MATRIX PROPERTIES OF MAGIC SQUARES

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Chapter 1
Introduction to Magic Squares

1

Historical Background

For many centuries numbers have been considered by some peoples to be endowed with various magic powers. Certain numbers were considered to have special properties. The number four, for example, often represented the earth, since the earth was considered to have four corners. Seven was often considered a lucky number, and thirteen was an unlucky number.

One such example of "magic" in numbers is the concept of a magic square. Magic squares first appeared in recorded history in ancient China. The story is told that around 2200 B. C. the emperor Yu observed a divine tortoise crawling out of the Yellow River. On the turtle's back was a three-by-three array of numbers, here arranged in matrix form:

$$\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}.$$

This square became known as the *lo-shu magic square*. One can see that the sum of the numbers in any row or column, or the diagonals formed by {4, 5, 6} or {8, 5, 2}, is 15. This story is recorded in the *I-king*, or *Book of Permutations* (5, p. 179). Mystical significance was attributed to this magic square. Even numbers were found in the corners and were thought to "symbolize the female-passive or *yin* and odd numbers the male-active or *yang*" (1, p. 42). The 5 in the center represented earth, surrounded by four major "elements" of metal (represented by

4 and 9), fire (2 and 7), water (1 and 6), and wood (3 and 8). All four elements contained both *yang* and *yin*, male and female (1, p. 42).

Later, magic squares appeared in India, then were known to the Arabs, who introduced them to the West. More research on the topic was done during the Renaissance by the mathematician Cornelius Agrippa (1486 – 1535) (1, p. 43), who constructed magic squares of orders 3 through 9 to represent various planets, the sun, and the moon (2, p. 194).

Another famous example of a magic square appeared in Albrecht Dürer's engraving *Melancholia, or the Genius of the Industrial Science of Mathematics*. On a wall behind an angel pondering the universe is the following figure:

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}.$$

The four-by-four magic square in this painting has 34 as the sum of each column, row, and corner-to-corner diagonal. The sum 34 can also be found in the four center squares, the four corner squares, the two squares in the middle of the top and bottom row, and the two entries in the middle of the left and right columns. Additionally, the date of the painting, 1514, appears in entries a_{42} and a_{43}, the bottom center squares (5, pp. 211-12; 17, pp. 40-42).

Magic squares were seen as having marvelous powers. They were carved onto amulets and silver tablets for decoration and protection against the plague in the sixteenth and seventeenth centuries. Certain fifth-order magic squares, called pandiagonal magic squares, were seen by Medieval Moslems as a special way of signifying God, especially if the center number was 1 (1, pp. 43, 48). One such magic square, constructed using the uniform step method outlined later, is

$$\begin{bmatrix} 25 & 6 & 17 & 3 & 14 \\ 2 & 13 & 24 & 10 & 16 \\ 9 & 20 & 1 & 12 & 23 \\ 11 & 22 & 8 & 19 & 5 \\ 18 & 4 & 15 & 21 & 7 \end{bmatrix} .$$

Very little in the literature concerning magic squares is said of any practical uses or applications for magic squares. For the most part, they may be thought of as interesting diversions. A third-order magic square is used on a shuffleboard court on cruise ships as an aid in keeping scores (2, p. 196). In a more serious vein, magic squares (or latin squares in general, which are defined below) "are an essential feature in statistical investigations of many kinds" (11, p. 16). They also possess an interesting array of mathematical properties.

Definitions and Notation

In addition to the mystic properties attributed to them by people in former times, magic squares possess a wide variety of interesting mathematical properties. Two apparent properties, relating to dot products of rows or columns and eigenvalues (developed in Chapters 3 and 4), are not mentioned in any of the literature reviewed. In order to develop these properties, certain definitions must be presented first.

A *magic square of order n* is a square matrix or array of n^2 numbers such that the sum of the elements of each row and column, as well as the main diagonal and main backdiagonal, is the same number, called the *magic constant* (or *magic sum,* or *line-sum*), sometimes denoted by $\sigma(M)$. Generally, the entries are thought of as the natural numbers 1, 2, ..., n^2, where each number is used exactly once; such magic squares will be referred to here as *normal magic squares* (5, p. 179), although they are sometimes also called *classical magic squares* (19, p. 109).

As in most common discussions of matrices, the *main diagonal* consists of the entries a_{11}, a_{22}, a_{33}, . . . , a_{nn} (or the entries from "northwest" to "southeast"). By *main backdiagonal* is meant the entries a_{1n}, $a_{2(n\text{-}1)}$, $a_{3(n\text{-}2)}$, . . . , $a_{(n\text{-}1)2}$, a_{n1}. (That is, the main backdiagonal is the entries on the diagonal from "northeast" to "southwest.")

The *trace* of the magic square M, represented by tr(M), is the sum of the elements on the main diagonal. The *backtrace of M*, represented by btr(M), is the sum of the elements on the main backdiagonal.

The set of all magic squares of order n may be represented by MS(n). The set of all magic squares of order n whose magic constant is m will be denoted mMS(n) (19, p. 109).

An arbitrary row of a magic square will be denoted by R with a subscript, such as R_i. Similarly, an arbitrary column of the magic square will be represented by a subscripted C, such as C_p. The symbol $\mathbb{N}$ will have the usual meaning of representing the set of natural numbers.

The *dot product*, or *inner product*, of two rows (columns) p and q is obtained by multiplying corresponding elements of p and q and summing the results. That is, for square A, the dot product of rows R_p and R_q is given by $R_p \cdot R_q = \sum_{i=1}^{n} a_{pi}a_{qi}$, and the dot product of columns C_p and C_q is given by $C_p \cdot C_q = \sum_{i=1}^{n} a_{ip}a_{iq}$.

Special Kinds of Magic Squares

Certain kinds of magic squares have been given more narrow definitions based on the kinds of additional properties they possess. Listed below are some of the more common ones. Not all of these will be discussed later. There are still other kinds of magic squares besides these, but the ones listed here are among those more commonly mentioned.

A *semimagic square* is an $n \times n$ matrix such that the sum of the elements on each row and column is equal. Nothing is required of the diagonals. An example of a semimagic square of order 5 is

$$\begin{bmatrix} 1 & 17 & 8 & 24 & 15 \\ 7 & 23 & 14 & 5 & 16 \\ 13 & 4 & 20 & 6 & 22 \\ 19 & 10 & 21 & 12 & 3 \\ 25 & 11 & 2 & 18 & 9 \end{bmatrix},$$

in which the rows, columns, and main diagonal have the magic sum of 65, but the main backdiagonal has a sum of 75. (This was constructed using the De la Hire method outlined below).

A *diabolic, pandiagonal,* or *perfect magic square* is a magic square with the additional property that the sum of any extended diagonal parallel to the main diagonal and backdiagonal is also $\sigma(M)$, the magic constant. An example will be constructed later in this chapter.

A *symmetric magic square*, in addition to being magic, has the property that “the sum of the two numbers in any two cells symmetrically placed with respect to the center cell is the same" (12, p. 529). A symmetric magic square is also called an *associative magic square* (11, p. 7; 2, p. 202). Multiplying $\frac{n}{2}$ by the sum of a pair of numbers symmetrically placed to the center square gives the magic sum (11, p. 7). King also points out that any magic square produced by the Hindu (stairstep) method described later will be symmetric.

A *concentric* (17, p. 55), or *bordered* (2, p. 200), magic square, is a magic square for which removing the top and bottom rows and the left and right columns (the “borders") results in another magic square. In the bordered square below, each of the three outer borders may be removed, leaving a square that is still magic (17, p. 55):

$$\begin{bmatrix} 4 & 5 & 6 & 43 & 39 & 38 & 40 \\ 49 & 15 & 16 & 33 & 30 & 31 & 1 \\ 48 & 37 & 22 & 27 & 26 & 13 & 2 \\ 47 & 36 & 29 & 25 & 21 & 14 & 3 \\ 8 & 18 & 24 & 23 & 28 & 32 & 42 \\ 9 & 19 & 34 & 17 & 20 & 35 & 41 \\ 10 & 45 & 44 & 7 & 11 & 12 & 42 \end{bmatrix}.$$

A *zero magic square* is a magic square whose magic constant is 0. The set of all such zero magic squares of order n is symbolized 0MS(n) (19, p. 109). Obviously a zero magic square cannot also be a normal magic square since it must contain negative entries. One such would be

$$\begin{bmatrix} 4 & 11 & -12 & -5 & 2 \\ 10 & -8 & -6 & 1 & 3 \\ -9 & -7 & 0 & 7 & 9 \\ -3 & -1 & 6 & 8 & -10 \\ -2 & 5 & 12 & -11 & -4 \end{bmatrix},$$

constructed using a modification of the Hindu method described later.

A *geometric,* or *multiplication magic square,* is a square matrix of numbers such that the product of the elements of each row, column, and corner-to-corner diagonal is a constant. An example with a magic product of 746,496, is given by King (11, p. 23):

$$\begin{bmatrix} 432 & 6 & 18 & 16 \\ 4 & 72 & 24 & 108 \\ 8 & 36 & 12 & 216 \\ 54 & 48 & 144 & 2 \end{bmatrix}.$$

An *addition-multiplication magic square* is a magic square in which both the sum and product in each row, column, and main diagonal and backdiagonals is a constant. An example of such a square of order 8 given by Dénes and Keedwell (4, p. 215) is

$$\begin{bmatrix} 162 & 207 & 51 & 26 & 133 & 120 & 116 & 25 \\ 105 & 152 & 100 & 29 & 138 & 243 & 39 & 34 \\ 92 & 27 & 91 & 136 & 45 & 38 & 150 & 261 \\ 57 & 30 & 174 & 225 & 108 & 23 & 119 & 104 \\ 58 & 75 & 171 & 90 & 17 & 52 & 216 & 161 \\ 13 & 68 & 184 & 189 & 50 & 87 & 135 & 114 \\ 200 & 203 & 15 & 76 & 117 & 102 & 46 & 81 \\ 153 & 78 & 54 & 69 & 232 & 175 & 19 & 60 \end{bmatrix}.$$

In this example, the magic sum is 840 and the magic product is 2,058,068,231,856,000. Neither geometric nor addition-multiplication squares will be considered further here.

Related to magic squares in a roundabout way is the concept of an *antimagic square*. In an antimagic square, no two rows, columns, or diagonals have the same sum. This work will not deal with antimagic squares.

Magic squares are related to another kind of square array known as a *latin square*. Latin squares are $n \times n$ arrays of n elements such that the same element appears exactly once in any given row or column. Proofs of some properties of magic squares, as well as some descriptions of methods of constructing magic squares, depend on the use of latin squares. One such example is a method of constructing a pandiagonal magic square of order 8 given later in this chapter. There are certain types of latin squares of interest in dealing with magic squares. A latin square is *diagonal* provided each element appears exactly once in the main diagonal and main backdiagonal. Two latin squares are said to be *orthogonal* provided that “if superimposed, every cell value of one square matches once, and once only, with every cell value of the other square" (11, p. 35). As an example, here are two orthogonal, diagonal latin squares of order 4 (11, p. 167):

$$\begin{bmatrix} 4 & 2 & 3 & 1 \\ 3 & 1 & 4 & 2 \\ 1 & 3 & 2 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 4 & 2 & 3 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \end{bmatrix}.$$

Methods of Constructing Magic Squares

There are many ways of producing magic squares. Several construction methods exist. Squares of odd order have different construction methods from squares of even order. Even-ordered squares may have different methods depending on whether or not the order is a multiple of four (called “doubly even") or not (called “singly even").

Odd-Ordered Squares

Several methods of constructing odd-ordered magic squares exist, some of them quite ancient. The first three, which deal with normal magic squares of odd order n, are from Schubert (17, pp. 44-47).

1. The Hindu method.—Start with 1 in the top center position (that is, $a_{1\,(n+1)/2}$). Put 2 in the bottom row in the column to the right, then continue diagonally upward (towards the “northeast") until reaching the right hand side. Then pick up on the left hand side on the next row up, entering the natural numbers through n, until reaching 1 again. The next number, which would be n + 1, goes directly under n. Continue in the same pattern until the square is filled. If the top is reached, the next number is placed in the bottom square of the next column. An example using the Hindu Rule for n = 9 follows:

$$\begin{bmatrix} 47 & 58 & 69 & 80 & 1 & 12 & 23 & 34 & 45 \\ 57 & 68 & 79 & 9 & 11 & 22 & 33 & 44 & 46 \\ 67 & 78 & 8 & 10 & 21 & 32 & 43 & 54 & 56 \\ 77 & 7 & 18 & 20 & 31 & 42 & 53 & 55 & 66 \\ 6 & 17 & 19 & 30 & 41 & 52 & 63 & 65 & 76 \\ 16 & 27 & 29 & 40 & 51 & 62 & 64 & 75 & 5 \\ 26 & 28 & 39 & 50 & 61 & 72 & 74 & 4 & 15 \\ 36 & 38 & 49 & 60 & 71 & 73 & 3 & 14 & 25 \\ 37 & 48 & 59 & 70 & 81 & 2 & 13 & 24 & 35 \end{bmatrix}$$

For the sake of convenience, we will refer to magic squares produced using this method as Hindu magic squares. Apparently the origin of this method is in doubt. Both King (11, p. 4) and Ball and Coxeter (2, p. 195) say that this method was developed by S. De La Loubère.

The Hindu, or staircase, method, need not use the integers 1 through n^2. Any standard arithmetic sequence can generate a magic square (11, pp. 5-7). In fact, other sequences may work as well, such as "an array of dates from a calendar with, added for completeness, a few notional days at the month end. The dates may also be regarded as . . . small, separate series (each row), but that is not important" (11, p. 7). As an example, we will construct a fifth-order square from an imaginary calendar beginning with 2. So we will use numbers from this array:

$$\left\{ \begin{matrix} 2 & 3 & 4 & 5 & 6 \\ 9 & 10 & 11 & 12 & 13 \\ 16 & 17 & 18 & 19 & 20 \\ 23 & 24 & 25 & 26 & 27 \\ 30 & 31 & 32 & 33 & 34 \end{matrix} \right\}$$

We start with 2 in the top center position and follow the usual method described above, using the order 2, 3, ... , reading the array from left to right, top row to bottom row, to produce square A below. We could also follow the order of the array from bottom to top, first column, to last column (using the order 2, 9, 16, 23, 30, 3, 10, ...) and produce magic square B below. Both have a magic constant of $\frac{5}{2}$ (2 + 34) = 90. The formula for finding the magic constant of a square is given at the end of the chapter.

$$A = \begin{bmatrix} 24 & 33 & 2 & 11 & 20 \\ 32 & 6 & 10 & 19 & 23 \\ 5 & 9 & 18 & 27 & 31 \\ 13 & 17 & 26 & 30 & 4 \\ 16 & 25 & 34 & 3 & 12 \end{bmatrix}; \quad B = \begin{bmatrix} 12 & 27 & 2 & 17 & 32 \\ 20 & 30 & 10 & 25 & 5 \\ 23 & 3 & 18 & 33 & 13 \\ 31 & 11 & 26 & 6 & 16 \\ 4 & 19 & 34 & 9 & 24 \end{bmatrix}$$

Another variation on this method is to start in the top center position with 0, then write the numbers through $n^2 - 1$ in base n; the resulting magic square is still magic in base 10 (2, pp. 195-96).

2. Method of Bachet de Méziriac.— This method follows the same general pattern as the Hindu Rule, but after each diagonal of n numbers, rather than moving one space down, the next number is placed two spaces to the right (or the equivalent after "wrapping around") The pattern does not begin in the center top position. An example for $n = 7$ is given below (17, p. 46):

$$\begin{bmatrix} 4 & 29 & 12 & 37 & 20 & 45 & 28 \\ 35 & 11 & 36 & 19 & 44 & 27 & 3 \\ 10 & 42 & 18 & 43 & 26 & 2 & 34 \\ 41 & 17 & 49 & 25 & 1 & 33 & 9 \\ 16 & 48 & 24 & 7 & 32 & 8 & 40 \\ 47 & 23 & 6 & 31 & 14 & 39 & 15 \\ 22 & 5 & 30 & 13 & 38 & 21 & 46 \end{bmatrix}$$

3. Method of Phillipe de la Hire.—To construct an odd-ordered normal magic square of order n, first construct two nth-order latin squares. The first latin square consists of the numbers 1 through n in each row and column. The elements of the rows and columns of the second latin square are 0 and the first $(n-1)$ multiples of n. (For example, for a fifth-order magic square, the second latin square would contain 0, 5, 10, 15, and 20 in each row and column.) The two latin squares must be orthogonal. The sum of these two latin squares is a magic square. This method can generate a number of distinct magic squares of the same order. According to Schubert, magic squares produced by this method are pandiagonal as well (17, p. 48). Magic squares $A, B,$ and D in Chapter 3 were produced by a computer program by Pizarro (15, p. 472) based on this method.

4. Uniform Step Method.—Related to the Hindu method is the uniform step method for producing normal magic squares of odd order n as explained by Lehmer (12, p. 530). The numbers 1 through n^2 are arranged as follows. Choose a position (p, q) in the square M so that 1 is the entry m_{pq}. Pick numbers α and β ($\alpha < n$, $\beta < n$), called "steps," to determine the desired position for the number 2. Then 2 is placed as entry $m_{p+\alpha,q+\beta}$, 3 is in $m_{p+2\alpha,q+2\beta}$, and so forth through n, putting each number k in the position $m_{p+(k-1)\alpha,q+(k-1)\beta}$, where $p+(k-1)\alpha$ and $q+(k-1)\beta$ are reduced modulo n. To keep the number $n+1$ from being in the same place as 1, introduce a "break step" (a, b) so that $n + 1$ becomes entry $m_{p+a,q+b}$, and so forth. In general, a number x is placed in entry m_{ij} where

$$i \equiv p + \alpha(x-1) + \left[\!\left[\tfrac{x-1}{n}\right]\!\right] \pmod{n} \quad \text{and} \quad j \equiv q + \beta(x-1) + \left[\!\left[\tfrac{x-1}{n}\right]\!\right] \pmod{n}.$$

(The symbol $[\![y]\!]$ represents the greatest integer function.) The square formed by this method is magic if and only if $a, b, \alpha, \beta,$ and $(\alpha b - \beta a)$ are each relatively prime to n (12, p. 535). Furthermore, this square is diabolic (pandiagonal) if and only if $\alpha \pm \beta$ and $a \pm b$ are relatively prime to n (12, p. 535). It is symmetric if and only if $2p \equiv \alpha + a + 1 \pmod{n}$ and $2q \equiv \beta + b + 1 \pmod{n}$ (12, p. 536). (Note: a symmetric square that is not a magic square can be produced in this manner by failing to fulfill the necessary conditions listed above for a uniform step square to be magic.)

The uniform step method can produce many magic squares for each order. It is a fairly simple task to produce a program in VAX BASIC which generates a uniform step magic square. As an example, here is a fifth-order square produced using the program in Appendix B:

$$\begin{bmatrix} 16 & 24 & 2 & 10 & 13 \\ 5 & 8 & 11 & 19 & 19 \\ 14 & 17 & 25 & 3 & 6 \\ 23 & 1 & 9 & 12 & 20 \\ 7 & 15 & 18 & 21 & 4 \end{bmatrix}$$

Even-Ordered Magic Squares

A more complicated proposition is the construction of even-ordered magic squares. They may be divided into two categories: those with $n \equiv 0 \pmod 4$, called doubly even, and those with $n \equiv 2 \pmod 4$, called singly even. Some of the processes involved are too complicated for this paper.

In some cases doubly even squares may be constructed using the De la Hire method (adding a pair of orthogonal diagonal latin squares called auxiliary squares). The method will not always work, but it sometimes does. For example, the construction of a 4th-order magic square follows (17, p. 51). Note that the resulting magic square is not pandiagonal, although a few broken diagonals do yield the magic constant of 34.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 8 & 12 \\ 8 & 12 & 0 & 4 \\ 12 & 8 & 4 & 0 \\ 4 & 0 & 12 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 11 & 16 \\ 12 & 15 & 2 & 5 \\ 14 & 9 & 8 & 3 \\ 7 & 4 & 13 & 10 \end{bmatrix}$$

Another way to construct a square in MS(4) is to write the natural numbers 1 through 16 horizontally in a 4×4 grid. Leave the four corners and four center squares alone. We note that the magic sum will be 34. To fill in the other entries, take 35 minus the original entry to obtain the new entry. The result is a magic square of order 4 (17, pp. 49-50).

A similar pattern may be followed to produce an eighth-order magic square. This time, the numbers left unchanged form a checkerboard pattern of 2×2 squares, beginning at the center of the square. The remaining entries are obtained by subtracting the original entry from 65 (one more than the magic sum). Even larger magic squares of order $n \equiv 0 \pmod 4$ using this form may be created.

Qian Jianping describes a complicated method (which is too long to describe here) for obtaining an even-ordered magic square (16, pp. 254-55). It involves starting with an odd-ordered magic square produced by certain transformations on latin squares, taking another function value, and subtracting a multiple of a certain matrix. There are seven cases of what the matrix can be.

Hendricks (7, pp. 55-58) shows the following manner of generating a pandiagonal magic square of order 8. Take an eighth-order diagonal latin square whose elements consist of the letters a through h. Reflect it across the main backdiagonal (thus forming an orthogonal pair), then combine the two squares so that each entry consists of two letters, the first from the first square and the second from the reflected square. Assign some value of 0 through 7 to each letter uniquely. Take these as base-8 numbers; convert them to the decimal system. Add 1 to each element. The result is a normal magic square, order 8. To make it pandiagonal, divide the magic square into four equal quadrants. Reflect the upper right-hand quadrant on its vertical axis, the lower left on its horizontal axis, rotate the lower right quadrant 180°, and leave the upper left quadrant alone. The result is a pandiagonal magic square. Hendricks hints that this method may be used for 12th- and 16th-order magic squares as well. A similar method is given by King (11, pp. 21-22); this is given as a method for constructing magic squares of doubly even order.

Some Basic Properties of Magic Squares

The following is a collection of some known properties of magic squares. Some are obvious; others are not. Some are even surprising. Many of them are given with only a minimum of proof. Not all of them will be useful in this work, but this gives an overview of some properties of magic squares that exist and are seen in the literature.

1. The sum of two magic squares of the same order is also a magic square. Suppose A and B are both MS(n) and $\sigma(A) = a$, $\sigma(B) = b$. Then for any row of $A + B$, $\sigma(A + B) = \sigma(A) + \sigma(B)$. The same clearly holds true for any column; the same property holds for the main diagonal and main backdiagonal.

2. If M is a magic square, then M^T (the transpose of M) is also a magic square. It is easy to see that the rows of M become the columns of M^{T} and the columns of M become the rows of M^{T}, so the row and column sums are preserved. Likewise, the diagonals and their sums are preserved in their new orientation in M^{T}.

3. If M is a magic square, and M' can be obtained from M by a rigid transformation (i. e., a rotation or reflection), then M' is also a magic square. Since the positions of the elements have not changed relative to each other, $\sigma(M)$ remains intact.

4. If A is a magic square, and each element of B is obtained by adding, subtracting, multiplying, or dividing the corresponding element of A by the same number (not 0 for multiplication or division), then B is a magic square.

5. For a normal magic square M of order n, $\sigma(M) = \frac{n}{2}(n^2 + 1)$. This result is given by most references on magic squares, among them Schubert (17, p. 44). A proof of this formula for odd values of n, based on construction using orthogonal diagonal latin squares, is given by Dénes and Keedwell (4, p. 208, Theorem 6.2.2). Other proofs of the formula rely on dealing with the sum of an arithmetic sequence.

6. For a magic square M formed by numbers from an arithmetic series,

$$\sigma(M) = \frac{n}{2}(\text{lowest cell value + highest cell value})$$

(11, pp. 6-7).

7. No normal magic square of order 2 exists. A trivial second-order magic square may be constructed in which all four elements are the same, but it is easy to see that no other possibilities exist for MS(2).

8. "[I]f a [pandiagonal magic] square is (mentally) divided between any two rows or columns, the two pieces thus formed may be interchanged without disturbing its pandiagonality" (11, p. 11). This property can also be thought of in the following manner. Take an nth-order pandiagonal magic square and make several copies of it in, say, a 2 by 2 array. From that array, take any n by n square. The resulting square is also pandiagonal (17, p. 48). It is fairly easy to see why the sum of the rows and columns would be unchanged, since the same numbers appear in each row and column of the new square as in some row and column of the original square. The fact that the diagonals still have the same sum lies in the square's being pandiagonal, so that any diagonal, broken or not, of the new square contains the same entries as some diagonal on the original square.

9. The determinant of a [normal] pandiagonal magic square of order 4 is 0. This is proved in an article with this property as its title by Hendricks (8).

10. In a Hindu magic square, the arithmetic mean of the elements of the square is the element in the center of the square. This is noted by King (11, p. 9).

11. An upper bound for the number of normal magic squares of order n can be given by $\frac{(n^2)!}{8(2n+1)!}$ (19, p. 111). There is only one distinct third-order normal magic square. However, there are 880 distinct fourth order squares, of which 48 are pandiagonal. There are 3600 fifth-order pandiagonal magic squares out of over 13,000,000 possible fifth order normal magic squares. There are no pandiagonal squares of order 6. There are over 38,000,000 seventh-order magic squares, and over 6,500,000,000,000 of the eighth order (2, pp. 202 and 204).

A Look Ahead

We have explored some of the historical background of magic squares and reviewed some of the associated terminology. Different kinds of magic squares are produced using various methods. Some of these methods will become important in discussing other properties magic squares appear to possess. We will treat three of them. In Chapter 2, we will prove that the set of all magic squares of a given order form a vector space. In Chapter 3, we will take pairs of rows or columns of magic squares and examine their dot products. Finally, Chapter 4 will take a look at the eigenvalues of magic squares.

Chapter 2
Vector Spaces and Magic Squares

Definition of a Vector Space

A vector space is a concept studied in several branches of mathematics, including (among others) linear algebra, matrix algebra, and topology. Exactly what kinds of things are defined as vectors is not as important as whether the things defined as vectors satisfy several given properties. Cohen and Bernard (3, p. 76) and van den Essen(18), among others, remark that the set of all magic squares of a given order satisfy the definition of a vector space. The proof that magic squares comprise a vector space, although requiring nine parts, is quite simple. The definition of a vector space used in the following proof is from Cohen and Bernard (3, p. 76).

Definition. We will let X, Y, and Z be magic squares of order n and a and b be real numbers. A *vector space* has the following properties:

1. $X+Y \in$ MS(n); $aX \in$ MS(n).
2. $X+Y=Y+X$
3. $X+(Y+Z)=(X+Y)+Z$
4. $\exists$ **0** such that X + **0** = **0** + X = X
5. $\exists$ X' such that $X + X' = X' + X =$ **0**
6. $a(X+Y)=aX+aY$
7. $(a+b)X=aX+bX$
8. $(ab)X=a(bX)$
9. $1X = X$

Theorem: Magic Squares Form Vector Spaces

Theorem 2.1. *For $n \in \mathbb{N}$, $n \neq 2$, MS(n) is a vector space.*

Proof: Throughout the following, let X, Y, and $Z \in$ MS(n). Let a, $b \in \mathbb{R}$. Use matrix notation to denote each magic square. That is, we may represent a matrix X by using a representative element $[x_{ij}]$. Then the following conditions for forming a vector space hold.

1a. $X + Y$ is a magic square: $X + Y = [x_{ij}] + [y_{ij}] = [x_{ij} + y_{ij}]$. For $X + Y$ to be a magic square, the sum of each row, column, and diagonal must be the same. Choose an arbitrary row i from rows 1 through n. Then

$\sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{n} (a_{ij} + b_{ij}) = \sigma(X+Y)$, using a well-known property of sums.

In a similar fashion we can see that the sum of the elements of any column of $X + Y$ is $\sigma(X + Y)$; a similar result holds for the main diagonal and main backdiagonal.

1b. aX is a magic square: We wish to show that mutliplying each element of the magic square X by a constant a, we get a new magic square. We see that $aX = a[x_{ij}] = [ax_{ij}]$. To show that aX is a magic square, then, we note that $\sum_{i=1}^{n} ax_{ij} = nax_{ij}$ for the rows, $\sum_{j=1}^{n} ax_{ij} = nax_{ij}$ for the columns, tr$(aX) = nax_{ij}$, and btr$(aX) = nax_{ij}$. Thus $aX \in$ MS(n).

2. $X + Y = Y + X$: To show that addition of magic squares is commutative, we see that $X+Y = [x_{ij} + y_{ij}] = [y_{ij} + x_{ij}] = Y + X$.

3. $(X + Y) + Z = X + (Y + Z)$: To show addition of magic squares is associative, we see that $(X + Y) + Z = ([x_{ij}] + [y_{ij}]) + [z_{ij}] = [x_{ij} + y_{ij}] + [z_{ij}] = [x_{ij} + y_{ij} + z_{ij}]$

$= [x_{ij}] + [y_{ij} + z_{ij}] = [x_{ij}] + ([y_{ij}] + [z_{ij}]) = X + Y + Z$.

4. There exists **0** *such that* X + **0** = **0** + X = X*:* Let **0** $\in$ MS(n) such that a_{ij} = 0 $\forall$ a_{ij}. Then X + **0** = [x_{ij} + 0] = [x_{ij}] = X; also, **0** + X = [0 + x_{ij}] = [x_{ij}] = X.

5. There exists X' *in MS(*n*) such that* X + X' = X' + X = **0***:* Let X' = (-1)X. Then X + X' = [x_{ij}] + [-x_{ij}] = [x_{ij} $-$ x_{ij}] = [0] = **0**. Similarly, we can see that X' + X = **0**.

6. a*(*X + Y*)* = aX + aY*:* To prove that scalar multiplication is distributive over magic square addition we see that $a(X+Y)$ = a[x_{ij} + y_{ij}] = [a(x_{ij} + y_{ij})] = [ax_{ij} + ay_{ij}] = aX + aY.

7. (a+b)X = aX + bX: To show that the sum of two scalars mutliplied by a magic square possesses a kind of right distributive property, we see that $(a+b)X =$ [$(a+b)x_{ij}$] = [ax_{ij} + bx_{ij}] = [ax_{ij}] + [bx_{ij}] = $aX + bX$.

8. (ab)X = a(bX): To show that the product of two scalars and a magic square is associative, we note that $(ab)X =$ [(ab)x_{ij}] = [a(bx_{ij})] = a[bx_{ij}] = a(bX).

9. 1X = X: To show that the scalar 1 is the scalar mutliplication identity, we show that 1X = [1x_{ij}] = [x_{ij}] = X.

Since all nine vector space properties hold, it follows that MS(n) is a vector space. ☐

Chapter 3
Dot Products of Magic Squares

Introduction

When working with matrices, sometimes only a single row or column of the matrix is considered. The rows or columns are individually referred to as row or column vectors. One operation performed on these vectors is taking the dot product, defined earlier. One might wonder what would result from taking dot products of various rows or columns of a magic square. A reasonable assumption might be that all the dot products would be the same, since the sums of all rows, columns, and diagonals are equal. Such is not the case, however. A computer program in BASIC was developed (see Appendix A) to check this assumption for magic squares of various sizes. After trying a number of various-sized magic squares with this program, it appears that the dot product of one pair of rows (columns) will be the same as the dot product of one other pair of rows (columns). The dot product of some pairs of rows (columns) had no “mates" at all. Such an observation, of course, does not constitute a proof. To prove this conjecture would require an efficient representation of a magic square of a particular order, or, better, yet, of any order. At this point, however, strange and interesting things begin to happen. A general proof of at this time in the study of this conjecture remains elusive.

Examples

As an example of the property that the dot products of many rows and columns of a magic square will have a “mate," consider the following seventh-order magic square taken from Schubert (17, p. 44). This square was constructed using the Hindu (staircase) method. The BASIC program in the Appendix was used to check to dot products. The magic sum of this particular square is 175. Solely for ease in notation, the dot product of rows or columns i and j in this example only will be written <i,j>. The dot products which have no “mate" are marked with an asterisk.

$$\begin{bmatrix} 30 & 39 & 48 & 1 & 10 & 19 & 28 \\ 38 & 47 & 7 & 9 & 18 & 27 & 29 \\ 46 & 6 & 8 & 17 & 26 & 35 & 37 \\ 5 & 14 & 16 & 25 & 34 & 36 & 45 \\ 13 & 15 & 24 & 33 & 42 & 44 & 4 \\ 21 & 23 & 32 & 41 & 43 & 3 & 12 \\ 22 & 31 & 40 & 49 & 2 & 11 & 20 \end{bmatrix}$$

To find the dot product of rows 1 and 2, for example, we compute
30 · 38 + 39 · 47 + 48 · 7 + 1 · 9 + 10 · 18 + 19 · 27 + 28 · 29 = 4823.
Computing all other combinations, we have the following:

Row dot products:	<1,2> = 4823	<1,3> = 3976	<1,4> = 3773	<1,5> = 3528
<1,6> = 3927	<1,7> = 4627*	<2,3> = 4725	<2,4> = 4074	<2,5> = 3724
<2,6> = 3675*	<2,7> = 3927	<3,4> = 4676	<3,5> = 4221*	<3,6> = 3724
<3,7> = 3528	<4,5> = 4676	<4,6> = 4074	<4,7> = 3773	<5,6> = 4725
<5,7> = 3976	<6,7> = 4823			

Column dot products:		<1,2> = 4662	<1,3> = 4018	<1,4> = 3647
<1,5> = 3843	<1,6> = 4263	<1,7> = 4613*	<2,3> = 4809	<2,4> = 3871

<2,5> = 3549	<2,6> = 3794*	<2,7> = 4263	<3,4> = 4711	<3,5> = 3822*
<3,6> = 3549	<3,7> = 3843	<4,5> = 4711	<4,6> = 3871	<4,7> = 3647
<5,6> = 4809	<5,7> = 4018	<6,7> = 4662		

In this example we see the following pairs of matching row dot products:

<1,2> and <6,7>	<1,3> and <5,7>	<1,4> and <4,7>
<1,5> and <3,7>	<1,6> and <2,7>	<2,3> and <5,6>
<2,4> and <4,6>	<2,5> and <3,6>	<3,4> and <4,5>

The same pairs of dot products also match for the columns, although the row dot products and the column dot products are not the same. It is interesting to note here that the sum of the subscripts in each pair is 16, which is $2(n+1)$ for this magic square. The “singleton" dot products — <1,7>, <2,6>, and <3,5> — all have subscript sums of $8 = n + 1$. For other examples, however, the matching dot product pairs did not all have the same subscript sum, so this result does not help us find a general pattern for *all* magic squares.

For magic squares of certain orders and types, there are general algebraic forms available to show us what form any such magic square will take. In some cases, taking the dot products of rows and columns of the algebraic forms will give us results showing which pairs will have the same dot product. Others do not. One such example that does shed some light is the magic square of order 3.

Third-Order Magic Squares

If we denote the entries of a third-order magic square by $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, it can be shown that any magic square of that order can be written in a more general form where each entry is solely expressed in terms of a, e, and h (3, p. 77). This allows us to generate a theorem concerning dot products for magic squares of that order.

Theorem 3.1. *If M is a magic square of order 3, then the dot product of rows 1 and 2 equals the dot product of rows 2 and 3, and the dot product of columns 1 and 2 equals the dot product of columns 2 and 3.*

Proof: Let $M \in$ MS(3). Then, according to Cohen and Bernard, M may be written in this form (3, p. 77):

$$\begin{bmatrix} -c+e+h & 2e-h & c \\ 2c-h & e & -2c-2e+h \\ -c+2e & h & c+e-h \end{bmatrix} .$$

Then $R_1 \cdot R_2$ $= (-c + e + h)(2c - h) + (2e - h)e + c(-2c + 2e + h)$
$= -2c^2 + 2ce + 2ch + ch + eh - h^2 + 2e^2 - eh - 2c^2 + 2ce + ch$
$= -4c^2 + 2e^2 - h^2 + 4ce + 4ch$

and $R_2 \cdot R_3 = (2c - h)(-c + 2e) + eh + (-2c + 2e + h)(c + e - h)$
$= -2c^2 + ch + 4ce - 2eh + eh - 2c^2 + 2ce + ch - 2ce + 2e^2 + eh + 2ch - 2eh - h^2$
$= -4c^2 + 2e^2 + - h^2 + 4ce + 4ch.$

So $R_1 \cdot R_2 = R_2 \cdot R_3$.

In a similar fashion, the dot product of the first two columns and the last two columns are the same:

$C_1 \cdot C_2$ = (-c + e + h)(2e − h) + (2c − h)e + (-c + 2e)h

= -2ce + 2e^2 + 2eh + ch − eh − h^2 + 2ce − eh − ch + 2eh

= 2e^2 − h^2 + 2eh.

$C_2 \cdot C_3$ = (2e − h)c + e(-2c + 2e + h) + h(c + e − h)

= 2ce − ch − 2ce + 2e^2 + eh + ch + eh − h^2

= 2e^2 − h^2 + 2eh.

So $C_1 \cdot C_2 = C_2 \cdot C_3$.

Thus the theorem is proved. ☐

In each case, it is interesting (although perhaps not significant) to note that the sum of the subscripts of the two equal pairs of rows (columns) is 8.

By way of contrast, it might bear noting that $R_1 \cdot R_3 = 2c^2 + 2e^2 - h^2 - 2ce - 2ch + 4eh$, and $C_1 \cdot C_3 = -6c^2 - h^2 + 6ce + 6ch - 4eh + 2e^2$, and neither of these dot products has a “mate."

Fourth-Order Magic Squares

At first glance, a similar property appears to exist for a fourth-order magic square. There are 880 possible fourth-order magic squares, though (2, p. 202). However, using an algebraic representation of a general fourth-order magic square does not show that certain dot products are equal. When working with some pandiagonal fourth-degree magic squares, we find that the dot product of rows (and columns) 1 and 2 equal to the dot product of rows (and columns) 3 and 4. In some cases we find that the product of rows (columns) 1 and 4 equals that of 2 and 3; other times we find that the dot product of rows (columns) 1 and 3 equal to that of rows 2 and 4. This fact could be frustrating if we wanted to have a property that is always true of dot products. An algebraic representation of a normal, pandiagonal fourth-degree magic square is available which does show some of the desired properties and leads to the following theorem.

Theorem 3.2. *If a magic square of degree 4 is normal and pandiagonal , then* $R_1 \cdot R_2 = R_3 \cdot R_4$ *and* $C_1 \cdot C_2 = C_3 \cdot C_4$.

Proof: According to Hendricks (6, p. 299), a pandiagonal magic square of order 4 may be written in the form

$$\begin{bmatrix} A+s & B+t & C-t & D-s \\ B-t & A-s & D+s & C+t \\ D+t & C+s & B-s & A-t \\ C-s & D-t & A+t & B+s \end{bmatrix}.$$

(a) We first show that the dot product of rows 1 and 2 matches that of rows 3 and 4:

$R_1 \cdot R_2 = (A + s)(B - t) + (B + t)(A - s) + (C - t)(D + s) + (D - s)(C + t)$

$= AB - At + Bs - st + AB - Bs + At - st + CD + Cs - st + CD - Cs + Dt - st$

$= 2AB + 2CD - 4st$, and

$R_3 \cdot R_4 = (D + t)(C - s) + (C + s)(D - t) + (B - s)(A + t) + (A - t)(B + s)$

$= CD - Ds + Ct - st + CD - Ct + Ds - st + AB + Bt - As - st + AB + As - Bt - st$

$= 2AB + 2CD - 4st$.

Thus $R_1 \cdot R_2 = R_3 \cdot R_4$.

(b) $C_1 \cdot C_2 = C_3 \cdot C_4$: To show that the dot product of the first two columns is equal to the dot product of the last two columns, we see that

$C_1 \cdot C_2 = (A+s)(B+t) + (B-t)(A-s) + (D+t)(C+s) + (C-s)(D-t)$

$= AB + At + Bs + st + AB - Bs - At + ts + CD + Ds + Ct + st + CD - Ct - Ds + st$

$= 2AB + 2CD + 4st$, and

$C_3 \cdot C_4 = (C-t)(D-s) + (D+s)(C+t) + (B-s)(A-t) + (A+t)(B+s)$

$= CD - Cs - Dt + st + CD + Dt + Cs + st + AB - Bt - As + st + AB + As + Bt + st$
$= 2AB + 2CD + 4st.$
Thus $C_1 \cdot C_2 = C_3 \cdot C_4$, and the theorem is proved. $\square$

In working with several pandiagonal fourth-order magic squares, we sometimes find other matching pairs of dot products, but they do not always turn out to be the same pairs. Using the same general form, we can perhaps see why this is so.
(a) We would hope to find that $R_1 \cdot R_3 = R_2 \cdot R_4$; however, the results give us the same terms with some differing signs:

$R_1 \cdot R_3 = (A + s)(D + t) + (B + t)(C + s) + (C - t)(B - s) + (D - s)(A - t)$
$= AD + At + Ds + st + BC + Bs + Ct + st + BC - Cs - Bt + st + AD - Dt - As + st$
$= 2AD - As + At + 2BC + Bs - Bt - Cs + Ct + Ds - Dt + 4st$, but
$R_2 \cdot R_4 = (B - t)(C - s) + (A - s)(D - t) + (D + s)(A + t) + (C + t)(B + s)$
$= BC - Bs - Ct + st + AD - At - Ds + st + AD + Dt + As + st + BC + Cs + Bt + st$
$= 2AD + As - At + 2BC - Bs + Bt + Cs - Ct - Ds + Dt + 4st.$
As a result, we cannot unequivocally say always that $R_1 \cdot R_3 = R_2 \cdot R_4$. A similar result occurs when we try, unsuccessfully, to show that $R_1 \cdot R_4 = R_2 \cdot R_3$: we get the same terms with some differing signs.
(b) As in (a) above, we would like to confirm that $C_1 \cdot C_3 = C_2 \cdot C_4$:
$C_1 \cdot C_3 = (A + s)(C - t) + (B - t)(D + s) + (D + t)(B - s) + (C - s)(A + t)$
$= AC - At + Cs - st + BD + Bs - Dt - st + BD - Ds + Bt - ts + AC + Ct - As - st$
$= 2AC - As - At + 2Bc + Bs + Bt + Cs + Ct - Ds - Dt - 4st$, but
$C_2 \cdot C_4 = (B + t)(D - s) + (A - s)(C + t) + (C + s)(A - t) + (D - t)(B + s)$
$= BD - Bs + Dt - st + AC + At - Cs - st + AC - Ct + As - st + BD + Ds - Bt - st$
$= 2AC + As + At + 2BD - Bs - Bt - Cs - Ct + Ds + Dt - 4st.$
Once again, the terms of the two dot products are the same, but some signs are different, and we cannot guarantee that $C_1 \cdot C_3 = C_2 \cdot C_4$. But we can say that the dot product of the first and second rows or columns equals the dot product of the third and fourth rows or columns.

The pattern for pandiagonal magic squares of order four is not the same for all of MS(4). For example, consider this fourth order-magic square, which is not pandiagonal, from Qin Jiang (16, p. 253):

$$\begin{bmatrix} 1 & 8 & 10 & 15 \\ 11 & 14 & 4 & 5 \\ 16 & 9 & 7 & 2 \\ 6 & 3 & 13 & 12 \end{bmatrix}.$$

The dot products $R_1 \cdot R_2 = R_3 \cdot R_4 = 238$, as in Theorem 3.2, and $R_1 \cdot R_4 = R_2 \cdot R_3 = 340$. But the dot product $C_1 \cdot C_2 = 324$, and $C_3 \cdot C_4 = 340$, contrary to our suspicions. Although we could not prove above that $C_1 \cdot C_3 = C_2 \cdot C_4$ in general, in this case both products are 244. The other four dot products in this example have no “mates." For this particular example, then, the dot product pairs of the rows are not even the same as the dot product pairs of the columns.

Fifth–Order Magic Squares and Beyond

For a fifth-order magic square, it appears that certain pairs of dot products will be equal. There does not seem to be a set pattern, however. Consider the following normal fifth-order magic squares. The first three were generated by a computer program by Pizarro (15, p. 472; also found on “MathDisk IV" from the National Council of Teachers of Mathematics). Magic squares $A, B,$ and D were generated by adding two auxiliary squares, both of which were produced by cyclic permutations: one consisted of the integers 1 through 5, the other of the multiples 0, 5, 10, 15, and 20. Magic square E was formed by the Hindu, or staircase, method.

$$A = \begin{bmatrix} 18 & 24 & 5 & 6 & 12 \\ 22 & 3 & 9 & 15 & 16 \\ 1 & 7 & 13 & 19 & 25 \\ 10 & 11 & 17 & 23 & 4 \\ 14 & 21 & 21 & 2 & 8 \end{bmatrix} ; \quad B = \begin{bmatrix} 18 & 25 & 4 & 6 & 12 \\ 22 & 3 & 10 & 14 & 16 \\ 1 & 7 & 13 & 20 & 24 \\ 9 & 11 & 17 & 23 & 5 \\ 15 & 19 & 21 & 2 & 8 \end{bmatrix} ;$$

$$D = \begin{bmatrix} 8 & 4 & 22 & 20 & 11 \\ 1 & 23 & 19 & 12 & 10 \\ 25 & 16 & 13 & 9 & 2 \\ 17 & 15 & 6 & 3 & 24 \\ 14 & 7 & 5 & 21 & 18 \end{bmatrix} ; \quad E = \begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix} .$$

For magic square A there are four matching pairs of dot products; the pairs are the same in rows and columns, although some of the products are different. Using the program in Appendix A again we see that $R_1 \cdot R_2 = R_4 \cdot R_5 = 795$; $R_1 \cdot R_3 = R_3 \cdot R_5 = 665$; $R_1 \cdot R_4 = R_2 \cdot R_5 = 715$; and $R_2 \cdot R_3 = R_3 \cdot R_4 = 845$. Similarly, $C_1 \cdot C_2 = C_4 \cdot C_5 = 895$; $C_1 \cdot C_3 = C_3 \cdot C_5 = 765$; $C_1 \cdot C_4 = C_2 \cdot C_5 = 715$; and $R_2 \cdot R_3 = R_3 \cdot R_4 = 845$.

Magic square B has only two matching dot product pairs, which are the same for rows and columns. We find that $R_1 \cdot R_2 = R_4 \cdot R_5 = 787$, and $R_1 \cdot R_3 = R_3 \cdot R_5 = 653$. In the columns, $C_1 \cdot C_2 = C_4 \cdot C_5 = 907$, and $C_1 \cdot C_3 = C_3 \cdot C_5 = 773$. No other pair of row or column dot products in this square has a “mate."

The dot products for magic square D are rather unusual: instead of pairs of dot products showing up, there are two triples of matching dot products for both rows and columns. In square D, it turns out that $R_1 \cdot R_2 = R_1 \cdot R_5 = R_4 \cdot R_5 = 868$, and $R_1 \cdot R_3 = R_2 \cdot R_4 = R_3 \cdot R_5 = 752$. For the columns, the dot product triples are $C_1 \cdot C_2 = C_1 \cdot C_5 = C_4 \cdot C_5 = 808$, and $C_1 \cdot C_3 = C_2 \cdot C_4 = C_3 \cdot C_5 = 692$.

For magic square E there are four matching pairs of dot products in both the rows and columns: <1,2> and <4,5>; <1,3> and <3,5>; <1,4> and <2,5>; and <2,3> and <3,4>. As in the example of the seventh–order Hindu magic square, the total of the subscripts adds up to $2(n+1)$, or 12 this time. The dot products which have no match are <1,5> and <2,4>; again, these subscripts add to $n+1$ or 6.

From these four examples we can see that not all fifth-order magic squares have the same pattern for matching dot products, but it appears that the sums of the subscripts *may* have some relationship to the dot products. We turn our consideration now to some other odd-ordered Hindu magic squares. For convenience, we may represent the Hindu magic square of order n as HMS(n). (There will be only one HMS(n) for any odd n since the arrangement of the square is fixed by the method, so we may speak of *the* HMS(n).)

For HMS(9), we find the following matching pairs of dot products in both the rows and columns. We will employ the notation used earlier, and will disregard the actual products:

<1,2> and <8,9>	<1,3> and <7,9>	<1,4> and <6,9>
<1,5> and <5,9>	<1,6> and <4,9>	<1,7> and <3,9>
<1,8> and <2,9>	<2,3> and <7,8>	<2,4> and <6,8>
<2,5> and <5,8>	<2,6> and <4,8>	<2,7> and <3,8>

<3,4> and <6,7>   <3,5> and <5,7>   <3,6> and <4,7>
<4,5> and <5,6>

There are no matches for the dot products <2,8>, <4,6>, <3,7>, or <1,9>. As with HMS(5) and HMS(7), the subscripts of the matching pairs add up to $2(n+1)$. Further examination reveals that the sum of the outer members of each pair is equal to the sum of the inner members of each pair, and that both equal $n+1$. The dot products with no matches are those for which the subscripts add to $n+1$.

An examination of Hindu magic squares of orders 11, 13, 15, 17, and 19 reveals the same pattern: The pairs of dot products in the nth order square which match are the ones for which both the inner and outer subscripts have a sum of n + 1. This examination does not guarantee that squares of a larger order follow the same pattern, but it would be reasonable assumption. We conclude this section with a re-statement of this pattern as a conjecture.

***Conjecture 3.3**. If M is an nth-order magic square (where n is an odd natural number greater than 1) constructed by the Hindu method, then the dot products of any two rows (or columns) a and b of M, where $a < n$, $b < n$, and $a \neq b$, will be equal provided the subscripts of the first pair are a and b and the subscripts of the second pair are $(n+1-b)$ and $(n+1-a)$, respectively. If $a+b=n+1$, then the dot product of rows a and b will not generally equal the dot product of any two other rows (columns).*

To illustrate this conjecture one more time, suppose we have a Hindu magic square of order 35. The conjecture predicts that the dot product of columns 11 and 15 will equal the dot product of columns 21 and 25. If the conjecture is true, though, the dot product of columns 12 and 24 will not necessarily equal the dot product of any other pair of columns. A proof of this pattern apparently awaits a more efficient representation of odd-ordered Hindu magic squares. By way of extension, this conjecture appears to be true for any odd-ordered magic squares constructed by the staircase method, even using sequences other than the natural numbers 1 through n. For example, magic squares F and G below are fifth-order Hindu squares which follow this pattern. Square F, taken from King (11, p. 6) was constructed using the arithmetic sequence 7, 10, 13, ..., 76, 79. Square G (11, p. 8) uses 5 series from a hypothetical calendar; that is, it contains 4, 5, 6, 7, 8, 11, 12, ..., 15, 18, ..., 22, 25, ... 29, 32, ..., 36.

$$F = \begin{bmatrix} 55 & 76 & 7 & 28 & 49 \\ 73 & 19 & 25 & 46 & 52 \\ 16 & 22 & 43 & 64 & 70 \\ 34 & 40 & 61 & 67 & 13 \\ 37 & 58 & 79 & 10 & 31 \end{bmatrix}; \quad G = \begin{bmatrix} 14 & 29 & 4 & 19 & 34 \\ 22 & 32 & 12 & 27 & 7 \\ 25 & 5 & 20 & 35 & 15 \\ 33 & 13 & 28 & 8 & 18 \\ 6 & 21 & 36 & 11 & 26 \end{bmatrix}$$

As mentioned earlier, not all magic squares of the same order follow the same pattern for matching pairs of dot products, but there were usually some matching pairs. One interesting exception was the following magic square of order 7 given by Schubert (17, p. 55). In this square, unlike any other tested for this paper, there were no pairs of matching dot products at all.

$$\begin{bmatrix} 4 & 5 & 6 & 43 & 39 & 38 & 40 \\ 49 & 15 & 16 & 33 & 30 & 31 & 1 \\ 48 & 37 & 22 & 27 & 26 & 13 & 2 \\ 47 & 36 & 29 & 25 & 21 & 14 & 3 \\ 8 & 18 & 24 & 23 & 28 & 32 & 42 \\ 9 & 19 & 34 & 17 & 20 & 35 & 41 \\ 10 & 45 & 44 & 7 & 11 & 12 & 46 \end{bmatrix}$$

Other Even — Ordered Squares

An examination of even-ordered magic square dot products yielded several different patterns. No pattern seemed as “nice" or orderly as the apparent pattern for Hindu squares. In some instances there seemed to be a pattern in the rows, only to find that there was a different pairing in the columns. No doubt the pattern, or lack thereof, is related in some way to the construction methods used. It would not be surprising if there is a difference between singly even and doubly even varieties, since their construction methods differ.

Chapter 4
Eigenvalues of Magic Squares

In the study of matrices, one topic often treated is finding eigenvalues of a matrix. One might wonder what kinds of eigenvalues a magic square might produce. Using a public-domain FORTRAN program called MatLab (13), such an investigation was undertaken. The program generates a single normal magic square of any order from 3 up to the limitations of the computer's memory. The odd-ordered squares generated are Hindu squares; it is not clear how MatLab generates its even-ordered squares.

The *eigenvalues* of a matrix **A** are found by solving the *characteristic equation* $|\mathbf{A} - \lambda\mathbf{I}| = 0$ for λ, where **I** is the familiar identity matrix consisting of 1's on the main diagonal and 0's elsewhere. For example, to find the eigenvalues of the classic third-order magic square, we would set

$$\begin{vmatrix} 8-\lambda & 1 & 6 \\ 3 & 5-\lambda & 7 \\ 4 & 9 & 2-\lambda \end{vmatrix} = 0,$$ obtaining the characteristic equation

$$(8-\lambda)(5-\lambda)(2-\lambda)+(3)(9)(6)+(1)(7)(4)-(4)(5-\lambda)(6)-(3)(1)(2-\lambda)-(9)(7)(8-\lambda)=0.$$

After some simplification, we would produce the following sets of equivalent equations leading to values for λ:

$$(40 - 13\lambda + \lambda^2)(2-\lambda) + 162 + 28 - 120 + 24\lambda - 504 + 63\lambda - 6 + 3\lambda = 0$$
$$80 - 26\lambda + 2\lambda^2 - 40\lambda + 13\lambda^2 - \lambda^3 + 70 + 24\lambda - 504 + 63\lambda - 6 + 3\lambda = 0$$
$$-\lambda^3 + 15\lambda^2 + 24\lambda - 360 = 0$$
$$-\lambda^2(\lambda - 15) + 24(\lambda - 15) = 0$$
$$(-\lambda^2 + 24)(\lambda - 15) = 0$$
$$\lambda^2 = 24 \quad \text{or} \quad \lambda = 15$$

So the eigenvalues for this magic square are $\lambda = 15$, $2\sqrt{6}$, and $-2\sqrt{6}$. The value $\lambda = 15$ is the same as $\sigma(M)$, the magic constant of 15. In fact, of the three eigenvalues, 15 is the largest in absolute value, also known as the *principal* or *dominant eigenvalue*.

Of course, calculating eigenvalues by this method would become almost prohibitively difficult for a square of much larger order. Fortunately, numerical methods exist for using a computer to do the calculation. Investigation using MatLab shows the pattern apparently continues. A short routine was written to calculate the eigenvalues of a magic square of each order from 3 through 36. For each nth order magic square, there are n (not necessarily distinct) eigenvalues. For each one generated by the program, the principal eigenvalue turned out to be the magic constant $\sigma(M)$. For example, for the 13th-order Hindu magic square

$$\begin{bmatrix} 93 & 108 & 123 & 138 & 153 & 168 & 1 & 16 & 31 & 46 & 61 & 76 & 91 \\ 107 & 122 & 137 & 152 & 167 & 13 & 15 & 30 & 45 & 60 & 75 & 90 & 92 \\ 121 & 136 & 151 & 166 & 12 & 14 & 29 & 44 & 59 & 74 & 89 & 104 & 106 \\ 135 & 150 & 165 & 11 & 26 & 28 & 43 & 58 & 73 & 88 & 103 & 105 & 120 \\ 149 & 164 & 10 & 25 & 27 & 42 & 57 & 72 & 87 & 102 & 117 & 119 & 134 \\ 163 & 9 & 24 & 39 & 41 & 56 & 71 & 86 & 101 & 116 & 118 & 133 & 148 \\ 8 & 23 & 38 & 40 & 55 & 70 & 85 & 100 & 115 & 130 & 132 & 147 & 162 \\ 22 & 37 & 52 & 54 & 69 & 84 & 99 & 114 & 129 & 131 & 146 & 161 & 7 \\ 36 & 51 & 53 & 68 & 83 & 98 & 113 & 128 & 143 & 145 & 160 & 6 & 21 \\ 50 & 65 & 67 & 82 & 97 & 112 & 127 & 142 & 144 & 159 & 5 & 20 & 35 \\ 64 & 66 & 81 & 96 & 111 & 126 & 141 & 156 & 158 & 4 & 19 & 34 & 49 \\ 78 & 80 & 95 & 110 & 125 & 140 & 155 & 157 & 3 & 18 & 33 & 48 & 63 \\ 79 & 94 & 109 & 124 & 139 & 154 & 169 & 2 & 17 & 32 & 47 & 62 & 77 \end{bmatrix},$$

the eigenvalues are found to be approximately 1105.0, $\pm$ 353.1, $\pm$ 181.8, $\pm$ 127.4, $\pm$ 102.7, $\pm$ 90.4, and $\pm$ 85.1. Using the formula for finding the magic sum of a normal magic square, we find $\sigma(M)$ to be $\frac{13}{2}$ $(13^2 + 1) = \frac{13}{2}$ (170) = 1105, which is indeed the principal eigenvalue of this square.

One might suspect at first that this would only be true for a normal magic square, but investigations with other magic squares also show the principal eigenvalue to be the magic constant. For example, the following fourth-order magic square (11, p. 128) is made up entirely of prime numbers:

$$\begin{bmatrix} 7 & 167 & 89 & 193 \\ 229 & 53 & 107 & 67 \\ 137 & 73 & 223 & 23 \\ 83 & 163 & 37 & 173 \end{bmatrix}$$

Its magic constant is 456, and so is its principal eigenvalue.

The only exception found was the zero magic square given in Chapter 1. Its magic constant was 0, but its principal eigenvalue was not 0. However, one of its eigenvalues is 0. Perhaps only magic squares with positive entries have the magic constant as the principal eigenvalue, and others will have the magic constant as some eigenvalue.

Conjecture. *The principal eigenvalue of a magic square composed of positive elements is its magic constant. If a magic square has some negative elements, then its magic constant is one of its eigenvalues.*

Why does this appear to be the case? Hruska (9, p. 188) noticed this result for third- and fourth-order magic squares and raised a similar question. Clearly, even if we had a good general form for a magic square, it would still be necessary to do an enormous amount of algebra to show how the principal eigenvalues turned out to be the magic constant. An examination of the method for computing eigenvalues, however, may shed some light.

The eigenvalues of M are computed by use of a determinant. The main diagonal of this determinant has $-\lambda$ added to each element. The sum of the elements on the main diagonal, tr(M), is equal to $\sigma(M)$ by definition. With a heavy dose of number theory it should be possible to see how $\sigma(M)$ shows up as λ. Unfortunately, this heavy dose of number theory is at least beyond the scope of this paper and probably beyond the current knowledge of the writer.

Conclusion

The idea for this paper was conceived upon hearing a remark that the set of magic squares forms a vector space. A proof of this property turns out to be about a fifteen-minute exercise. However, thinking about a vector space led to thinking about row and column vectors of matrices, which somehow led to the thought of taking dot products. It is surprising that no references to dot products of magic squares were to be found in the literature—and several dozen references were checked. Before the advent of electronic calculators and computers, the task of taking dot products would have been a horrendously tedious process. Even with just a calculator, the process is long enough to be somewhat distasteful. It is unfortunate that no proof of the conjecture about dot products for Hindu magic squares was apparent. Perhaps such a proof is possible, but to turn this conjecture into an elegant general theorem may necessitate as much work as Fermat's Last Theorem. If the conjecture is true, perhaps the general proof is waiting to be scrawled in an adequately wide margin of a book on magic squares.

After taking a course in matrix algebra, it seemed natural to consider magic squares as matrices and to investigate eigenvalues. Once again, it is surprising that only one reference to eigenvalues of magic squares was found, and that was only a passing remark that the one of the eigenvalues of third and fourth order squares seems to be the magic constant. The conjecture has turned out correct for over forty different squares of varying types of construction, but that does not constitute a proof. Perhaps an investigation by someone else will yield a simple proof.

Although there are a few applications of magic squares, they perhaps best belong to the category of recreational mathematics. For those who dabble in mathematics for enjoyment, magic squares are rich with mathematical properties related to many branches of mathematics.

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Appendix A

BASIC Program to Find Dot Products of Magic Squares

The following program was written in VAX BASIC, but is easily transported to other dialects of BASIC. An original version was written in Applesoft BASIC for an Apple // series computer. It was not designed to be extremely user-friendly, as that was not its purpose. It was written purely to check out the hypothesis about dot products of magic squares described in Chapter 3.


```
10 rem **** DOT PRODUCTS OF MAGIC SQUARES ****

20 rem by Daryl Stephens

30 rem Graduate teaching assistant

40 rem Texas Woman's University

50 rem June 23, 1992

60 rem ---------------------------------

70 rem Program to find dot products of row vectors

71 rem and column vectors of magic squares

100 input "How many rows = columns do you want?";n

110 dim a(n,n)

190 REM Enter magic square entries

200 for i = 1 to n

210 for j = 1 to n

220 print "Enter entry ";i;",";j;": "

225 input a(i,j)

230 next j

240 next i

250 gosub 2000

300 REM Calculate dot products

310 PRINT "Row dot products:"

320 for i = 1 to n-1
```

```
325 for row = i+1 to n
330   dot = 0
340   for j = 1 to n
350      p = a(i,j) * a(row,j)
360      dot = dot + p
370   next j
380 print i; "& "; row; ": "; dot;
385 next row
390 next i
400 print
410 print "Column dot products:"
420 for j = 1 to n-1
425   for col = j+1 to n
430      dot = 0
440      for i = 1 to n
450         p = a(i,j) * a(i,col)
460         dot = dot + p
470      next i
480       Print j; "&";col;": ";dot;
485   next col
490 next j
500 print
1999 goto 9999
2000 rem subroutine for printing out the matrix for checking purposes
2005 print "The matrix to be checked is:"
2010 for i = 1 to n
2020    for j = 1 to n
2030      print a(i,j),
```

```
2040   next j
2050   print
2060 next i
2070 print
2072 print
2080 rem Check to see that all is correct.
2090 print "Do any entries need correcting (y/n)";
2100 input yn$
2110 if (yn$ <> "y") and (yn$ <> "Y") then 2200
2120 input "Which entry (i,j) needs changing?";i,j
2130 Print "Enter new value for entry ";i;",";j;":  "
2140 input a(i,j)
2150 print "Do any other entries need changing (y/n)";
2160 input yn$
2170 if (yn$ <> "y") and (yn$ <> "Y") then 2200 else 2120
2200 rem Find magic sum.
2210 for i = 1 to n
2220   sum = sum + a(i,1)
2230 next i
2240 print "The magic sum is ";sum;"."
2250 print
2260 return
9999 end
```

Appendix B

BASIC Program to Create Uniform Step Magic Squares

The following VAX BASIC program was written to quickly generate some odd-ordered magic squares by the uniform step method as described in Chapter 1. Like the program in Appendix A, it was written as a utility program assuming the user was familiar with the particulars of the uniform step method. It was not designed to be extremely user-friendly simply because that was not necessary for the purposes of this paper. (It would be a relatively simple task to make the program user-friendly, however.)

To translate this program into BASIC for an IBM compatible computer, the statements of the form X = MOD (A,B) would need to be changed to X = A MOD B.


```
100 rem Program for creating magic squares using the

110 rem Uniform Step Method

120 rem by Daryl Stephens

130 rem March 26, 1993

200 print "Give two numbers P and Q for the coordinates of 1:"

210 print "Value for P:";

220 input p

230 print "Value for Q:";

240 input q

250 print "Give two numbers alpha and beta for the steps:"

260 print "Value for alpha:";

270 input alpha

280 print "Value for beta:";

290 input beta

300 print "Give two numbers A and B for the break steps:"

310 print "Value for A:";

320 input a

330 print "Value for B:";

340 input b
```

```
350 Print "What order square do you want?  (An odd natural number, please!)"

360 input n

370 dim ms(n,n)

380 d = alpha * b - beta * a

390 print "Is "; d; " relatively prime to "; n; "? (y/n)"

400 input yn$

410 if (yn$ = "n") or (yn$ = "N") then print "Try another. " else 430

420 goto 250

430 for x = 1 to n^2

440 gix = int ( (x-1)/n )

450 aco = p + alpha * (x - 1) + a * gix

460 bco = q + beta * (x - 1 ) + b * gix

470 ace = mod (aco,n) + 1

480 bce = mod (bco,n) + 1

490 print "The number "; x; " goes in cell ("; ace; ","; bce; ")."

500 ms(ace,bce) = x

510 next x

1000 rem Print the resulting square.

1010 print "Here is the resulting square:"

1020 print

1030 for i = 1 to n

1040   for j = 1 to n

1050     print ms(i,j),

1060   next j

1070   print

1080 next i

9999 end
```