Chapter 5. Trigonometric Functions

5.3. Properties of the Trigonometric Functions

Note. In preparation for this section, you may need to review Appendix A Section A.5, Section 2.1, and Section 2.3.

Recall. Let $\theta$ be an angle in standard position and let $P = (x, y)$ be the point on the unit circle that corresponds to $\theta$. Then the trigonometric functions are defined as:

\[
\begin{align*}
\sin \theta &= y \\
\cos \theta &= x \\
\tan \theta &= \frac{y}{x}, x \neq 0 \\
\csc \theta &= \frac{1}{y}, y \neq 0 \\
\sec \theta &= \frac{1}{x}, x \neq 0 \\
\cot \theta &= \frac{x}{y}, y \neq 0
\end{align*}
\]

Since the equation for the unit circle is $x^2 + y^2 = 1$, then $x \in [-1, 1]$ and $y \in [-1, 1]$. Any angle $\theta$ can be drawn to determine a point $P$. However, the corresponding coordinates $(x, y)$ may have $x = 0$ (when $\theta$ is an odd integer multiple of $\pi/2$) or $y = 0$ (when $y$ is an integer multiple of $\pi$). Therefore we have the following domains of the trigonometric functions:
### Table: Properties of the Trigonometric Functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td>All real numbers</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>All real numbers</td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td>All real numbers except odd integer multiples of ( \pi/2 )</td>
</tr>
<tr>
<td>( \csc \theta )</td>
<td>All real numbers except integer multiples of ( \pi )</td>
</tr>
<tr>
<td>( \sec \theta )</td>
<td>All real numbers except odd integer multiples of ( \pi/2 )</td>
</tr>
<tr>
<td>( \cot \theta )</td>
<td>All real numbers except integer multiples of ( \pi )</td>
</tr>
</tbody>
</table>

**Note.** Since \( x \in [-1, 1] \) and \( y \in [-1, 1] \), then \(-1 \leq \cos \theta \leq 1\) and \(-1 \leq \sin \theta \leq 1\). From this it also follows that: \( \csc \theta \leq -1 \) or \( \csc \theta \geq 1 \), and \( \sec \theta \leq -1 \) or \( \sec \theta \geq 1 \). These bounds give the ranges of these functions.

**Note.** The tangent of an angle in standard position is the slope of the terminal side (the terminal side passes through the two points \((0, 0)\) and \((x, y)\), and so its slope is \(y/x\)). Therefore the range of \( \tan \theta \) (and also \( \cot \theta \)) is all real numbers.

**Definition.** A function \( f \) is *periodic* if there is a positive number \( p \) such that, whenever \( \theta \) is in the domain of \( f \), so is \( \theta + p \), and \( f(\theta + p) = f(\theta) \). If there is a smallest such number \( p \), this smallest value is called the *period* of \( f \).
Note. Each of the trigonometric functions is periodic and they satisfy the following:

\[
\begin{align*}
\sin(\theta + 2\pi) &= \sin \theta & \cos(\theta + 2\pi) &= \cos \theta & \tan(\theta + \pi) &= \tan \theta \\
\csc(\theta + 2\pi) &= \csc \theta & \sec(\theta + 2\pi) &= \sec \theta & \cot(\theta + \pi) &= \cot \theta
\end{align*}
\]

That is, the functions \(\sin \theta\), \(\cos \theta\), \(\csc \theta\), and \(\sec \theta\) have period \(2\pi\), and functions \(\tan \theta\) and \(\cot \theta\) have period \(\pi\).


Note. We can also determine the sign for each trigonometric function by considering the quadrant in which the terminal side lies. We have:

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>(\sin \theta), (\csc \theta)</th>
<th>(\cos \theta), (\sec \theta)</th>
<th>(\tan \theta), (\cot \theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Positive</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>II</td>
<td>Positive</td>
<td>Negative</td>
<td>Negative</td>
</tr>
<tr>
<td>III</td>
<td>Negative</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>IV</td>
<td>Negative</td>
<td>Positive</td>
<td>Negative</td>
</tr>
</tbody>
</table>

Example. Page 367 number 32.
5.3 Properties of the Trigonometric Functions

**Note.** From the definition of the trigonometric functions, we have:

*Reciprocal Identities.* \( \csc \theta = \frac{1}{\sin \theta} \), \( \sec \theta = \frac{1}{\cos \theta} \), \( \cot \theta = \frac{1}{\tan \theta} \)

*Quotient Identities.* \( \tan \theta = \frac{\sin \theta}{\cos \theta} \), \( \cot \theta = \frac{\cos \theta}{\sin \theta} \)

**Example.** Page 367 number 36.

**Note.** Since the unit circle has equation \( x^2 + y^2 = 1 \), then we immediately have the relationship \( \cos^2 \theta + \sin^2 \theta = 1 \) for all \( \theta \). If we divide both sides of the equation by \( \cos^2 \theta \), then we get \( 1 + \tan^2 \theta = \sec^2 \). If we, instead, divide both sides by \( \sin^2 \theta \), then we get \( 1 + \cot^2 \theta = \csc^2 \theta \). These three identities are called the *Pythagorean identities*.

**Example.** Page 367 number 56.

**Note.** The trigonometric functions have the following even/odd properties:

\[
\sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta \quad \tan(-\theta) = -\tan \theta \\
\csc(-\theta) = -\csc \theta \quad \sec(-\theta) = \sec \theta \quad \cot(-\theta) = -\cot \theta
\]

**Example.** Page 368 number 114.