Physics 4617/5617: Quantum Physics Course Lecture Notes

Dr. Donald G. Luttermoser
East Tennessee State University

Edition 5.1

Abstract

These class notes are designed for use of the instructor and students of the course Physics 4617/5617: Quantum Physics. This edition was last modified for the Fall 2006 semester.

II. The Wave Function

A. The Schrödinger Equation.

1. As mentioned in §I of the notes, quantum mechanics approaches the trajectory problem of Newtonian mechanics quite differently. On a microscopic level, particles do not follow trajectories, but instead are characterized by their **wave function**, $\Psi(x,t)$, where x is the 1-dimensional position (we will worry about 3-dimensions later) of the wave function at time t.

a) The wave function is determined from **Schrödinger's Equation**:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \ . \qquad \text{(II-1)}$$

i) Here, $i = \sqrt{-1}$ and

$$\hbar = \frac{h}{2\pi} = 1.054573 \times 10^{-34} \text{ J s} \ . \qquad \text{(II-2)}$$

ii) Whereas Newton's Second Law, $F = ma$, is the most important equation in all of classical physics, Eq. (II-1) is the most important equation in all of quantum physics.

b) Given suitable initial conditions [typically, $\Psi(x,0)$], the Schrödinger equation determines $\Psi(x,t)$ for all future times, just as, in classical mechanics, Newton's Second Law determines $x(t)$ for all future times.

2. What exactly is the *wave function*, and what does it do for you once you got it?

a) Whereas a particle is localized at a point in classical mechanics, a wave function is spread out in space $\Longrightarrow$ it is a function of x for any given time t.

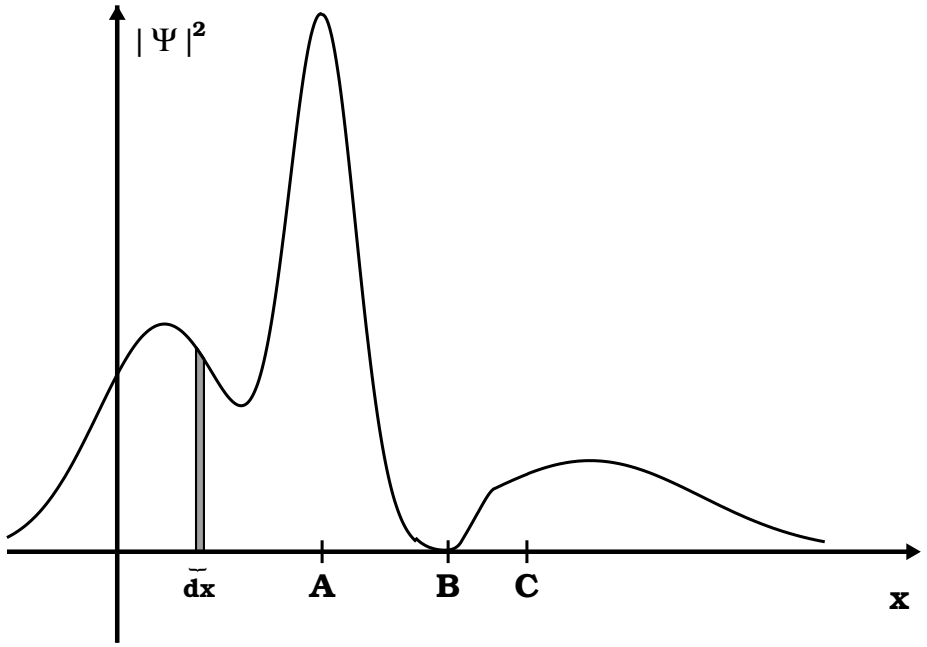

Figure II–1: A hypothetical wave function. The particle would be relatively likely to be found near A, and unlikely to be found near B. The shaded area represents the probability of finding the particle in the range dx.

b) Born came up with a **statistical interpretation** of the wave function, which says that $|\Psi(x,t)|^2$ gives the *probability* of finding the particle at point x, at time t, or more precisely,

$$|\Psi(x,t)|^2\, dx = \left\{\begin{array}{l}\text{probability of finding the particle} \\ \text{between } x \text{ and } (x+dx) \text{ at time } t.\end{array}\right\} \tag{II-3}$$

c) The wave function itself is complex, but $|\Psi|^2 = \Psi^*\Psi$ (where Ψ^* is the complex conjugate of Ψ) is real and non-negative — as a probability must be.

d) For the hypothetical wave function in Figure (II-1), you would be quite likely to find the particle in the vicinity of point A, and relatively unlikely to find it near point B.

3. From the concept of the wave function, it becomes easier to see how the Heisenberg Uncertainty Principle arises in nature. The wave function will not allow you to predict with certainty the outcome of a simple experiment to measure a particle's position — all quantum mechanics has to offer is *statistical* information about the *possible* results.

B. Philosophical Interpretations.

1. The **Realist** Position:

a) We view the microscopic world as probabilistic due to the fact that quantum mechanics is an *incomplete* theory.

b) The particle really was at a specific position (say point C in Figure II-1), yet quantum mechanics was unable to tell us so.

c) To the realist, indeterminacy is not a fact of nature, but a reflection of our ignorance.

d) If this scenario is, in fact, the correct one, then Ψ is not the whole story — some additional information (known as a **hidden variable**) is needed to provide a complete description of the particle.

2. The **orthodox** position $\Longrightarrow$ the **Copenhagen interpretation**:

a) The particle isn't really anywhere in space. The act of the measurement forces the particle to *take a stand* — though how and why we dare not ask!

b) Observations not only *disturb* what is to be measured, they *produce* it.

c) Bohr and his followers put forward this interpretation of quantum mechanics.

d) It is the most widely accepted position of the interpretation of quantum mechanics in physics.

3. The **agnostic** position:

a) Refuse to answer! What sense can there be in making assertions about the status of a particle *before* a measurement, when the only way of knowing whether you were right is precisely to conduct the measurement, in which case what you get is no longer *before the measurement.*

b) This has been used as a *fall-back* position used by many physicists if one is unable to convince another of the orthodox position.

4. In 1964, John Bell astonished the physics community by showing that it makes an *observable* difference if the particle had a precise (although unknown) position prior to its measurement.

a) This discovery effectively eliminated the *realist* position.

b) **Bell's Theorem** showed that the *orthodox* position is the correct interpretation of quantum mechanics by proving that any local *hidden variable* theory is incompatible with quantum mechanics (see Bell, J.S. 1964, *Physics*, **1**, 195).

c) We won't get into the details of Bell's Theorem at this point in time. Suffice it to say that a particle does not have a precise position prior to the measurement, any more than ripples in a pond do $\Longrightarrow$ it is the measurement process that insists upon one particular number, and thereby in a sense *creates* the specific result.

5. The act of the measurement **collapses** the wave function to a delta function (*e.g.*, a sharp peak) at some position — Ψ soon spreads out again after the measurement in accordance to the Schrödinger equation.

C. Probability and Normalization.

1. Because of the statistical interpretation, **probability**, P, plays a central role in quantum mechanics.

a) A probability value is the likelihood of a sample point occurring in a given *distribution* of points, where a *sample point* is defined here as a possible outcome of an experiment.

b) A distribution of points can either be a set of *discrete* values or a *continuous* set of values.

2. **Discrete Measurements**. Below are a few definitions concerning discrete measurements.

a) The total number of particles (or measurements) in a system is

$$N = \sum_{j=1}^{\infty} N(j) \ , \tag{II-4}$$

where $N(j)$ is the number of particles (or measurements) in state j.

b) The probability of a particle being in state j is

$$P(j) = \frac{N(j)}{N} \ , \tag{II-5}$$

whereas the sum of all the probabilities is

$$\sum_{j=1}^{\infty} P(j) = 1 \ . \tag{II-6}$$

c) The **average** (or **mean**) of a particle being found in state j is

$$\langle j \rangle = \frac{\sum jN(j)}{N} = \sum_{j=1}^{\infty} jP(j) \ , \qquad \text{(II-7)}$$

whereas the **most probable** value of j is $\mathbf{MAX}(N(j))$.

d) The average of the *square* of a particle being found in state j is

$$\langle j^2 \rangle = \frac{\sum j^2 N(j)}{N} = \sum_{j=1}^{\infty} j^2 P(j) \ . \qquad \text{(II-8)}$$

e) In general, the average value of some *function* of j is given by

$$\langle f(j) \rangle = \sum_{j=1}^{\infty} f(j) P(j) \ . \qquad \text{(II-9)}$$

f) The numerical measure of the amount of *spread* in a distribution with respect to the average is

$$\Delta j = j - \langle j \rangle \ . \qquad \text{(II-10)}$$

g) The **variance** of the distribution is defined as

$$\sigma^2 \equiv \langle (\Delta j)^2 \rangle \ , \qquad \text{(II-11)}$$

where σ is called the **standard deviation** of the measurement.

h) It can be proven that

$$\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2 \ . \qquad \text{(II-12)}$$

Note that

$$\langle j^2 \rangle \geq \langle j \rangle^2 \ . \qquad \text{(II-13)}$$

3. Continuous Measurements. For continuous distributions of data or measurements, it is often convenient to define the **probability density**, $\rho(x)$, as

$$\left\{\begin{array}{l}\text{probability that a random measurement}\\ \text{lies between } x \text{ and } (x+dx)\end{array}\right\} = \rho(x)\,dx\ . \quad \text{(II-14)}$$

a) For these continuous distributions, the probability that x lies between a and b (a *finite* interval) is given by

$$P_{ab} = \int_a^b \rho(x)\,dx\ . \quad \text{(II-15)}$$

b) The following equations are also valid:

$$\int_{-\infty}^{+\infty} \rho(x)\,dx = 1\ , \quad \text{(II-16)}$$

$$\langle x\rangle = \int_{-\infty}^{+\infty} x\rho(x)\,dx \quad \text{(II-17)}$$

$$\langle f(x)\rangle = \int_{-\infty}^{+\infty} f(x)\rho(x)\,dx \quad \text{(II-18)}$$

$$\sigma^2 \equiv \langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2\ . \quad \text{(II-19)}$$

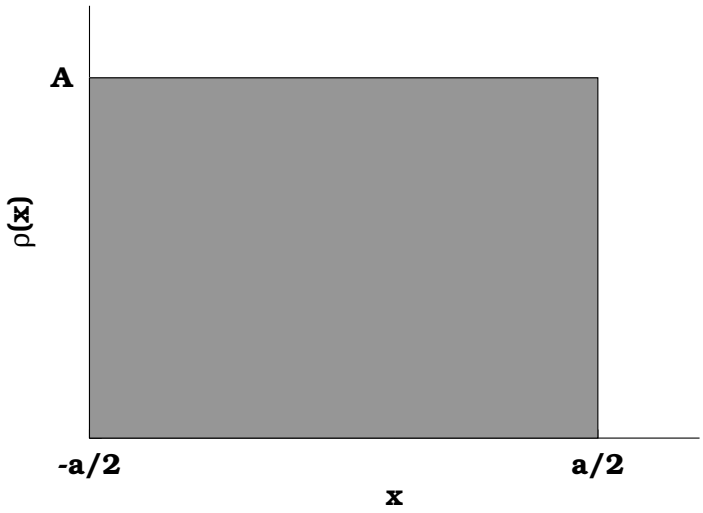

Figure II–2: Probability density function for a uniform distribution.

4. A **uniform distribution** (as shown in Figure II-2) is used when there is an equal probability that all of the possible measured values will occur. If we set the probability density to a constant value of $\rho(x) = A$ between the limits of $-a/2$ to $a/2$ (hence the total width of the distribution is a), the probability integral can then be used to find the amplitude A with respect to the width a:

$$P = \int_{-a/2}^{a/2} A\, dx = Aa = 1 \ , \tag{II-20}$$

hence the amplitude must be

$$A = \frac{1}{a} \tag{II-21}$$

for this probability function to be **normalizable**.

5. Meanwhile, a **normal distribution** (*i.e.*, a **Gaussian distribution**) can be used to describe the distribution of random events or observations. This distribution function is shown in Figure (II-3) and described by the equation

$$\rho(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-(x-\mu)^2/2\sigma^2} \ . \tag{II-22}$$

a) This distribution is centered around the *mean*, μ.

b) Here σ is the standard deviation of the distribution.

c) The *full-width-at-half-maximum* (FWHM), Γ, is related to the standard deviation by

$$\Gamma = 2.354\,\sigma \ . \tag{II-23}$$

d) The *probable error* (P.E.) of a normalized distribution is defined to be the absolute value of the deviation $|x - \mu|$ such that the probability for the deviation of any random observation $|x_i - \mu|$ to be less is equal to $1/2 \Longrightarrow$ that is,

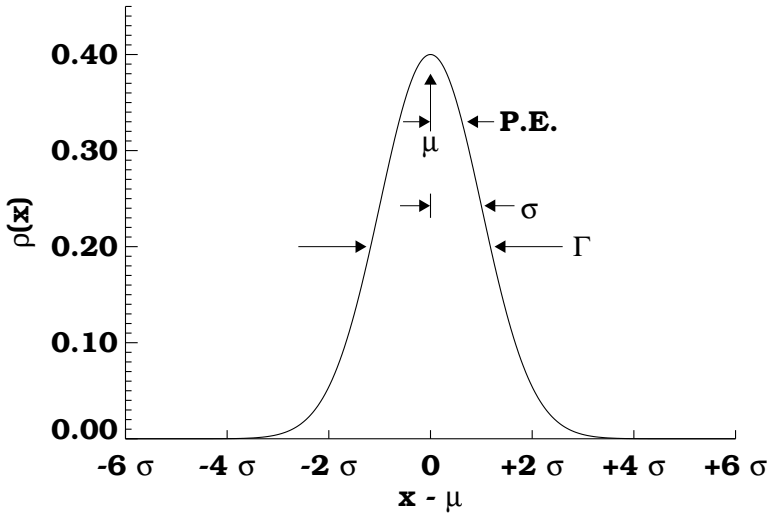

Figure II–3: Probability density function for a normal distribution.

half of the observations of an event are expected to fall within the boundaries denoted by $\mu \pm$ P.E.

$$\text{P.E.} \ = 0.6745\,\sigma = 0.2865\,\Gamma \ . \qquad \text{(II-24)}$$

6. The statistical interpretation of the wave function (*e.g.*, Eq. II-3), says that $|\Psi(x,t)|^2$ is the probability density for finding the particle at point x, at time t. This dictates the following normalization:

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx = 1 \ . \qquad \text{(II-25)}$$

Without this, the statistical interpretation would be nonsense.

a) But is this normalization consistent with Schrödinger's equation (*i.e.*, Eq. II-1)? That is, is it really valid for all time?

b) Let us take the time derivative of the LHS of Eq. (II-25), then

$$\frac{d}{dt} \int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx = \int_{-\infty}^{+\infty} \frac{\partial}{\partial t} |\Psi(x,t)|^2 \, dx \ . \qquad \text{(II-26)}$$

c) Note that the integral is a function only of t, since the x term(s) will disappear when the limits are applied. As such, a total derivative (d/dt) is taken for the solution to the integral on the LHS of Eq. (II-26), but the integrand (*i.e.*, the function inside the integral) is a function of x as well as t, so a partial derivative ($\partial/\partial t$) must be used when the derivative is taken inside the integral on the RHS of Eq. (II-26).

d) By the product rule, we get

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \Psi^*\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi \ . \qquad \text{(II-27)}$$

e) Schrödinger's equation says that

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi \ , \qquad \text{(II-28)}$$

and hence also using Schrödinger's equation for the complex conjugate of Eq. (II-27) gives

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^* \ . \qquad \text{(II-29)}$$

f) Using these equations in Eq. (II-27) gives

$$\begin{aligned} \frac{\partial}{\partial t}|\Psi|^2 &= \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\Psi\right) \\ &= \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right] \ . \end{aligned} \qquad \text{(II-30)}$$

g) The integral in Eq. (II-25) can now be evaluated explicitly:

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{+\infty} \ . \qquad \text{(II-31)}$$

h) $\Psi(x,t)$ must go to zero as $x \to \pm\infty$, otherwise the wave function would not be normalizable. It follows that

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = 0 \ , \qquad \text{(II-32)}$$

and hence that the integral on the left is *constant* (independent of time) $\Longrightarrow$ if Ψ is normalized at $t = 0$, it *stays* normalized for all future time. QED

Example II–1. Consider the wave function

$$\Psi(x,t) = Ae^{-\beta^2 x^2/2} e^{-iEt/\hbar} \ .$$

(a) Find the value of the constant A. (b) If $\Psi = 0.5$ at $x = 0$ and $t = 0$, what is the value of β?

Solution (a):

The total probability must equal unity, as such

$$\int_{-\infty}^{+\infty} |\Psi|^2 \, dx = \int_{-\infty}^{+\infty} \Psi^* \Psi \, dx = 1 \ .$$

The complex conjugate of our wave function is

$$\Psi^*(x,t) = Ae^{-\beta^2 x^2/2} e^{iEt/\hbar} \ ,$$

so our normalization equation becomes

$$\int_{-\infty}^{+\infty} \Psi^*(x,t)\Psi(x,t) \, dx = A^2 \int_{-\infty}^{+\infty} e^{-\beta^2 x^2} \, dx = A^2 \sqrt{\frac{\pi}{\beta^2}} = 1 \ ,$$

or

$$A = \left(\frac{\beta^2}{\pi}\right)^{1/4} \ .$$

So our wave function is

$$\Psi(x,t) = \left(\frac{\beta^2}{\pi}\right)^{1/4} e^{-\beta^2 x^2/2} e^{-iEt/\hbar} \ .$$

Solution (b):

Set $\Psi(0,0) = 0.5$ in the equation above and solve for β:

$$\Psi(0,0) = \left(\frac{\beta^2}{\pi}\right)^{1/4} e^0 e^0 = \frac{1}{2} \ ,$$

or

$$\frac{\beta^2}{\pi} = \frac{1}{16}, \qquad \beta^2 = \frac{\pi}{16}, \qquad \beta = \frac{\sqrt{\pi}}{4} \; .$$

D. Momentum

1. For a particle in state Ψ, the expectation value of x is

$$\langle x \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t)\, x\, \Psi(x,t)\, dx \; . \tag{II-33}$$

The expectation value is the average of repeated measurements on an ensemble of identically prepared systems, not the average of repeated measurements on one and the same system.

2. The rate of change of this expectation value is

$$\frac{d\langle x \rangle}{dt} = \frac{d}{dt} \int_{-\infty}^{+\infty} \Psi^*\, x\, \Psi\, dx = \int_{-\infty}^{+\infty} \left(\Psi^*\, x\, \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\, x\, \Psi \right) dx \; . \tag{II-34}$$

a) Substituting for $\partial\Psi/\partial t$ and $\partial\Psi^*/\partial t$ from the Schrödinger equation (*e.g.*, Eq. II-1) gives

$$\begin{aligned} \frac{d\langle x \rangle}{dt} \;\; = \;\; & -\frac{i}{\hbar} \int_{-\infty}^{+\infty} \left[\Psi^*\, x \left(-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi \right) \right. \\ & \left. - \left(-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + V\Psi^* \right) x\, \Psi \right] dx \; . \end{aligned} \tag{II-35}$$

b) The terms involving V cancel out, and we have

$$\frac{d\langle x \rangle}{dt} = \frac{i\hbar}{2m} \int_{-\infty}^{+\infty} \left(\Psi^*\, x\, \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2}\, x\, \Psi \right) dx \; . \tag{II-36}$$

c) Separate out the second term of the integrand, and integrate by parts as follows:

$$\int_{-\infty}^{\infty} \underbrace{x\Psi}_{u} \underbrace{\frac{\partial^2 \Psi^*}{\partial x^2}\, dx}_{dv} = \left[\underbrace{x\Psi}_{u} \underbrace{\frac{\partial \Psi^*}{\partial x}}_{v} \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \underbrace{\frac{\partial \Psi^*}{\partial x}}_{v} \underbrace{\frac{\partial (x\,\Psi)}{\partial x}\, dx}_{du} \; . \tag{II-37}$$

d) Since Ψ must be in the form of a group of limited spatial extent in order that the uncertainty in the x coordinate be relatively small, both the wave function and its derivatives must go to zero faster than $x \to \pm\infty$. Consequently the integrand term is equal to zero, and we have

$$\int_{-\infty}^{\infty} x\Psi \frac{\partial^2 \Psi^*}{\partial x^2}\, dx = -\int_{-\infty}^{+\infty} \underbrace{\frac{\partial (x\,\Psi)}{\partial x}}_{u} \underbrace{\frac{\partial \Psi^*}{\partial x}\, dx}_{dv} \ . \qquad \text{(II-38)}$$

e) Integrating by parts again,

$$\int_{-\infty}^{+\infty} \frac{\partial (x\,\Psi)}{\partial x} \frac{\partial \Psi^*}{\partial x}\, dx = -\left[\frac{\partial (x\,\Psi)}{\partial x}\Psi^*\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \Psi^* \frac{\partial^2 (x\Psi)}{\partial x^2}\, dx \ . \qquad \text{(II-39)}$$

Reducing this again gives the following result for Eq. (II-38):

$$\int_{-\infty}^{\infty} x\Psi \frac{\partial^2 \Psi^*}{\partial x^2}\, dx = \int_{-\infty}^{+\infty} \Psi^* \frac{\partial^2 (x\Psi)}{\partial x^2}\, dx \ . \qquad \text{(II-40)}$$

f) Putting this back into Eq. (II-36), we have

$$\frac{d\langle x\rangle}{dt} = \frac{i\hbar}{2m} \int_{-\infty}^{+\infty} \Psi^* \left[x\, \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 (x\Psi)}{\partial x^2} \right] dx \ . \qquad \text{(II-41)}$$

Consider the bracket in the integrand, it can be written

$$\begin{aligned} x\, \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 (x\,\Psi)}{\partial x^2} &= x\, \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial}{\partial x}\left(x\, \frac{\partial \Psi}{\partial x} + \Psi \right) \\ &= x\, \frac{\partial^2 \Psi}{\partial x^2} - x\, \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi}{\partial x} \\ &= -2 \frac{\partial \Psi}{\partial x} \ . \end{aligned} \qquad \text{(II-42)}$$

g) Consequently,

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{m} \int_{-\infty}^{+\infty} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) dx \ . \qquad \text{(II-43)}$$

h) At this point, we will postulate that the *expectation value of the velocity of the particle is equal to the time derivative of the expectation value of the position of the particle*:

$$\langle v \rangle = \frac{d\langle x \rangle}{dt} \ . \tag{II-44}$$

i) Eq. (II-43) tells us, then, how to calculate $\langle v \rangle$ directly from Ψ.

3. Actually, it is customary to work with momentum ($p = mv$), rather than velocity:

$$\langle p \rangle \equiv m\frac{d\langle x \rangle}{dt} = -i\hbar \int_{-\infty}^{+\infty} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) dx \ . \tag{II-45}$$

a) Let's write expressions for $\langle x \rangle$ and $\langle p \rangle$ in a more suggestive way

$$\langle x \rangle = \int_{-\infty}^{+\infty} \Psi^*(x) \Psi \, dx \tag{II-46}$$

$$\langle p \rangle = \int_{-\infty}^{+\infty} \Psi^* \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi \, dx \ . \tag{II-47}$$

b) We say that the **operator** x *represents* position, and the operator $(\hbar/i)(\partial/\partial x)$ *represents* momentum, in quantum mechanics $\Longrightarrow$ to calculate expectation values, we *sandwich* the appropriate operator between Ψ^* and Ψ and integrate.

4. All such dynamic variables can be written in terms of position and momentum. **Kinetic energy** is

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{-\hbar^2}{2m} \int_{-\infty}^{+\infty} \Psi^* \left(\frac{\partial^2}{\partial x^2} \right) \Psi \, dx \ . \tag{II-48}$$

E. The Uncertainty Principle

1. The **uncertainty** in the measurement of an event is nothing more than the standard deviation $\sigma = \sqrt{\sigma^2}$ of the measurement (*e.g.*, Eq. II-19).

2. Hence, the **Heisenberg Uncertainty Principle** can be rewritten in the form

$$\sigma_x \, \sigma_p \geq \frac{\hbar}{2} \; , \tag{II-49}$$

where σ_x is the standard deviation in the x-position and σ_p is the standard deviation in the corresponding momentum.

3. We shall see from this point forward, that wave functions that describe real particles always obey Eq. (II-49). We shall prove this relation in §IV of the notes.

Example II–2. Consider the wave function of Example II-1:

$$\Psi(x,t) = \left(\frac{\beta^2}{\pi}\right)^{1/4} e^{-(\beta^2 x^2/2 + iEt/\hbar)} \; .$$

Show that this wave function satisfies the Heisenberg Uncertainty Relationship.

Solution:

First, calculate the various expectation values.

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{+\infty} \sqrt{\frac{\beta^2}{\pi}} \, e^{-(\beta^2 x^2/2 - iEt/\hbar)} \, x \, e^{-(\beta^2 x^2/2 + iEt/\hbar)} \, dx \\ &= \sqrt{\frac{\beta^2}{\pi}} \int_{-\infty}^{+\infty} x \, e^{-\beta^2 x^2} \, dx = 0 \; , \end{aligned}$$

since we are integrating an odd function over an even interval.

$$\begin{aligned} \langle x^2 \rangle &= \int_{-\infty}^{+\infty} \sqrt{\frac{\beta^2}{\pi}} \, e^{-(\beta^2 x^2/2 - iEt/\hbar)} \, x^2 \, e^{-(\beta^2 x^2/2 + iEt/\hbar)} \, dx \\ &= \sqrt{\frac{\beta^2}{\pi}} \int_{-\infty}^{+\infty} x^2 \, e^{-\beta^2 x^2} \, dx = 2\sqrt{\frac{\beta^2}{\pi}} \int_{0}^{+\infty} x^2 \, e^{-\beta^2 x^2} \, dx \\ &= 2\sqrt{\frac{\beta^2}{\pi}} \, \frac{1}{4\beta^2} \sqrt{\frac{\pi}{\beta^2}} = \frac{1}{2\beta^2} \; . \end{aligned}$$

$$
\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{+\infty} \sqrt{\frac{\beta^2}{\pi}}\, e^{-(\beta^2 x^2/2 - iEt/\hbar)} \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) e^{-(\beta^2 x^2/2 + iEt/\hbar)}\, dx \\
&= \sqrt{\frac{\beta^2}{\pi}}\,\frac{\hbar}{i} \int_{-\infty}^{+\infty} e^{-\beta^2 x^2/2}\,(-\beta^2 x)\, e^{-\beta^2 x^2/2}\, dx \\
&= -\sqrt{\frac{\beta^2}{\pi}}\,\frac{\hbar}{i}\,\beta^2 \int_{-\infty}^{+\infty} x\, e^{-\beta^2 x^2}\, dx = 0 \ ,
\end{aligned}
$$

again, since we are integrating and odd function over an even interval.

$$
\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{+\infty} \sqrt{\frac{\beta^2}{\pi}}\, e^{-(\beta^2 x^2/2 - iEt/\hbar)} \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 e^{-(\beta^2 x^2/2 + iEt/\hbar)}\, dx \\
&= -\sqrt{\frac{\beta^2}{\pi}}\,\hbar^2 \int_{-\infty}^{+\infty} e^{-\beta^2 x^2/2}\,\frac{\partial^2}{\partial x^2}\, e^{-\beta^2 x^2/2}\, dx \\
&= -\sqrt{\frac{\beta^2}{\pi}}\,\hbar^2 \int_{-\infty}^{+\infty} e^{-\beta^2 x^2/2}\,\frac{\partial}{\partial x}\left(-\beta^2 x\, e^{-\beta^2 x^2/2}\right) dx \\
&= \sqrt{\frac{\beta^2}{\pi}}\,\hbar^2\,\beta^2 \int_{-\infty}^{+\infty} e^{-\beta^2 x^2/2}\left[e^{-\beta^2 x^2/2} + x\,\left(-\beta^2 x\right)\, e^{-\beta^2 x^2/2}\right] dx \\
&= \sqrt{\frac{\beta^2}{\pi}}\,\hbar^2\,\beta^2 \int_{-\infty}^{+\infty} e^{-\beta^2 x^2}\left(1 - \beta^2 x^2\right) dx \\
&= \sqrt{\frac{\beta^2}{\pi}}\,\hbar^2\,\beta^2 \int_{-\infty}^{+\infty} \left(e^{-\beta^2 x^2} - \beta^2 x^2\, e^{-\beta^2 x^2}\right) dx \\
&= \sqrt{\frac{\beta^2}{\pi}}\,\hbar^2\,\beta^2 \left\{\int_{-\infty}^{+\infty} e^{-\beta^2 x^2}\, dx - \int_{-\infty}^{+\infty} \beta^2 x^2\, e^{-\beta^2 x^2}\, dx\right\} \\
&= \sqrt{\frac{\beta^2}{\pi}}\,\hbar^2\,\beta^2 \left\{2\int_{0}^{+\infty} e^{-\beta^2 x^2}\, dx - 2\beta^2 \int_{0}^{+\infty} x^2\, e^{-\beta^2 x^2}\, dx\right\} \\
&= \sqrt{\frac{\beta^2}{\pi}}\,\hbar^2\,\beta^2 \left\{2\cdot\frac{1}{2\beta}\sqrt{\pi} - 2\beta^2\cdot\frac{1}{4\beta^2}\sqrt{\pi}\beta^2\right\} \\
&= \sqrt{\frac{\beta^2}{\pi}}\,\hbar^2\,\beta^2 \left\{\sqrt{\frac{\pi}{\beta^2}} - \frac{1}{2}\sqrt{\frac{\pi}{\beta^2}}\right\} \\
&= \sqrt{\frac{\beta^2}{\pi}}\,\hbar^2\,\beta^2 \left\{\frac{1}{2}\sqrt{\frac{\pi}{\beta^2}}\right\} = \frac{1}{2}\,\hbar^2\,\beta^2 \ .
\end{aligned}
$$

Finally, using these values in the definition of the standard deviations:

$$\begin{aligned}\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\ &= \sqrt{\frac{1}{2\beta^2} - 0} = \sqrt{\frac{1}{2\beta^2}} = \frac{1}{\sqrt{2}\beta} = \frac{\sqrt{2}}{2\beta}\end{aligned}$$

and

$$\begin{aligned}\sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\ &= \sqrt{\frac{1}{2}\hbar^2\,\beta^2 - 0} = \frac{1}{\sqrt{2}}\hbar\,\beta = \frac{\sqrt{2}\,\hbar\,\beta}{2}\ .\end{aligned}$$

So

$$\sigma_x\,\sigma_p = \frac{\sqrt{2}}{2\beta}\cdot\frac{\sqrt{2}\,\hbar\,\beta}{2} = \frac{2\hbar\,\beta}{4\beta} = \frac{1}{2}\hbar\ .\quad\surd$$
