Chapter 2. Limits and Continuity

2.3 The Precise Definition of a Limit

(Read this section of the text! These are the most important 7 pages in this 1200 page book!)

Definition. Formal Definition of Limit

Let \( f(x) \) be defined on an open interval about \( x_0 \), except possibly at \( x_0 \) itself. We say that \( f(x) \) approaches the limit \( L \) as \( x \) approaches \( x_0 \) and write \( \lim_{x \to x_0} f(x) = L \), if, for every number \( \epsilon > 0 \), there exists a corresponding number \( \delta > 0 \) such that for all \( x \),

\[
0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon.
\]

Figure 1.11 from Edition 10, page 93
Example. Prove for \( f(x) = mx + b, \ m \neq 0 \), that \( \lim_{x \to a} f(x) = f(a) \).

Examples. Page 82 number 12, page 83 numbers 20 and 40.

Theorem 1. Sum Rule. If

\[
\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M,
\]

then \( \lim_{x \to c} (f(x) + g(x)) = L + M \).

Proof. We wish to prove \( \lim_{x \to c} (f(x) + g(x)) = L + M \) under the assumptions \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \). Let \( \epsilon > 0 \) be given. Then \( \epsilon/2 > 0 \) and there exists \( \delta_1 > 0 \) such that for all \( x \) with \( 0 < |x - c| < \delta_1 \) we have \( |f(x) - L| < \epsilon/2 \). Similarly, there exists \( \delta_2 > 0 \) such that for all \( x \) with \( 0 < |x - c| < \delta_2 \) we have \( |g(x) - M| < \epsilon/2 \). Therefore we choose \( \delta = \min\{\delta_1, \delta_2\} \). Then for \( 0 < |x - c| < \delta \) we have

\[
|\( f(x) + g(x) \) - (L + M)| \leq |(f(x) - L) + (g(x) - M)|
\]

\[
\leq |f(x) - L| + |g(x) - M|
\]

\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2}
\]

\[
= \epsilon.
\]

This proves the result. \( Q.E.D. \)

Example. Page 85 number 58.