Chapter 3. Differentiation

3.6 Implicit Differentiation

**Definition.** The function $f(x)$ is *implicit* to the equation $F(x, y) = 0$ if the substitution $y = f(x)$ into the equation yields an identity.

**Example.** The functions $f(x) = \sqrt{1-x^2}$ and $g(x) = -\sqrt{1-x^2}$ are implicit to the equation $x^2 + y^2 = 1$. Can you find other functions implicit to this equation?

**Note.** If $y$ is a function implicit to $F(x, y) = 0$, then we can generate an equation containing $dy/dx$ by differentiating “implicitly.” This follows by applying the Chain Rule.
Example. Suppose $y = f(x)$ is implicit to $x^2 + y^2 = 1$. Then differentiating implicitly:

\[
\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[1] \\
2x + 2y \left[ \frac{dy}{dx} \right] = 0 \\
\frac{dy}{dx} = -\frac{x}{y}.
\]

Notice that $dy/dx$ involves both $x$ and $y$. This is because we cannot find the slope of a line tangent to the graph of $F(x, y) = 0$ without knowing the $x$ and $y$ coordinates of the point of tangency.

Example. Find the slope of the line tangent to $x^2 + y^2 = 1$ at $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$. Do the same for the point $(x, y) = (\sqrt{2}/2, -\sqrt{2}/2)$.

Definition. A line is normal to a curve at a point if it is perpendicular to the curve’s tangent line. The line is called the normal to the curve at the point.

Examples. Page 209 number 32, page 210 number 54, and page 211 number 70.

If $n$ is a rational number, then $x^n$ is differentiable at every interior point of the domain of $x^{n-1}$, and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

**Proof.** Let $p$ and $q$ be integers with $q > 0$ and suppose that $y = \sqrt[n]{x^p} = x^{p/q}$. Then $y^q = x^p$. By the Power Rule for Integer Exponents,

$$\frac{d}{dx}[y^q] = \frac{d}{dx}[x^p]$$

$$qy^{q-1}\left[\frac{dy}{dx}\right] = px^{p-1}.$$  

If $y \neq 0$, then we can solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q} \frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q} x^{(p/q)-1}.$$  

**QED**