2\mu}{\ell}\, dA \\ \int_0^\tau dt &= \frac{2\mu}{\ell} \int_0^A dA \\ \tau &= \frac{2\mu}{\ell}\, A \ , \end{aligned} \qquad \text{(X-51)}$$

where the entire area of the ellipse A is swept out in one orbital period τ.

i) Since $A = \pi ab$ for an ellipse, we can write the orbital period in terms of the energy of a particle in orbit a

$$\begin{aligned} \tau &= \frac{2\mu}{\ell}\pi ab = \frac{2\mu}{\ell}\frac{k}{2|E|}\frac{\ell}{\sqrt{2\mu|E|}} \\ &= \pi k\sqrt{\frac{\mu}{2}}\,|E|^{-3/2} \ . \end{aligned} \tag{X-52}$$

j) Likewise, we can use Eqs. (X-42) and (X-47) to write

$$\tau = \frac{2\mu}{\ell}\pi ab = \frac{2\mu}{\ell}\pi a\sqrt{\alpha a} = \frac{2\pi\mu}{\ell}\sqrt{\frac{\ell^2}{\mu k}}\,a^{3/2} \ , \tag{X-53}$$

or

$$\boxed{\tau^2 = \frac{4\pi^2\mu}{k}\,a^3 = \frac{4\pi^2}{G(m_1+m_2)}\,a^3 \ ,} \tag{X-54}$$

which is the analytic proof to **Kepler's Third Law of Planetary Motion**. Note that for planets in the solar system, $m_1 = M_\odot \gg m_2$ (where m_2 is the mass of a planet), giving

$$\tau^2 \cong \frac{4\pi^2}{GM_\odot}\,a^3 = K_\odot\,a^3 \ ,$$

where $K_\odot$ is a constant (as long as $m_2 \ll M_\odot$) $\Longrightarrow$ this is the 3rd law form as Kepler stated it.

7. Kepler's Equation.

a) We have written an equation of $r(\theta)$ (see Eq. X-44). We will now develop an equation for θ, called the **true anomaly**, as a function of time.

b) We will start by setting up the ratio

$$\frac{A}{\tau} = \frac{\pi ab}{\tau} = \frac{dA}{dt} \ . \tag{X-55}$$

i) Let $\theta = 0$ at $t = 0$, where $dA = \frac{1}{2}r^2 d\theta$, then

$$\frac{\pi ab}{\tau}\int_0^t dt' = \int_0^A dA' = \frac{1}{2}\int_0^\theta r^2\,d\theta \ . \tag{X-56}$$

ii) Using Eq. (X-44), we can write r in terms of θ as

$$r = \frac{\alpha}{1 + \varepsilon \cos\theta} \ , \qquad \text{(X-57)}$$

and hence

$$\begin{aligned} \frac{\pi a b}{\tau} t &= \frac{\alpha^2}{2} \int_0^\theta \frac{d\theta}{(1+\varepsilon\cos\theta)^2} \\ &= \frac{\alpha^2}{2(1-\varepsilon^2)} \left[\frac{2}{\sqrt{1-\varepsilon^2}} \tan^{-1}\left(\frac{(1-\varepsilon)\tan(\theta/2)}{\sqrt{1-\varepsilon^2}} \right) \right. \\ &\qquad \left. - \frac{\varepsilon \sin\theta}{1+\varepsilon\cos\theta} \right] \ . \end{aligned}$$

iii) Note that

$$ab = \alpha^2 \left(1 - \varepsilon^2\right)^{-3/2} , \qquad \text{(X-58)}$$

so we can write

$$\begin{aligned} \frac{2\pi t}{\tau} &= 2\tan^{-1}\left[\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \tan\frac{\theta}{2} \right] \\ &\qquad - \frac{\varepsilon\sqrt{1-\varepsilon^2}\,\sin\theta}{1+\varepsilon\cos\theta} \ . \end{aligned} \qquad \text{(X-59)}$$

c) As can be seen, retrieving $\theta(t)$ from Eq. (X-59) will not be easy! Due to this difficulty, we will approach the problem through analytic geometry.

i) We will start by circumscribing the ellipse (representing an orbit) with a circle and define a coordinate system with the origin set at one of the foci of the ellipse as shown in Figure X-5.

ii) The equation for the ellipse is

$$\frac{(x + a\varepsilon)^2}{a^2} + \frac{y^2}{b^2} = 1 \ . \qquad \text{(X-60)}$$

- The circle has radius of 'a.'

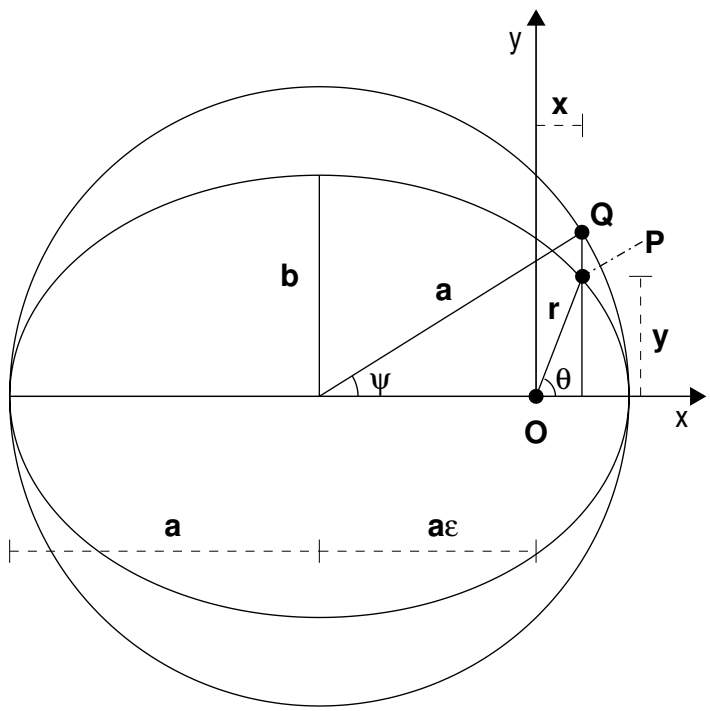


Figure X–5: The geometry used in the derivation of Kepler's Equation. Note that the Sun is located at one of the foci labeled as the origin 'O'.

- Point 'P' lies on the ellipse, 'Q' on the circle.

d) We now define a new angle ψ (called the **eccentric anomaly**) as the angle between the circle's radius vector and the x axis. In relation to Eq. (X-60), the following definitions are made for the trigonometric functions of the eccentric anomaly:

$$\cos\psi \equiv \frac{x + a\varepsilon}{a} \tag{X-61}$$

$$\sin\psi \equiv \frac{y}{b}\,. \tag{X-62}$$

Investigating Figure X-5, it is easy to see from trigonometry how the cosine equation (Eq. X-61) arises. However the sine equation is not as easily justified. Unfortunately, I have been unable to prove Eq. (X-62) with geometry or unable to find a reference that describes the proof for this equation. As such, we will just take it as an assumption.

A little algebra can be used to derive:

$$x = a(\cos\psi - \varepsilon) \tag{X-63}$$

$$y = b\sin\psi = a\sqrt{1-\varepsilon^2}\,\sin\psi\ . \tag{X-64}$$

e) Now looking at the smaller θ triangle in Figure X-5, we use the equations above in the Pythagorean theorem and carry out a bit of algebra to derive:

$$\begin{aligned}
r^2 &= x^2 + y^2 = a^2\,(\cos\psi - \varepsilon)^2 + a^2\,\left(1-\varepsilon^2\right)\sin^2\psi \\
&= a^2\,\left(\cos^2\psi - 2\varepsilon\cos\psi + \varepsilon^2\right) + a^2\,\sin^2\psi - a^2\varepsilon^2\sin^2\psi \\
&= a^2\cos^2\psi - 2a^2\varepsilon\cos\psi + a^2\varepsilon^2 + a^2\sin^2\psi - a^2\varepsilon^2\sin^2\psi \\
&= a^2\,\left(\cos^2\psi + \sin^2\psi\right) + a^2\varepsilon^2\,\left(1-\sin^2\psi\right) - 2a^2\varepsilon\cos\psi \\
&= a^2 + a^2\varepsilon^2\cos^2\psi - 2a^2\varepsilon\cos\psi \\
&= a^2\,\left(1 - 2\varepsilon\cos\psi + \varepsilon^2\cos^2\psi\right) \\
&= a^2(1-\varepsilon\cos\psi)^2\ , \quad \text{or} \\
r &= a(1-\varepsilon\cos\psi)\ .
\end{aligned} \tag{X-65}$$

f) Now rewrite Eq. (X-57) as

$$\varepsilon r\cos\theta = a(1-\varepsilon^2) - r\ , \tag{X-66}$$

followed by adding εr to both sides which gives

$$\varepsilon r(1+\cos\theta) = (1-\varepsilon)[a(1+\varepsilon) - r]\ . \tag{X-67}$$

g) Substitute Eq. (X-65) for the last term in this equation giving

$$\varepsilon r(1+\cos\theta) = (1-\varepsilon)[a(1+\varepsilon) - a(1-\varepsilon\cos\psi)]$$

or

$$r(1+\cos\theta) = a(1-\varepsilon)(1+\cos\psi)\ . \tag{X-68}$$

h) Subtracting εr from both sides of Eq. (X-66) [after using Eq. (X-65), as we did before] results in

$$r(1-\cos\theta) = a(1+\varepsilon)(1-\cos\psi)\ . \tag{X-69}$$

i) Dividing Eq. (X-69) by Eq. (X-68) gives:

$$\frac{1-\cos\theta}{1+\cos\theta} = \frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{1-\cos\psi}{1+\cos\psi} \ . \qquad \text{(X-70)}$$

j) If we make use of the trigonometric identity

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \ ,$$

we can take the square root of Eq. (X-70) to write

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\ \tan\frac{\psi}{2} \ . \qquad \text{(X-71)}$$

k) Now, $\theta(t)$ can be found directly from $\psi(t)$.

i) Differentiating Eq. (X-71) gives

$$d\theta = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\ \frac{\cos^2(\theta/2)}{\cos^2(\psi/2)}\, d\psi \ . \qquad \text{(X-72)}$$

ii) Rewriting Eq. (X-68) gives

$$\begin{aligned} r &= a(1-\varepsilon)\frac{1+\cos\psi}{1+\cos\theta} \\ &= a(1-\varepsilon)\frac{\cos^2(\psi/2)}{\cos^2(\theta/2)} \ . \qquad \text{(X-73)} \end{aligned}$$

l) We will now make use of this expression for r in Eq. (X-56):

$$\frac{\pi ab}{\tau}\, t = \frac{1}{2}\int_0^\theta r^2\, d\theta \ .$$

i) However for this integral, let $r^2 = (r_1)(r_2)$, where r_1 is given by Eq. (X-65) and r_2 is given by Eq. (X-73).

ii) Then, using Eq. (X-72) for $d\theta$ we get

$$r^2\, d\theta = [a(1-\varepsilon\cos\psi)]\left[a(1-\varepsilon)\frac{\cos^2(\psi/2)}{\cos^2(\theta/2)}\right] \cdot$$

$$\left[\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \cdot \frac{\cos^2(\theta/2)}{\cos^2(\psi/2)}\, d\psi\right]$$
$$= \; a^2\sqrt{1-\varepsilon^2}\,(1-\varepsilon\cos\psi)\, d\psi \; .$$

iii) So

$$\frac{\pi ab}{\tau}\, t = \frac{a^2\sqrt{1-\varepsilon^2}}{2} \int_0^{\psi} (1-\varepsilon\cos\psi)\, d\psi \; .$$

iv) Since $b = a\sqrt{1-\varepsilon^2}$, $ab = a^2\sqrt{1-\varepsilon^2}$, and

$$\frac{2\pi t}{\tau} = \psi - \varepsilon\sin\psi \; . \qquad \text{(X-74)}$$

v) Let $M = 2\pi t/\tau =$ **mean anomaly** $\Longrightarrow$ measures angular deviation of a body moving in a circular orbit of period τ (note that $M = 2\pi$ when the mass completes one orbit, $t = \tau$), then

$$\boxed{M = \psi - \varepsilon\sin\psi \; ,} \qquad \text{(X-75)}$$

which is referred to as **Kepler's Equation**.

m) In order to find $\psi(t)$, Kepler's Equation must be inverted by some approximation procedure which usually involves expanding the sine term in a Taylor's series (see E.W. Brown 1931, *Monthly Notices of the Royal Astronomical Society*, **92**, 104). Then, Eq. (X-71) relates ψ and θ and hence the time dependence of the *true* anomaly (*i.e.*, the anomaly for an elliptical orbit) can be found.

8. Following the same geometric arguments that we made to derive Kepler's Equation, we can develop a velocity equation as a function of r. The orbital velocity can be found with

$$v^2 = \dot{x}^2 + \dot{y}^2 \; , \qquad \text{(X-76)}$$

or

$$\begin{aligned} v^2 &= a^2\dot{\psi}^2 \sin^2\psi + a^2(1-\varepsilon^2)\dot{\psi}^2\cos^2\psi \\ &= a^2\dot{\psi}^2(1-\varepsilon^2\cos^2\psi) \ . \end{aligned} \tag{X-77}$$

a) Differentiating Eq. (X-74) gives

$$\frac{2\pi}{\tau} = \dot{\psi}(1-\varepsilon\cos\psi) \ . \tag{X-78}$$

b) Solving this equation for $\dot{\psi}$ and inserting this value in Eq. (X-77) gives

$$\begin{aligned} v^2 &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{1-\varepsilon^2\cos^2\psi}{(1-\varepsilon\cos\psi)^2} \\ &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{1+\varepsilon\cos\psi}{1-\varepsilon\cos\psi} \\ &= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{2-(1-\varepsilon\cos\psi)}{1-\varepsilon\cos\psi} \ . \end{aligned} \tag{X-79}$$

c) Note that

$$\frac{r}{a} = 1-\varepsilon\cos\psi \ ,$$

so we can write

$$v^2 = \left(\frac{2\pi}{\tau}\right)^2 a^3 \left(\frac{2}{r}-\frac{1}{a}\right) \ . \tag{X-80}$$

d) Using Kepler's 3rd Law:

$$v^2 = \frac{Gm_1m_2}{\mu}\left(\frac{2}{r}-\frac{1}{a}\right)$$

or

$$\boxed{v^2 = G\left(m_1+m_2\right)\left(\frac{2}{r}-\frac{1}{a}\right) \ .} \tag{X-81}$$

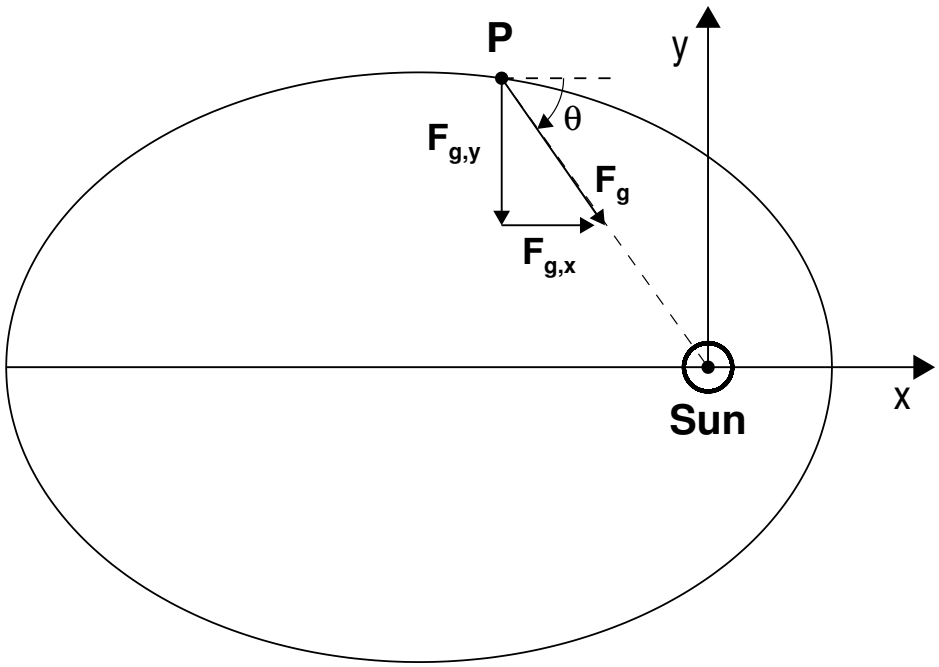


Figure X–6: Coordinate system for describing the motion of a planet (or comet or asteroid) in orbit about the Sun. The Sun is at the origin and the planet is located at coordinate (x, y).

B. Programming Orbital Motion in the Solar System.

1. As we have seen, the analytic solution of planetary motion is extremely complicated. This problem is much easier to solve numerically.

2. Here, we will limit ourselves to motion of a mass ($m_2 = m_p = m$) in orbit about the Sun ($m_1 = M_\odot$). Using these masses in Eq. (XII-6) Newton's law of gravity becomes

$$\vec{F}_g = -\frac{GM_\odot m}{r^2}\,\hat{r}\ , \qquad \text{(X-82)}$$

once again, the negative sign indicates that this is an attractive force.

a) As we did for trajectories (see Page IX-13), we will break this force up into an x component and a y component as shown in Figure X-6. Then, using Newton's 2nd law we

have

$$\frac{d^2x}{dt^2} = \frac{F_{g,x}}{m} \tag{X-83}$$

$$\frac{d^2y}{dt^2} = \frac{F_{g,y}}{m} , \tag{X-84}$$

where $F_{g,x}$ and $F_{g,y}$ are the x and y components of the gravitational force.

b) It is standard convention to define the x axis along the major axis of the ellipse and the y axis parallel to the minor axis with its origin centered at the primary mass location.

c) From Figure X-6 we have

$$F_{g,x} = -\frac{GM_\odot m}{r^2}\cos\theta = -\frac{GM_\odot mx}{r^3} \tag{X-85}$$

and

$$F_{g,y} = -\frac{GM_\odot m}{r^2}\sin\theta = -\frac{GM_\odot my}{r^3} . \tag{X-86}$$

d) As can be seen by the equations above, when x is positive, the force in the x direction will drive the orbiting mass towards the y axis, and $y > 0$, $F_{g,y}$ towards x axis. When $x < 0$, $F_{g,x} > 0$ which will force the mass back towards the y axis with a similar effect for $F_{g,y}$ forcing the mass backs towards the x axis, hence keeping the mass in orbit about the primary mass.

3. We will break the second-order differential equations given in Eqs. (X-85) and (X-86) into a pair of two first-order differential equations:

$$\frac{dv_x}{dt} = -\frac{GM_\odot x}{r^3} \tag{X-87}$$

$$\frac{dx}{dt} = v_x \tag{X-88}$$

$$\frac{dv_y}{dt} = -\frac{GM_\odot y}{r^3} \tag{X-89}$$

$$\frac{dy}{dt} = v_y \ . \tag{X-90}$$

4. Next, we convert these differential equation into difference equations using a forward-difference scheme:

$$v_{x,i+1} = v_{x,i} - \frac{GM_\odot x_i}{r_i^3}\,\Delta t \tag{X-91}$$

$$x_{i+1} = x_i + v_{x,i+1}\,\Delta t \tag{X-92}$$

$$v_{y,i+1} = v_{y,i} - \frac{GM_\odot y_i}{r_i^3}\,\Delta t \tag{X-93}$$

$$y_{i+1} = y_i + v_{y,i+1}\,\Delta t \ . \tag{X-94}$$

5. To carry out an orbit calculation, one must first define the semi-major axis size 'a' and the eccentricity ε of the orbit as input parameters. In addition, the masses of the primary (*e.g.*, the Sun for solar orbits) and of the secondary must be supplied.

6. Finally, we need to set the initial values for x, y, v_x, and v_y, and the initial time interval Δt. This is usually done by starting at the pericenter point with $\theta = \theta_\circ = 0$.

a) Then Eq. (X-49) gives us the value of r:

$$r_\circ = r_{\rm min} = a(1-\varepsilon) \ .$$

b) From this, the starting Cartesian displacements are

$$x_\circ = r_\circ \cos\theta_\circ = r_\circ = r_{\rm min} \tag{X-95}$$

$$y_\circ = r_\circ \sin\theta_\circ = 0 \ . \tag{X-96}$$

c) The velocity of an object in an elliptical orbit is given by Eq. (X-81). For $r = r_{\min}$, we have

$$v_\circ = \sqrt{G\left(M_\odot + m\right)\left(\frac{2}{r_{\min}} - \frac{1}{a}\right)} \ . \tag{X-97}$$

As such, the initial velocities in Cartesian coordinates are given by

$$v_{x,\circ} = 0 \tag{X-98}$$

$$v_{y,\circ} = v_\circ \tag{X-99}$$

since at $\theta = 0$ (*i.e.*, $r = r_{\min}$), the orbital velocity vector points entirely in the $+y$ direction (see Figure IX-10). Note that this will cause the mass to move in the counter-clockwise direction as drawn in Figure X-6.

7. One then numerically solves Eqs. (X-91) through (X-94) using the Euler-Cromer or 4th order Runge-Kutta (RK4) methods. The best results are usually obtained with RK4 with an adaptive grid (*i.e.*, time interval). See a *Numerical Recipes* (by Cambridge Publishing) book for details.