Points missed: _____   Student's Name: _____

Total score: _____/100 points

East Tennessee State University – Department of Computer and Information Sciences
CSCI 2710 (Tarnoff) – Discrete Structures
TEST 1 for Fall Semester, 2004

Read this before starting!

- This test is closed book and closed notes
- You may **NOT** use a calculator
- All answers must have a box drawn around them. This is to aid the grader (who might not be me!) Failure to do so might result in no credit for answer.
- If you perform work on the back of a page in this test, indicate that you have done so in case the need arises for partial credit to be determined.
- Statement regarding academic misconduct from Section 5.7 of the East Tennessee State University Faculty Handbook, June 1, 2001:

Addition principle of cardinality:
- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Properties of characteristic functions:
- $f_{A \cap B} = f_A f_B$; that is $f_{A \cap B}(x) = f_A(x) f_B(x)$ for all x.
- $f_{A \cup B} = f_A + f_B - f_A f_B$; that is $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) f_B(x)$ for all x.
- $f_{A \oplus B} = f_A + f_B - 2 f_A f_B$; that is $f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2 f_A(x) f_B(x)$ for all x.

Properties of integers:
- If n and m are integers and $n > 0$, we can write $m = qn + r$ for integers q and r with $0 \le r < n$.
- Properties of divisibility:

  If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$ If $a \mid b$ or $a \mid c$, then $a \mid bc$

  If $a \mid b$ and $a \mid c$, where $b > c$, then $a \mid (b - c)$ If $a \mid b$ and $b \mid c$, then $a \mid c$
- Every positive integer $n > 1$ can be written uniquely as $n = p_1^{k1} p_2^{k2} p_3^{k3} p_4^{k4} \ldots p_s^{ks}$ where $p_1 < p_2 < p_3 < p_4 < \ldots < p_s$ are distinct primes that divide n and the k's are positive integers giving the number of times each prime occurs as a factor of n.

Properties of operations for propositions

Commutative Properties
1. $p \vee q \equiv q \vee p$
2. $p \wedge q \equiv q \wedge p$

Associative Properties
3. $p \vee (q \vee r) \equiv (p \vee q) \vee r$
4. $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Distributive Properties
5. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
6. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Idempotent Properties
7. $p \vee p \equiv p$
8. $p \wedge p \equiv p$

Properties of Negation
9. $\sim(\sim p) \equiv p$
10. $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
11. $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

Problems 1 through 6 refer to the Venn Diagram shown to the right.
The shaded Venn Diagrams that are shown here with the answers were NOT part of the required answers.
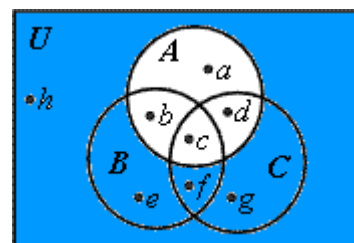They are provided here only to help understand how the answers were arrived at.

1. True or False: $f \in A \cap B$.

    The shaded area of the Venn Diagram to the right represents $A \cap B$, and since f is not contained in this area, the answer is FALSE.
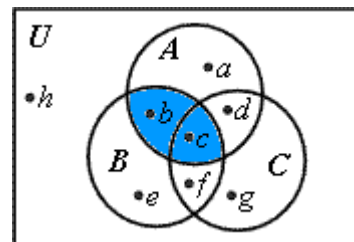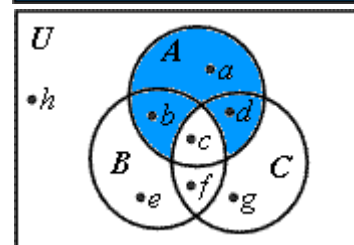
2. True or False: $c \notin \overline{A}$.

    The shaded area of the Venn Diagram to the right represents $\overline{A}$, and since c is not contained in this area, the answer is TRUE.
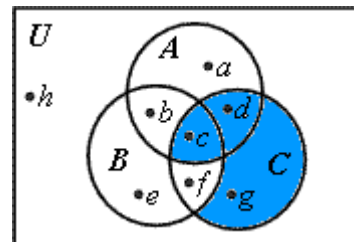
3. True or False: $b \in A - (A \cap B \cap C)$.

    The shaded area of the Venn Diagram to the right represents $A - (A \cap B \cap C)$, and since b is contained in this area, the answer is TRUE.

4. True or False: $(A \cap B \cap C) \subseteq C - (B - A)$

    The shaded area of the Venn Diagram to the right represents $C - (B - A)$. $(A \cap B \cap C)$ is the small area in the center that contains c. Since this smaller set is contained entirely within the shaded area, the answer is TRUE.

5. Using unions, intersections, and complements of sets A, B, and C, write an expression to describe the set that contains b and e, but none of the other labeled points. (3 points)

    The shaded area of the Venn Diagram to the right shows the region that we are interested in, specifically the sections that contain b and e, but no other points. If we can write an expression to describe this region, then we have answered the question.

    It looks like the region can be described as everything in B that is not in C, i.e., $B–C$.
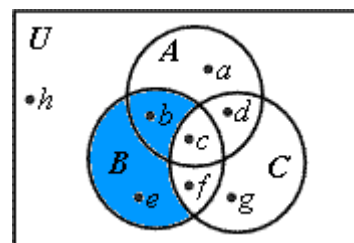
6. Using unions, intersections, and complements of sets *A*, *B*, and *C*, write an expression to describe the set that contains *c, f,* and *g*, but none of the other labeled points. (3 points)

The shaded area of the Venn Diagram to the right shows the region that we are interested in, specifically the sections that contain *c, f,* and *g*, but no other points. If we can write an expression to describe this region, then we have answered the question.
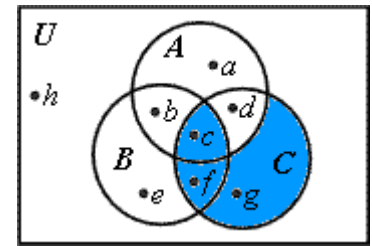
It looks like the region can be described as everything in *C* that is not in the area of *A* not covered by *B*, i.e., *C*–(*A*–*B*).

There are other answers that are valid here. For example, I would also accept (*B*∩*C*)∪(*C*–*A*).

7. Give the set corresponding to the sequence *asdseeseddes*.

{*a, s, d, e*}

8. If set *A* has 32 elements, set *B* has 40 elements, and they have 5 elements in common, how many elements are a member of either *A* or *B* but not both? (3 points)

Using the property of the characteristic function from the cover sheet, $f_{A \oplus B} = f_A + f_B - 2f_A f_B$; that is $f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x)f_B(x)$ for all x, we can see that the number of elements in the exclusive OR is equal to the number of elements in *A* plus the number of elements in *B* minus twice the number of elements in both.

Another way of looking at it is that the number of elements in *A* or *B* but not both is equal to the sum of the elements of *A* and *B* minus those elements in *A* that are also in *B* and those elements in *B* that are also in *A*, i.e., 2 times |*A*∩*B*|.

Answer: 32 + 40 – 2*5 = 62.

9. Identify one of the two cases where it is possible to have |A ∪ B ∪ C| = |A| + |B| + |C|.

From the expression for the addition principle of cardinality for three sets:

a.) If A, B, and C have no common elements, i.e., |A ∩ B ∩ C| = 0 or
b.) If |A ∩ B| + |A ∩ C| + |B ∩ C| = |A ∩ B ∩ C|
c.) If any two of the sets are empty (this is a special case of both a and b)

10. True or False: Given the set A = {*ab*, *ca*, *bc*}, the string *abababc* belongs to the set *A**.

FALSE – the string can be divided into elements of A except for the last c which does not belong to A. ab/ab/ab/c

11. True or False: The string *abb* belongs to the set corresponding to the regular expression *a*b*c*.

FALSE – the strings satisfying the regular expression a*b*c must end in c.

12. True or False: The string *c* belongs to the set corresponding to the regular expression *a*b*c*.

TRUE – Since both *a** and *b** contain the empty string, then we can have the catenation ∧∧*c* which equals *c*.

13. True or False: The string *aaac* belongs to the set corresponding to the regular expression $(a* \lor c)*$.

TRUE – Inside the parenthesis, we must select either an element from $a*$ or the single element c. $(a* \lor c)*$ however, allows us to have any combination of elements from $a*$ and the single element c, and therefore, *aaac* belongs to the set corresponding to the regular expression.

14. Write three elements that are members of the regular set corresponding to the regular expression $(0(01)*) \lor (1(10)*)$. (3 points)

Elements may be selected from either side of the '$\lor$', but no combinations may be made from both. Therefore, the three elements may be from either of the sets shown below:

$(0(01)*)$: $\{0, 001, 00101, 0010101, …\}$
$(1(10)*)$: $\{1, 110, 11010, 1101010, …\}$

15. Let $S = \{0,1\}$. Give the regular expression corresponding to the regular set $\{00, 010, 0110, 01110, 011110,…\}$. (3 points)

Each element of the set of strings begins and ends with a 0. Between these zeros is a string consisting of any number of 1's. Therefore, the regular expression should be:

$$01*0$$

16. Write the expansion in base 7 of 218_{10}. (3 points)

$7 \div 218 = 31$ with a remainder $= 1$
$7 \div 31 = 4$ with a remainder $= 3$
$7 \div 4 = 0$ with a remainder $= 4$

Therefore, by reading the resulting remainders from last to first, we get $218_{10} = \mathbf{431_7}$.

17. Write the expansion in base 6 of 221_{10}. (3 points)

$6 \div 221 = 36$ with a remainder $= 5$
$6 \div 36 = 6$ with a remainder $= 0$
$6 \div 6 = 1$ with a remainder $= 0$
$6 \div 1 = 0$ with a remainder $= 1$

Therefore, by reading the resulting remainders from last to first, we get $221_{10} = \mathbf{1005_6}$.

18. True or False: 53436_7 is a valid base 7 number.

TRUE – Since no digit in the number is greater than 6, 534367 can represent a base 7 value.

19. True or False: If A is a matrix, to compute A^p, p is a positive integer, $p \geq 2$, A must be a square matrix.

TRUE

20. True or False: Every matrix has a transpose.

TRUE

21. True or False: Every matrix has a multiplicative inverse.

FALSE – Just as not every integer has a multiplicative inverse, i.e., 0, some matrices don't have an inverse. An example would be a 2x2 matrix. Each element of the inverse an element from the original matrix divided by $ad - bc$ where a, b, c, and d are elements of the original matrix. If $ad = bc$, then the inverse does not exist.

For problems 22, 23, and 24, use the matrices A, B, and C shown below.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

22. True or False: It is possible to compute $A + B^T$.

TRUE – The transpose of B has the same number of rows and columns as A, and therefore, they can be added.

23. True or False: It is possible to compute AC.

FALSE – For matrix multiplication, the number of columns from the first matrix must equal the number of rows for the second. A has 3 columns and C has 2 rows, therefore, they can not be multiplied.

24. True or False: It is possible to compute $AB + C$.

TRUE – The number of columns of A equals the number of rows of B, so they can be multiplied resulting in a matrix containing the same number of rows as A and the same number of columns as B, i.e., a 2x2 matrix. Since C is also 2x2, it can be added to the result of AB. Therefore, this is a possible computation.

25. Give the negation of the statement "It will rain tomorrow or it will snow tomorrow."

This is the not of an or, i.e., $\sim(p \vee q)$. From the formula sheet on the front page, we see that $\sim(p \vee q)$ $\equiv (\sim p) \wedge (\sim q)$. Therefore, the negation of the statement is, **"It will not rain tomorrow and it will not snow tomorrow."** You could also say this as, **"It will not rain or snow tomorrow."** You cannot, however say, "It will not rain tomorrow or it will not snow tomorrow," as this has the form $\sim(p) \vee \sim(q)$.

For problems 26 and 27, find the truth value of each proposition if p **and** r **are true** and q **is false**.

26. $\sim p \wedge \sim q$ Answer: _____

$\sim p \wedge \sim q = \sim$(true) $\wedge \sim$(false) $=$ false $\wedge$ true $=$ **FALSE**

27. $\sim(p \vee q) \wedge r$ Answer: _____

$\sim(p \vee q) \wedge \sim r = \sim$(true $\vee$ false) $\wedge$ true $= \sim$(true) $\wedge$ true $=$ false $\wedge$ true $=$ **FALSE**

For problems 28 and 29, convert the sentence given to an expression in terms of p, q, r, and logical connectives if p: I studied; q: I'm having fun; and r: This test is difficult.

28. I didn't study, but I'm having fun.   Answer: $\sim p \wedge q$

29. Either I studied or this test is easy.   Answer: $p \vee \sim r$

Medium-ish Problems (5 points each)

30. Convert the base 5 number 342_5 to base 10.

    Each digit of a number in base n reflects a different power of n starting with n^0 in the rightmost position, n^1 in the next position, n^2 in the next position, and so on. Therefore, 342_5 represents a number with 2 increments of $5^0 = 1$, 4 increments of $5^1 = 5$, and 3 increments of $5^2 = 25$. This gives us an answer:

    $(2 * 1) + (4 * 5) + (3 * 25) = 2 + 20 + 75 = \mathbf{97}_{10}$.

31. Find the greatest common divisor of 126 and 420.

    There are a couple of ways to do this. First, you could break both 126 and 420 down to their multiples of prime numbers. This would give us:

    $126 = 2 * 63 = 2 * 3 * 21 = 2 * 3^2 * 7$
    $420 = 2 * 210 = 2^2 * 105 = 2^2 * 5* 21 = 2^2 * 3 * 5* 7$

    This shows us that 126 and 420 have common prime number divisors of one power each of 2, 3, and 7. Multiplying these together gives us the GCD:

    $$2 * 3 * 7 = 42$$

    Another way to do this is to do an iterative application of the principle of integers $m = qn + r$ using m and n to hold the values we are trying to find for the GCD where m≥n.

    $420 = \underline{3} \cdot 126 + \underline{42}$
    $126 = \underline{3} \cdot 42 + \underline{0}$

    The remainder of zero in the last expression shows that the n=42 is our GCD.

    Answer: $\mathbf{42}_{10}$

32. Use any method you wish to find the least common multiple of 150 and 70.

    Once again, there are a couple of ways to do this. The first way involves breaking both 150 and 70 down to their multiples of prime numbers. This would give us:

    $150 = 2 * 75 = 2 * 3 * 25 = 2 * 3 * 5^2$
    $70 = 2 * 35 = 2 * 5 * 7$

Finding the LCM uses these prime multiples differently than the process for finding GCD's does. In the case of the LCM, we need to take the highest power for each of the prime multiples and multiply them together for the LCM.

$$2 * 3 * 5^2 * 7 = 1050$$

Another way to do this is to find the GCD, then substitute it into the equation $GCD(m,n) \cdot LCM(m,n) = m \cdot n$.

$$150 = 2 \cdot 70 + 10$$
$$70 = 7 \cdot 10 + 0$$

The remainder of zero in the last expression shows that the n=10 is our GCD(150,70). Substituting it into the equation for the LCM gives us:

$$LCM(m,n) = m \cdot n \div GCD(m,n) = 150 \cdot 70 \div 10 = 150 \cdot 7 = 1050$$

Answer: $\mathbf{1050_{10}}$

33. Use truth tables to show that $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$. Show **_all_** intermediate steps.

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

34. Use truth tables to show that $\sim(p \Rightarrow q) \Rightarrow p$ is a tautology. Show **_all_** intermediate steps.

p	q	$(p \Rightarrow q)$	$\sim(p \Rightarrow q)$	p (duplicate)	$\sim(p \Rightarrow q) \Rightarrow p$
T	T	T	F	T	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	F	T

35. For the matrix A shown to the right, calculate A^3. Show **_all_** work.

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + 3 \cdot 3 & 2 \cdot 3 + 3 \cdot 1 \\ 3 \cdot 2 + 1 \cdot 3 & 3 \cdot 3 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 13 & 9 \\ 9 & 10 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 13 & 9 \\ 9 & 10 \end{bmatrix} * \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 13 \cdot 2 + 9 \cdot 3 & 13 \cdot 3 + 9 \cdot 1 \\ 9 \cdot 2 + 10 \cdot 3 & 9 \cdot 3 + 10 \cdot 1 \end{bmatrix} = \begin{bmatrix} \mathbf{53} & \mathbf{48} \\ \mathbf{48} & \mathbf{37} \end{bmatrix}$$

36. Using matrices A and B shown below, verify the theorem $(AB)^T = B^T A^T$. Show **_all_** work.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$(AB) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} * \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 3 & 1 \cdot 0 + 2 \cdot 1 \\ 0 \cdot 2 + 2 \cdot 3 & 0 \cdot 0 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 6 & 2 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 8 & 6 \\ 2 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \quad B^T = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 3 \cdot 2 & 2 \cdot 0 + 3 \cdot 2 \\ 0 \cdot 1 + 1 \cdot 2 & 0 \cdot 0 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 2 & 2 \end{bmatrix}$$

Since the result for $(AB)^T$ = the result for $B^T A^T$, then we have verified this theorem for these two matrices.