2.2 Calculating Limits Using the Limit Laws

Chapter 2. Limits and Continuity

2.2. Calculating Limits Using the Limit Laws

Theorem 1. Limit Rules.

If $L$, $M$, $c$, and $k$ are real numbers and

\[
\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M,
\]

then

1. **Sum Rule**: \( \lim_{x \to c} (f(x) + g(x)) = L + M. \)

2. **Difference Rule**: \( \lim_{x \to c} (f(x) - g(x)) = L - M. \)

3. **Product Rule**: \( \lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M. \)

4. **Constant Multiple Rule**: \( \lim_{x \to c} (k \cdot f(x)) = k \cdot L. \)

5. **Quotient Rule**: \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0. \)

6. **Power Rule**: If $r$ and $s$ are integers with no common factor and $s \neq 0$,

\[
\lim_{x \to c} (f(x))^{r/s} = L^{r/s}
\]

provided that $L^{r/s}$ is a real number AND $L > 0$ when $s$ is even.
2.2 Calculating Limits Using the Limit Laws

**Example.** Page 84 number 38.

**Theorem 2.** Limits of Polynomials Can Be Found by Substitution.

If \( P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 \) then

\[
\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.
\]

**Theorem 3.** Limits of Rational Functions Can Be Found by Substituting IF the Limit of the Denominator Is Not Zero.

If \( P(x) \) and \( Q(x) \) are polynomials and \( Q(c) \neq 0 \), then

\[
\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.
\]

**Example.** Page 83 numbers 4 and 10.

**Theorem. Dr. Bob’s Theorem.** (NOT IN 11TH EDITION!)

If \( f(x) = g(x) \) for all \( x \) in an open interval containing \( c \), except possibly \( c \) itself, then

\[
\lim_{x \to c} f(x) = \lim_{x \to c} g(x)
\]

provided these limits exist.
Note. We have to be careful in our dealings with functions! Notice that 
\[ f(x) = \frac{x(x-1)}{x-1} \] and \( g(x) = x \) are **NOT** the same functions! They do not even have the same domains. Therefore we cannot in general say 
\[ \frac{x(x-1)}{x-1} = x. \] However, this equality holds if \( x \) lies in the domains of the functions. We *can* say:

\[ \frac{x(x-1)}{x-1} = x \text{ IF } x \neq 1. \]

We can also say \( f(x) = g(x) \text{ IF } x \neq 1. \) If we are concerned with limits as \( x \) approaches 1, then from the definition, \( x \) **IS NOT EQUAL TO** 1 (but near 1). Therefore we can say \( \lim_{x \to 1} f(x) = \lim_{x \to 1} g(x). \) We have not said that the functions are equal, but that their limits are.

Example. Page 83 numbers 28 and 32.
Theorem 4. Sandwich Theorem.

Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x$ in some open interval containing $c$, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Figure 2.9, page 82

Example. Page 84 number 52.

Example. Page 85 number 55.