CSCI 1900
Discrete Structures

Properties of Relations

Reading: Kolman, Section 4.1-4.2

Cartesian Product

- If $A_1, A_2, \ldots, A_m$ are nonempty sets, then the ***Cartesian Product*** of them is the set of all ordered m-tuples $(a_1, a_2, \ldots, a_m)$, where $a_i \in A_i$, i = 1, 2, … m.
- Denoted $A_1 \times A_2 \times \ldots \times A_m = \{(a_1, a_2, \ldots, a_m) \mid a_i \in A_i, i = 1, 2, \ldots m\}$

Cartesian Product Example

- If A = {1, 2, 3} and B = {a, b, c}, find A × B
- A × B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c)}

Using Matrices to Denote Cartesian Product

- For Cartesian Product of two sets, you can use a matrix to find the sets.
- Example: Assume A = {1, 2, 3} and B = {a, b, c}. The table below represents A × B.

	a	b	c
1	(1, a)	(1, b)	(1, c)
2	(2, a)	(2, b)	(2, c)
3	(3, a)	(3, b)	(3, c)

Cardinality of Cartesian Product

The cardinality of the Cartesian Product equals the product of the cardinality of all of the sets:

$$| A_1 \times A_2 \times \ldots \times A_m | = | A_1 | \cdot | A_2 | \cdot \ldots \cdot | A_m |$$

Subsets of the Cartesian Product

- Many of the results of operations on sets produce subsets of the Cartesian Product set
- Relational database
  - Each column in a database table can be considered a set
  - Each row is an m-tuple of the elements from each column or set
  - No two rows should be alike

Relations

- A relation, R, is a subset of a Cartesian Product that uses a definition to state whether an m-tuple is a member of the subset or not
- Terminology: ***Relation R from A to B***
- $R \subseteq A \times B$
- Denoted "x R y" where $x \in A$ and $y \in B$ and x has a relation with y
- If x does not have a relation with y, denoted

$$x \not R y$$

Relation Example

- A is the set of all students and B is the set of all courses
- A relation R may be defined as the course is required

Paul Giblock R CSCI 2710

Danny Camper $\not R$ CSCI 2710

Relations Across Same Set

- Relations may be from one set to the same set, i.e., A = B
- Terminology: ***Relation R on A***
$R \subseteq A \times A$

Relation on a Single Set Example

- A is the set of all courses
- A relation R may be defined as the course is a prerequisite
- CSCI 2150 R CSCI 3400
- R = {(CSCI 2150, CSCI 3400), (CSCI 1710, CSCI 2910), (CSCI 2800, CSCI 2910), …}

Example: Features of Digital Cameras

- Megapixels = {<2, 3 to 4, >5}
- battery life = {<200 shots, 200 to 400 shots, >400 shots}
- optical zoom = {none, 2X to 3X, 4X or better}
- storage capacity = {<32 MB, 32MB to 128MB, >128MB}
- price = Z+

Digital Camera Example (continued)

Possible relations might be:

- Priced below $X
- above a certain megapixels
- a combination of price below $X and optical zoom of 4X or better

Theorems of Relations

- Let R be a relation from A to B, and let A_1 and A_2 be subsets of A
  - If $A_1 \subseteq A_2$, then $R(A_1) \subseteq R(A_2)$
  - $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$
  - $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$
- Let R and S be relations from A to B. If R(a) = S(a) for all a in A, then R = S.

Matrix of a Relation

- We can represent a relation between two finite sets with a matrix
- $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$$

Example of Using a Matrix to Denote a Relation

- Using the previous example where A = {1, 2, 3} and B = {a, b, c}. The matrix below represents the relation R = {(1, a), (1, c), (2, c), (3, a), (3, b)}.

	a	b	c
1	1	0	1
2	0	0	1
3	1	1	0

Digraph of a Relation

- Let R be a relation on A
- We can represent R pictorially as follows
  - Each element of A is a circle called a vertex
  - If a_i is related to a_j, then draw an arrow from the vertex a_i to the vertex a_j
- In degree = number of arrows coming into a vertex
- Out degree = number of arrows coming out of a vertex

Representing a Relation

The following three representations depict the same relation on A = {1, 2, 3}.

R = {(1, 1), (1, 3), (2, 3), (3, 2), (3, 3)}

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$