1 Introduction

For this project, you are to solve the set of equations of motion for a system of coupled harmonic oscillators. However, before describing the physics for such a system, let’s start by describing simple harmonic motion (SHM) for a single mass connected to a spring. Such motion results from the famous Hooke’s Law:

\[ \mathbf{F} = -k \mathbf{x}, \]  

for one-dimensional motion. Coupling this with Newton’s second law of motion, \( \ddot{x} = m \ddot{a} = m \dddot{x} \), and dropping the vector notation since the motion is entirely in one dimension, we can write the following differential equations to represent the equations of motion:

\[ \dot{x} = \frac{dx}{dt} = v, \quad \ddot{x} = \frac{dv}{dt} = \frac{1}{m} F. \]  

1.1 Simple Harmonic Motion

For a simple harmonic oscillator (i.e., one mass and one spring) oscillating horizontally along a frictionless surface, the equation of motion for the mass is:

\[ m \ddot{x} = -k x. \]  

The solution to this equation is

\[ x = A \cos(\omega t + \theta), \quad \omega^2 = k/m, \]  

where \( \omega \) is the angular frequency of the oscillation. The two constants (\( A \) and \( \theta \)) depend on the initial values \( x_0 \) and \( \dot{x}_0 \). The period of oscillation for this mass is

\[ \tau = \frac{2\pi}{\omega}. \]  

1.2 Coupled Harmonic Motion

Consider the system of 2 blocks of mass \( m_1 \) and \( m_2 \) connected to each other and immovable walls by 3 springs with spring constants \( k_1, k_2, \) and \( k_3 \). The position of the blocks, relative to the left
Figure 1: A two-mass coupled harmonic oscillator with the origin set at the position of the left wall.

wall, are $x_1$ and $x_2$. The distance between the walls is $L_w$ and the unstretched lengths of the springs are $L_1$, $L_2$, and $L_3$ (note that $L_1 + L_2 + L_3 = L_w$). Assume the blocks are of negligible width.

The equation of motion for block $i$ is

$$\dot{x}_i = \frac{dx_i}{dt} = v_i, \quad \ddot{x}_i = \frac{dv_i}{dt} = \frac{1}{m_i} F_i,$$

where $F_i$ is the net force on block $i$. At the steady state, the velocities, $v_i$, and the net forces, $F_i$, are zero. This is just static equilibrium. When working with coupled oscillators, one must define a frame of reference from which the measurements are made. Here we will choose two different frames of reference which we will label Case A and Case B.

For Case A we define the reference point to be the left wall of the system as shown in Figure 1. For this reference frame the equations of motion become

$$m_1 \ddot{x}_1 = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2)$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1 - L_3) + k_3(L_w - x_2 - L_3).$$

For Case B we choose the equilibrium position of each block as the reference as shown in Figure 2, and write the equations of motion as

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 + x_2)$$

$$m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_1 + x_2).$$

2 Computer Coding Techniques

The technique that should be used for solving the equation of motion for both of these cases involves setting up your equations in matrix format and inverting the matrices, say by Gaussian Elimination
or some other technique. Writing Eqs. (7) and (8) for Case A in matrix form, we get

\[
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
= 
\begin{bmatrix}
-k_1 - k_2 & k_2 \\
k_2 & -k_2 - k_3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

or

\[
F = K_A \cdot x - b .
\]

In matrix form, Case B takes the form of

\[
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
= 
\begin{bmatrix}
-k_1 - k_2 & k_2 \\
k_2 & -k_2 - k_3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

or in shorthand notation

\[
F = K_B \cdot x .
\]

Students who choose to carry out this project are free to choose between either Case A or Case B and write your code to solve either of the equations above. Your code should have the ability to save the output data in an external file which can then be read by a plotting program to display the results (see the Analysis section below). **Honors Section and Graduate Students** will need to write a code that carries out calculations for both cases.

### 3 Analysis

You are to write and run a code that has either of the two model cases built into it (or both for honors and graduate students). For those students who write a code with both cases in it, the input to your code should be set up to allow the user to select between one of the 2 cases. The spring constants, masses, and spring lengths also should be available as input. You are to run your code with the inputs of Table 1.
Table 1: Input Parameters for Harmonic Oscillators

<table>
<thead>
<tr>
<th>Model</th>
<th>Spring Constant (newtons/m)</th>
<th>Spring Length (meters)</th>
<th>Mass (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>((k_1, k_2, k_3))</td>
<td>((L_1, L_2, L_3))</td>
<td>((m_1, m_2))</td>
</tr>
<tr>
<td>a</td>
<td>1, 2, 3</td>
<td>0.1, 0.1, 0.1</td>
<td>0.01, 0.01</td>
</tr>
<tr>
<td>b</td>
<td>1, 1, 1</td>
<td>0.1, 0.2, 0.3</td>
<td>0.05, 0.03</td>
</tr>
<tr>
<td>c</td>
<td>1, 2, 3</td>
<td>0.3, 0.2, 0.1</td>
<td>0.01, 0.02</td>
</tr>
</tbody>
</table>

Your code should produce an output table that contains header information concerning the input listed in Table 1 followed by the mass positions as a function of time. Time intervals should be small enough to allow for the proper solution to the above equations. With IDL (or some other plotting software), develop a code to read in your output and plot \(x_i(t)\). The analysis and final report should address the following:

- A sample of the printed output and graphs displaying this data.
- A discussion of the equations you integrated.
- A discussion of your integration and solution scheme.
- A discussion of the numerical accuracy of your code and other numerical tests, problems, or insights encountered along the way.

The final computer project report must be written in \LaTeX! I will have a \LaTeX template file called `template.tex` on the course web page. Happy computing!

**References**