Chapter 3. Differentiation

3.2 The Derivative as a Function

Definition. Derivative Function.

The derivative of the function $f(x)$ with respect to the variable $x$ is the function $f'$ whose value at $x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

Note. Motivated by Sections 2.1 and 3.1, we see that $f'(x)$ is the slope of the line tangent to $y = f(x)$ as a function of $x$.

Note. There are a number of ways to denote the derivative of $y = f(x)$:

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}[f].$$
Example. Page 128 Example 2(a). Notice the text uses the “alternative formula” of the derivative.


Note. We can also study “one-sided derivatives” at a point defined as follows:

Right-hand derivative at $a$:
$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

Left-hand derivative at $b$:
$$\lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h}$$

Example. Page 133 number 40.
Note. The function in the previous example is not differentiable at \( x = 1 \). There are a number of reasons as to why a function might not have a derivative at a point. Some of these reasons are illustrated here:

1. a *corner*, where the one-sided derivatives differ.

2. a *cusp*, where the slope of \( PQ \) approaches \( \infty \) from one side and \( -\infty \) from the other.

3. a *vertical tangent*, where the slope of \( PQ \) approaches \( \infty \) from both sides or approaches \( -\infty \) from both sides (here, \( -\infty \)).

4. a *discontinuity* (two examples shown).

From page 130
Theorem 1. Differentiability Implies Continuity

If \( f \) has a derivative at \( x = c \), then \( f \) is continuous at \( x = c \).

**Proof.** By definition, we need to show that \( \lim_{x \to c} f(x) = f(c) \), or equivalently that \( \lim_{h \to 0} f(c + h) = f(c) \). Then

\[
\lim_{h \to 0} f(c + h) = \lim_{h \to 0} \left( f(c) + \frac{f(c + h) - f(c)}{h} \cdot h \right) \\
= \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \cdot \lim_{h \to 0} h \\
= f(c) + f'(c) \cdot 0 \\
= f(c).
\]

Therefore \( f \) is continuous at \( x = c \). \( \text{QED} \)

**Examples.** Page 155 numbers 48 and 54.