Note. Recall that the graph of a one-to-one function $f$ and its inverse $f^{-1}$ are mirror images of each other about the line $y = x$. 

Figure 3.44 page 213
**Theorem 5.** If \( f \) has an interval \( I \) as domain and \( f'(x) \) exists and is never zero on \( I \), then \( f^{-1} \) is differentiable at every point in its domain. The value of \( (f^{-1})' \) at a point \( b \) in the domain of \( f^{-1} \) is the reciprocal of the value of \( f' \) at the point \( a = f^{-1}(b) \):

\[
\frac{df^{-1}}{dx} \bigg|_{x=b} = \frac{1}{\frac{df}{dx} \bigg|_{x=f^{-1}(b)}}.
\]

**Proof.** By definition of inverse function, \( f^{-1}(f(x)) = x \) for all \( x \in I \). Differentiating this equation, we have by the Chain Rule:

\[
\frac{d}{dx} \left[ f^{-1}(f(x)) \right] = \frac{d}{dx} [x]
\]

\[
f^{-1}(f(x))f'(x) = 1
\]

\[
f^{-1}(f(x)) = \frac{1}{f'(x)}.
\]

Plugging in \( x = f^{-1}(b) \), we get the theorem. \( Q.E.D. \)

**Example.** Page 221 number 8.
Theorem. For $x > 0$ we have

$$\frac{d}{dx} [\ln x] = \frac{1}{x}.$$

If $u = u(x)$ is a differentiable function of $x$, then for all $x$ such that $u(x) > 0$ we have

$$\frac{d}{dx} [\ln u] = \frac{d}{dx} [\ln u(x)] = \frac{1}{u} \left[ \frac{du}{dx} \right] = \frac{1}{u(x)} [u'(x)].$$

Proof. We know that $f(x) = e^x$ is differentiable for all $x$, so we can apply Theorem 5 to find the derivative of $f^{-1}(x) = \ln x$:

$$\frac{d}{dx} [\ln x] = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

By the Chain Rule

$$\frac{d}{dx} [\ln u(x)] = \frac{d}{du} [\ln u] \frac{du}{dx} = \frac{1}{u} \frac{du}{dx}.$$

Q.E.D.

Note. We can apply the previous theorem to show that $\frac{d}{dx} [\ln |x|] = \frac{1}{x}$. 
Recall. For any numbers $a > 0$ and for any real $x$, $a^x = e^{x \ln a}$.

**Theorem.** If $a > 0$ and $u$ is a differentiable function of $x$, then $a^u$ is a differentiable function of $x$ and

$$\frac{d}{dx} [a^u] = a^u \ln a \left[ \frac{du}{dx} \right].$$

**Proof.** First

$$\frac{d}{dx} [a^x] = \frac{d}{dx} [e^{x \ln a}]$$

$$= e^{x \ln a} \left[ \frac{d}{dx} [x \ln a] \right]$$

$$= a^x \ln a.$$

Combining this result with the Chain Rule yields the theorem.  \(Q.E.D.\)

**Note.** Notice that the previous theorem implies that $\frac{d}{dx} [a^x] = a^x \ln a$. With $a = e$, we have the special case $\frac{d}{dx} [e^x] = e^x(1) = e^x$. This is what is natural about $e$ When you first meet the natural exponential and logarithmic functions in algebra, it is hard to understand what is NATURAL about them. That is because the “natural-ness” is a calculus property (namely this differentiation property).
Note. We saw in section 3.2 that \( \frac{d}{dx}[a^x] = a^x \left( \lim_{h \to 0} \frac{a^h - 1}{h} \right) \). We said then that the limit exists. We now see that the limit is \( \lim_{h \to 0} \frac{a^h - 1}{h} = \ln a \).

In particular, for \( a = e \), \( \lim_{h \to 0} \frac{e^h - 1}{h} = \ln e = 1 \).

Example. Page 222 number 70.

Definition. For any \( a > 0 \), \( a \neq 1 \), define \( \log_a x = \frac{\ln x}{\ln a} \). (This is called the change of base formula. See page 53.)

Theorem. Differentiating a logarithm base \( a \) gives:

\[
\frac{d}{dx} [\log_a u] = \frac{1}{\ln a} \frac{1}{u} \frac{du}{dx}.
\]

Proof. This follows easily:

\[
\frac{d}{dx} [\log_a x] = \frac{d}{dx} \left[ \frac{\ln x}{\ln a} \right] = \frac{1}{\ln a} \frac{d}{dx} [\ln x] = \frac{1}{\ln a} \frac{1}{x}.
\]

Combining this result with the Chain Rule gives the theorem. \( Q.E.D. \)

Examples. Page 222 numbers 74, and 80.
Note. We can, in fact, take the logarithm of a complicated function before differentiating it and then implicitly differentiate the result. This process is called logarithmic differentiation. It allows us to use the laws of logarithms instead of some of the complicated rules of differentiation.

Example. Page 222 number 90.

Theorem. Power Rule (General Form). If $u$ is a positive differentiable function of $x$ and $n$ is any real number, then $u^n$ is a differentiable function of $x$ and

$$
\frac{d}{dx} [u^n] = nu^{n-1} \frac{du}{dx}.
$$

Proof. First,

$$
\frac{d}{dx} [x^n] = \frac{d}{dx} [e^{n \ln x}]
= e^{n \ln x} \frac{d}{dx} [n \ln x] \text{ by the Chain Rule}
= \frac{x^n n}{x}
= n x^{n-1}.
$$

Combining this with the Chain Rule gives the result. Q.E.D.
Example. Page 222 Example 72.

Theorem 6. We can find $e$ as a limit:

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof. Let $f(x) = \ln x$. Then $f'(x) = 1/x$ and $f'(1) = 1$. Now by the definition of derivative:

$$f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{(1 + x) - f(1)}{x} = \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1 + x) = \lim_{x \to 0} \ln(1 + x)^{1/x}$$

since $\ln x$ is continuous.

Therefore since $f'(1) = 1$ we have

$$\ln \left( \lim_{x \to 0} (1 + x)^{1/x} \right) = 1.$$

Since $\ln e = 1$ and $\ln x$ is one-to-one,

$$\lim_{x \to 0} (1 + x)^{1/x} = e.$$

Q.E.D.

Note. We can use the previous theorem to find that

$$e \approx 2.71828 1828 45 90 45 9.$$