Chapter 2. Limits and Continuity

2.5 Infinite Limits and Vertical Asymptotes

Definition. Infinity, Negative Infinity as Limits

1. We say that \( f(x) \) approaches infinity as \( x \) approaches \( x_0 \), and we write

\[
\lim_{x \to x_0} f(x) = \infty,
\]

if for every positive real number \( B \) there exists a corresponding \( \delta > 0 \) such that for all \( x \)

\[
0 < |x - x_0| < \delta \implies f(x) > B.
\]

2. We say that \( f(x) \) approaches negative infinity as \( x \) approaches \( x_0 \), and we write

\[
\lim_{x \to x_0} f(x) = -\infty,
\]

if for every negative real number \( -B \) there exists a corresponding \( \delta > 0 \) such that for all \( x \)

\[
0 < |x - x_0| < \delta \implies f(x) < -B.
\]
Note. Informally, \( \lim_{x \to x_0} f(x) = \infty \) if \( f(x) \) can be made arbitrarily large by making \( x \) sufficiently close to \( x_0 \) (and similarly for \( f \) approaching negative infinity). We can also define one-sided infinite limits in an analogous manner (see page 118 number 51).
2.5 Infinite Limits and Vertical Asymptotes

Definition. Vertical Asymptotes.

A line $x = a$ is a vertical asymptote of the graph if either

$$\lim_{x \to a^+} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm \infty.$$ 

Note. Recall that we look for the vertical asymptotes of a rational function where the denominator is zero (though just because the denominator has zero at a point, the function does not necessarily have a vertical asymptote at that point). We make things more precise in the following result:

Dr. Bob’s Second Theorem. Let $f(x) = \frac{p(x)}{q(x)}$. Suppose $\lim_{x \to x_0} p(x) = L \neq 0$, $\lim_{x \to x_0} q(x) = 0$, and $q(x)$ is of the same sign in some open interval containing $x_0$. Then $\lim_{x \to x_0^+} f(x) = \pm \infty$. We can say something similar for one-sided limits.

Note. We can simplify Dr. Bob’s Second Theorem by applying it to rational functions. It then becomes: “Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function. Suppose $\lim_{x \to x_0^+} p(x) = L \neq 0$ and $\lim_{x \to x_0^+} q(x) = 0$. Then $\lim_{x \to x_0^+} f(x) = \pm \infty$.” We can say something similar for limits from the left and for two-sided limits.