Chapter 3. Differentiation

3.8. Derivatives of Inverse Functions and Logarithms

**Note.** Recall that the graph of a one-to-one function $f$ and its inverse $f^{-1}$ are mirror images of each other about the line $y = x$. 

Figure 3.36 page 177
Theorem 3. The Derivative Rule for Inverses

If \( f \) has an interval \( I \) as domain and \( f'(x) \) exists and is never zero on \( I \), then \( f^{-1} \) is differentiable at every point in its domain. The value of \((f^{-1})'\) at a point \( b \) in the domain of \( f^{-1} \) is the reciprocal of the value of \( f' \) at the point \( a = f^{-1}(b) \):

\[
\frac{df^{-1}}{dx}\bigg|_{x=b} = \frac{1}{\frac{df}{dx}\bigg|_{x=f^{-1}(b)}}.
\]

Proof. By definition of inverse function, \( f^{-1}(f(x)) = x \) for all \( x \in I \). Differentiating this equation, we have by the Chain Rule:

\[
\frac{d}{dx} \left[ f^{-1}(f(x)) \right] = \frac{d}{dx} [x]
\]

\[
f^{-1'}(f(x))[f'(x)] = 1
\]

\[
f^{-1'}(f(x)) = \frac{1}{f'(x)}.
\]

Plugging in \( x = f^{-1}(b) \), we get the theorem. \( Q.E.D. \)

Example. Page 184 number 8.
Theorem. For \( x > 0 \) we have
\[
\frac{d}{dx} [\ln x] = \frac{1}{x}.
\]

If \( u = u(x) \) is a differentiable function of \( x \), then for all \( x \) such that \( u(x) > 0 \) we have
\[
\frac{d}{dx} [\ln u] = \frac{d}{dx} [\ln u(x)] = \frac{1}{u} \left( \frac{du}{dx} \right) = \frac{1}{u(x)} [u'(x)].
\]

Proof. We know that \( f(x) = e^x \) is differentiable for all \( x \), so we can apply Theorem 3 to find the derivative of \( f^{-1}(x) = \ln x \):
\[
\frac{d}{dx} [\ln x] = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}.
\]

By the Chain Rule
\[
\frac{d}{dx} [\ln u(x)] = \frac{d}{du} [\ln u] \left( \frac{du}{dx} \right) = \frac{1}{u} \left( \frac{du}{dx} \right).
\]

Q.E.D.

Note. We can apply the previous theorem to show that \( \frac{d}{dx} [\ln |x|] = \frac{1}{x} \).
Recall. For any numbers $a > 0$ and for any real $x$, $a^x = e^{x \ln a}$.

**Theorem.** If $a > 0$ and $u$ is a differentiable function of $x$, then $a^u$ is a differentiable function of $x$ and

$$\frac{d}{dx} [a^u] = a^u \ln a \left[ \frac{du}{dx} \right].$$

**Proof.** First

$$\frac{d}{dx} [a^x] = \frac{d}{dx} [e^{x \ln a}]$$

$$= e^{x \ln a} \left[ \frac{d}{dx} [x \ln a] \right]$$

$$= a^x \ln a.$$

Combining this result with the Chain Rule yields the theorem. \textit{Q.E.D.}

**Note.** Notice that the previous theorem implies that $\frac{d}{dx} [a^x] = a^x \ln a$.

With $a = e$, we have the special case $\frac{d}{dx} [e^x] = e^x(1) = e^x$. This is what is \textit{natural} about $e$ When you first meet the natural exponential and logarithmic functions in algebra, it is hard to understand what is NATURAL about them. That is because the “natural-ness” is a calculus property (namely this differentiation property).
Note. We saw in section 3.3 that \( \frac{d}{dx}[a^x] = a^x \left( \lim_{h \to 0} \frac{a^h - 1}{h} \right) \). We said then that the limit exists. We now see that the limit is \( \lim_{h \to 0} \frac{a^h - 1}{h} = \ln a \).

In particular, for \( a = e \), \( \lim_{h \to 0} \frac{e^h - 1}{h} = \ln e = 1 \).

Example. Page 185 number 70.

Definition. For any \( a > 0 \), \( a \neq 1 \), define \( \log_a x = \frac{\ln x}{\ln a} \). (This is called the change of base formula. See page 45.)

Theorem. Differentiating a logarithm base \( a \) gives:

\[
\frac{d}{dx}[\log_a u] = \frac{1}{\ln a u} \left( \frac{du}{dx} \right).
\]

Proof. This follows easily:

\[
\frac{d}{dx}[\log_a x] = \frac{d}{dx} \left[ \frac{\ln x}{\ln a} \right] = \frac{1}{\ln a} \frac{d}{dx} [\ln x] = \frac{1}{\ln a} \cdot \frac{1}{x}.
\]

Combining this result with the Chain Rule gives the theorem. Q.E.D.

Examples. Page 185 numbers 74, and 80.
3.8 Derivatives of Inverse Functions and Logarithms

Note. We can, in fact, take the logarithm of a complicated function before differentiating it and then implicitly differentiate the result. This process is called **logarithmic differentiation**. It allows us to use the laws of logarithms instead of some of the complicated rules of differentiation.

Example. Page 185 number 90.

Definition. For any $x > 0$ and for any real number $n$, define $x^n = e^{n \ln x}$.
(Here, we have finally formally defined what it means to exponentiate with irrational exponents.)

Theorem. General Power Rule for Derivatives.
For $x > 0$ and any real number $n$,

$$
\frac{d}{dx} [x^n] = nx^{n-1}.
$$

If $x < 0$, then the formula holds whenever the derivative $x^n$, and $x^{n-1}$ all exist.
3.8 Derivatives of Inverse Functions and Logarithms

**Proof.** We have
\[
\frac{d}{dx} [x^n] = \frac{d}{dx} [e^{n \ln x}]
\]
\[
= e^{n \ln x} \frac{d}{dx} [n \ln x] \text{ by the Chain Rule}
\]
\[
= x^n \frac{n}{x}
\]
\[
= n x^{n-1}.
\]

*Q.E.D.*

**Example.** Page 185 Example 72.

**Theorem 4. The Number e as a Limit**

We can find e as a limit:

\[
e = \lim_{x \to 0} (1 + x)^{1/x}.
\]

**Proof.** Let \( f(x) = \ln x \). Then \( f'(x) = 1/x \) and \( f'(1) = 1 \). Now by the definition of derivative:

\[
f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{(1 + x) - f(1)}{x}
\]
\[
= \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1 + x)
\]
\[
= \lim_{x \to 0} \ln(1 + x)^{1/x}
\]
\[
= \ln \left( \lim_{x \to 0} (1 + x)^{1/x} \right) \text{ since } \ln x \text{ is continuous.}
\]
Therefore since \( f'(1) = 1 \) we have

\[
\ln \left( \lim_{x \to 0} (1 + x)^{1/x} \right) = 1.
\]

Since \( \ln e = 1 \) and \( \ln x \) is one-to-one,

\[
\lim_{x \to 0} (1 + x)^{1/x} = e.
\]

\( Q.E.D. \)

**Note.** We can use the previous theorem to find that

\[
e \approx 2.71828 1828 45 90 45 9.
\]