

Points missed: \_\_\_\_\_

Student's Name: \_\_\_\_\_

Total score: \_\_\_\_\_/100 points

East Tennessee State University – Department of Computer and Information Sciences  
CSCI 2710 (Tarnoff) – Discrete Structures  
TEST 2 for Spring Semester, 2005

**Read this before starting!**

- This test is closed book and closed notes
- You may **NOT** use a calculator
- All answers must have a box drawn around them. This is to aid the grader (who might not be me!) Failure to do so might result in no credit for answer.
- If you perform work on the back of a page in this test, indicate that you have done so in case the need arises for partial credit to be determined.
- Statement regarding academic misconduct from Section 5.7 of the East Tennessee State University Faculty Handbook, June 1, 2001:

"Academic misconduct will be subject to disciplinary action. Any act of dishonesty in academic work constitutes academic misconduct. This includes plagiarizing, the changing or falsifying of any academic documents or materials, cheating, and the giving or receiving of unauthorized aid in tests, examinations, or other assigned school work. Penalties for academic misconduct will vary with the seriousness of the offense and may include, but are not limited to: a grade of 'F' on the work in question, a grade of 'F' of the course, reprimand, probation, suspension, and expulsion. For a second academic offense the penalty is permanent expulsion."

**A short list of some tautologies:**

- |   |  |
|---|--|
| 1. $(p \wedge q) \Rightarrow p$                           | 2. $(p \wedge q) \Rightarrow q$  |
| 3. $p \Rightarrow (p \vee q)$                             | 4. $q \Rightarrow (p \vee q)$  |
| 5. $\sim p \Rightarrow (p \Rightarrow q)$                 | 6. $\sim(p \Rightarrow q) \Rightarrow p$   |
| 7. $((p \Rightarrow q) \wedge p) \Rightarrow q$           | 8. $((p \vee q) \wedge \sim p) \Rightarrow q$                                    |
| 9. $((p \Rightarrow q) \wedge \sim q) \Rightarrow \sim p$ | 10. $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$ |

**Mathematical induction:**

If  $P(n_0)$  is true and assuming  $P(k)$  is true implies  $P(k+1)$  is true, then  $P(n)$  is true for all  $n \geq n_0$

**Permutations and Combinations:**

$${}_n P_r = \frac{n!}{(n-r)!} \qquad {}_n C_r = \frac{n!}{r!(n-r)!}$$

**Properties of operations for propositions**

Commutative Properties

1.  $p \vee q \equiv q \vee p$
2.  $p \wedge q \equiv q \wedge p$

Associative Properties

3.  $p \vee (q \vee r) \equiv (p \vee q) \vee r$
4.  $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Distributive Properties

5.  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
6.  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Idempotent Properties

7.  $p \vee p \equiv p$
8.  $p \wedge p \equiv p$

Properties of Negation

9.  $\sim(\sim p) \equiv p$
10.  $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
11.  $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

*Short answers – 2 points each unless otherwise noted*

For problems 1 through 4, indicate whether the phrase is a statement or not.

1. "Does class begin at 9:45 AM?"  Statement  Not a statement
2. "5 is greater than 25."  Statement  Not a statement
3. "Read the section before coming to class."  Statement  Not a statement
4. "It snowed more than usual this past February."  Statement  Not a statement
5. Give the negation of the statement " $24 \leq 5$ ."  $24 > 5$
6. Give the negation of the statement "I will exercise and eat right." (3 points)

This problem is based on DeMorgan's Theorem which says that the negation of two AND'ed statements is the OR of the negation of each of the statements.  $(\sim(p \wedge q) \equiv (\sim p) \vee (\sim q))$  This means the answer is "I will not exercise OR I will not eat right."

For problems 7 and 8, find the truth value of each proposition if **p is false** and **q and r are true**.

7.  $p \vee \sim q$  Answer: false  $\vee$   $\sim$ true = false  $\vee$  false = false
8.  $\sim(p \vee r) \wedge q$  Answer:  $\sim$ (false  $\vee$  true)  $\wedge$  true = false  $\wedge$  true = false

For problems 9 and 10, convert the sentence given to an expression in terms of p, q, r, and logical connectives if p: I drove; q: I found parking; and r: I am on time.

9. I am on time and I found parking. Answer:  $r \wedge q$
10. I am on time if and only if I don't drive. Answer:  $r \Leftrightarrow \sim p$

Each of the following six arguments uses one of the tautologies listed on the coversheet. (See table under the heading, "a short list of some tautologies.") For each of the four arguments, identify which tautology was used from this list **by entering a value 1 through 10 in the space provided**.

11. Either this is easy or I studied  
This isn't easy  
I must have studied

Answer: 8

12. If I bought a Model T, it is black  
I bought a Model T  
My Model T is black

Answer: 7

13. If I work hard, I will succeed  
I didn't succeed  
I must not have worked hard

Answer: 9

14. After March 31, ETSU turns off the heat  
The heat is still on at ETSU  
It must be before March 31

Answer: 9

15. It is cold and it is snowing  
It is snowing

Answer: 1 or 2

16. It's hot in here  
Either its hot in here or I'm tired

Answer: 3 or 4

For the next four arguments, *indicate which are valid and which are invalid.*

17. Casey is the name of my pet  
I only own dogs for pets  
-----  
Casey must be a dog

Valid                       Invalid

18. If you are driving, I am walking  
If I am walking, then I am on time  
-----  
I'm not on time, therefore, you didn't drive

Valid                       Invalid

19. If I win the lottery, I will invest wisely  
If I invest wisely, I will be rich  
-----  
I am rich, therefore, I won the lottery

Valid                       Invalid

20. If I live in DC, driving is a hassle  
Driving is a hassle  
-----  
I must live in DC

Valid                       Invalid

- In 17, if Casey is the name of my pet, and the only pets I own are dogs, then Casey must be a dog.
- In 18, this is the tautology  $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$ .
- In 19, the argument is  $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (r \Rightarrow p)$ . Do the truth table and you will see that this is *not* a tautology. Another way of looking at it is that I could be rich because I inherited millions.
- In 20, the argument is  $(p \Rightarrow q) \wedge q \Rightarrow p$ . Do the truth table and you will see that this is *not* a tautology. For example, I could live in LA and driving would still be a hassle.

The following seven problems present seven situations where  $r$  items are selected from a set of  $n$  items. **Select the formula,  $n^r$ ,  ${}_nP_r$ ,  ${}_nC_r$ , or  ${}_{(n+r-1)}C_r$ , that will compute the number of different, valid sequences and identify the values of  $r$  and  $n$ .** (4 points each)

To answer these problems, you simply need to remember which formula pertains to which situation: ordered/unordered and duplicates allowed/duplicates not allowed. Each situation is either ordered with duplicates allowed,  $n^r$ , ordered with no duplicates allowed,  ${}_nP_r$ , unordered with no duplicates allowed,  ${}_nC_r$ , or unordered with duplicates allowed,  ${}_{(n+r-1)}C_r$ . From there, it's just a matter of setting  $n$  to the number of items in the set being selected from and  $r$  to the number of items being selected.

21. Compute the number of possible license plates with 6 digits that can be either letters or numbers.

a.)  $n^r$       b.)  ${}_nP_r$       c.)  ${}_nC_r$       d.)  ${}_{(n+r-1)}C_r$        $n = \underline{36}$        $r = \underline{6}$

22. Compute the number of combinations of 5 marbles you could pull from a bag containing 10 different colored marbles.

a.)  $n^r$       b.)  ${}_nP_r$       c.)  ${}_nC_r$       d.)  ${}_{(n+r-1)}C_r$        $n = \underline{10}$        $r = \underline{5}$

23. How many subsets are there of the set  $A = \{a, b, c, d, e\}$ ?

This is a tricky one. We did it in class. Basically, each element is either a member of the subset or not a member of the subset. This can be represented with a five digit binary number. For example, the binary number 11111 would represent the subset that contained all elements of A, i.e.,  $\{a, b, c, d, e\}$ . The binary number 10100 would represent the subset  $\{a, c\}$ . Therefore, since there are  $2^5$  possible 5-digit binary numbers, the answer is  $2^5$ .

a.)  $n^r$       b.)  ${}_nP_r$       c.)  ${}_nC_r$       d.)  ${}_{(n+r-1)}C_r$        $n = \underline{2}$        $r = \underline{5}$

24. How many five-digit numbers are there in base-16? Assume leading zeros are included as digits.

a.)  $n^r$       b.)  ${}_n P_r$       c.)  ${}_n C_r$ ,      d.)  ${}_{(n+r-1)} C_r$        $n = \underline{16}$        $r = \underline{5}$

25. How many shades of color can be created by mixing 5 parts from red, green, and blue?

a.)  $n^r$       b.)  ${}_n P_r$       c.)  ${}_n C_r$ ,      d.)  ${}_{(n+r-1)} C_r$        $n = \underline{3}$        $r = \underline{5}$

26. How many ways can the letters in the word "COMPUTER" be arranged without omitting or duplicating a letter?

a.)  $n^r$       b.)  ${}_n P_r$       c.)  ${}_n C_r$ ,      d.)  ${}_{(n+r-1)} C_r$        $n = \underline{8}$        $r = \underline{8}$

27. How many different dominos are there in a package? (Note: Each domino is a pair of numbers from the values 0, 1, 2, 3, 4, 5, and 6. A number can be paired with itself, e.g., 3 and 3 is allowed, but there is no order, e.g., a 3 paired with a 4 is the same as a 4 paired with a 3.)

a.)  $n^r$       b.)  ${}_n P_r$       c.)  ${}_n C_r$ ,      d.)  ${}_{(n+r-1)} C_r$        $n = \underline{7}$        $r = \underline{2}$

28.  True or false:  $r$  must always be less than or equal to  $n$  when determining the number of ways  $r$  items can be selected from a set of  $n$  items when order matters and duplicates are not allowed.

Without allowing duplicates, the most elements that can be selected from a set of  $n$  elements is  $n!$

Therefore,  $r$  must always be less than or equal to  $n$ .

29.  True or false:  ${}_n C_1$  is always equal to  ${}_n C_{(n-1)}$ .

The simple way would have been to have simply tried a couple of examples to have seen how the two expressions are always equal. You could have also done the mathematical proof. (By the way, you did know that the equation for  ${}_n C_r$  was on the front of the test, right?)

$${}_n C_1 = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!1!} = \frac{n!}{(n-1)!(n-(n-1))!} = {}_n C_{n-1}$$

30. Which of the following expressions describes how to calculate the number of ways that a committee of 2 students and 4 faculty members can be formed from sets of 10 students and 16 faculty members?

a.)  ${}_{10} C_2 \cdot {}_{16} C_4$       b.)  ${}_{(10+16-1)} C_6$       c.)  ${}_{10} P_2 \cdot {}_{16} P_4$       d.)  ${}_{(10+16-1)} P_6$   
 e.)  ${}_{(26+6-1)} C_6$       f.)  $10^2 \cdot 16^4$       g.)  $(10! \cdot 16!) \div (2! \cdot 4!)$       h.) None of the above

First, choose 2 students from a set of 10, then choose 4 faculty members from a set of 16. There are  ${}_{10} C_2$  ways to do the first thing then  ${}_{16} C_4$  ways to do the second. Multiplying them together produces the final result.

31. Which of the following expressions describes how to calculate the number of ways that drawing 5 cards from a deck of 52 can result in 4 of a kind with any other card being the fifth card?

a.)  ${}_{52} C_5 \div 13$       b.)  ${}_{52} P_4 \cdot {}_{48} P_1$       c.)  ${}_{52} C_1 \cdot {}_{48} C_1$       d.)  ${}_{52} C_4 \cdot {}_{48} C_1$   
 e.)  ${}_{13} C_1 \cdot {}_{48} C_1$       f.)  ${}_{13} C_1 \cdot {}_4 P_1 \cdot {}_{48} C_1$       g.)  ${}_{52} P_4 \cdot {}_{48} C_1$       h.) None of the above

First, choose from the set {2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A} to figure out which "kind" to have four of. This is  ${}_{13} C_1$ . From the remaining 48 cards, choose one to make it a hand of five. This is  ${}_{48} C_1$ . Multiplying them together produces the final result.

**Medium answers – 4 points each unless otherwise noted**

32. Assume that a lottery allows you to pick 5 numbers from a group of 62. What is the probability that you will pick all five right? Don't bother performing multiplications or divisions. Just leave expanded.

Remember that a probability is the count of the number of ways you are interested in divided by the count of the number of possible results. There is exactly one way to get all five numbers right. This is the count of the number of ways we are interested in. The total number of ways is the number of ways that 5 numbers can be picked from 62 without duplicates and where order doesn't matter, i.e.,  ${}_{62}C_5$ .

$$\frac{1}{{}_{62}C_5}$$

33. What is the probability that you will get a royal flush (four possible ways to do this) from drawing 5 cards from a deck of 52? Don't bother performing multiplications or divisions. Just leave expanded.

Once again, probability is the count of the number of ways you are interested in divided by the count of the number of possible results. As stated in the problem, there are four ways to get a royal flush. So how many ways can we pull 5 cards from 52?  ${}_{52}C_5$ .

$$\frac{4}{{}_{52}C_5}$$

34. Use truth tables to show that  $p \Rightarrow (p \vee q)$  is a tautology. Show **all** intermediate steps. Be sure to label columns.

$p$	$q$	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

35. Use truth tables to show that  $(p \Leftrightarrow q) \Leftrightarrow ((q \Rightarrow p) \wedge (p \Rightarrow q))$  is a tautology. Show **all** intermediate steps. Be sure to label columns.

$p$	$q$	$p \Leftrightarrow q$	$q \Rightarrow p$	$p \Rightarrow q$	$(q \Rightarrow p) \wedge (p \Rightarrow q)$	$(p \Leftrightarrow q) \Leftrightarrow ((q \Rightarrow p) \wedge (p \Rightarrow q))$
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	F	F	T	F	T
F	F	T	T	T	T	T

**Mathematical induction problem – 7 points**

36. **Select only one** of the following statements to prove true using mathematical induction.

a.)  $2 + 4 + 6 + \dots + 2n = n(n + 1)$

b.)  $1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 = \frac{n(2n + 1)(2n - 1)}{3}$

c.)  $5 + 10 + 15 + \dots + 5n = \frac{5n(n + 1)}{2}$

a.)  $2 + 4 + 6 + \dots + 2n = n(n + 1)$

Base case:  $n = 1$

$2 = 1 \cdot (1 + 1) = 2$  ← It works for  $n = 1$ !

Assume the  $k$  case is true:

$2 + 4 + 6 + \dots + 2k = k(k + 1)$

From it, derive the  $k+1$  case which is  $2 + 4 + 6 + \dots + 2k + 2(k+1) = (k + 1) \cdot (k + 1 + 1)$

$2 + 4 + 6 + \dots + 2k + 2(k + 1) = k(k + 1) + 2(k + 1)$       Add  $2(k + 1)$  to both sides

$= (k + 1) \cdot (k + 2)$       Pull out the  $(k + 1)$  from both terms

$= (k + 1) \cdot (k + 1 + 1)$       Set 2 equal to  $1 + 1$ .

Since the last line equals the  $k+1$  case, we've proven the formula for all values  $n \geq 1$ .

b.)  $1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 = \frac{n(2n + 1)(2n - 1)}{3}$

Base case:  $n = 1$

$(2 \cdot 1 - 1)^2 = 1^2 = \frac{1(2 \cdot 1 + 1)(2 \cdot 1 - 1)}{3} = (1 \cdot 3 \cdot 1) / 3 = 1$  ← It works for  $n = 1$ !

Assume the  $k$  case is true:

$1^2 + 3^2 + 5^2 + \dots + (2k - 1)^2 = \frac{k(2k + 1)(2k - 1)}{3}$

From it, we need to derive the  $k+1$  case which is:

$1^2 + 3^2 + 5^2 + \dots + (2k - 1)^2 = \frac{(k+1)(2(k+1) + 1)(2(k+1) - 1)}{3} = \frac{(k+1)(2k+3)(2k+1)}{3}$

$$\begin{aligned} 1^2 + 3^2 + 5^2 + \dots + (2k - 1)^2 + (2(k + 1) - 1)^2 &= \frac{k(2k + 1)(2k - 1)}{3} + (2(k + 1) - 1)^2 \\ &= \frac{k(2k + 1)(2k - 1)}{3} + \frac{3(2(k + 1) - 1)^2}{3} \\ &= \frac{k(2k + 1)(2k - 1) + 3(2(k + 1) - 1)^2}{3} \\ &= \frac{k(2k + 1)(2k - 1) + 3(2k + 1)^2}{3} \\ &= \frac{(2k + 1) [k(2k - 1) + 3(2k + 1)]}{3} \\ &= \frac{(2k + 1) [2k^2 - k + 6k + 3]}{3} \\ &= \frac{(2k + 1) [2k^2 + 5k + 3]}{3} \\ &= \frac{(k+1)(2k+3)(2k+1)}{3} \end{aligned}$$

$$c.) 5 + 10 + 15 + \dots + 5n = \frac{5n(n+1)}{2}$$

Base case:  $n = 1$

$$5 = \frac{5 \cdot 1 \cdot (1+1)}{2} = 5 \quad \leftarrow \text{It works for } n = 1!$$

Assume the  $k$  case is true:

$$5 + 10 + 15 + \dots + 5k = \frac{5k(k+1)}{2}$$

From the  $k$  case, prove the  $k+1$  case which is shown below:

$$5 + 10 + 15 + \dots + 5k + 5(k+1) = \frac{5(k+1)(k+1+1)}{2}$$

$$5 + 10 + 15 + \dots + 5k + 5(k+1) = \frac{5k(k+1)}{2} + 5(k+1) \quad \leftarrow \text{Add } 5(k+1) \text{ to both sides.}$$

$$5 + 10 + 15 + \dots + 5k + 5(k+1) = \frac{5k(k+1)}{2} + \frac{10(k+1)}{2}$$

$$5 + 10 + 15 + \dots + 5k + 5(k+1) = \frac{5k(k+1) + 10(k+1)}{2} \quad \leftarrow \text{Pull out } 5 \text{ and } (k+1)$$

$$5 + 10 + 15 + \dots + 5k + 5(k+1) = \frac{5(k+1)[k+2]}{2}$$

$$5 + 10 + 15 + \dots + 5k + 5(k+1) = \frac{5(k+1)(k+1+1)}{2}$$

Since the last line equals the  $k+1$  case, we've proven the formula for all values  $n \geq 1$ .