Chapter 3. Differentiation

3.8. Inverse Trigonometric Functions

Recall. The six inverse trigonometric functions are defined as follows:

1. \( y = \cos^{-1} x \) if and only if \( \cos y = x \) and \( y \in [0, \pi] \).
2. \( y = \sin^{-1} x \) if and only if \( \sin y = x \) and \( y \in [-\pi/2, \pi/2] \).
3. \( y = \tan^{-1} x \) if and only if \( \tan y = x \) and \( y \in (-\pi/2, \pi/2) \).
4. \( y = \sec^{-1} x \) if and only if \( \sec y = x \) and \( y \in [0, \pi/2) \cup (\pi/2, \pi] \).
5. \( y = \csc^{-1} x \) if and only if \( \csc y = x \) and \( y \in [-\pi/2, 0) \cup (0, \pi/2] \).
6. \( y = \cot^{-1} x \) if and only if \( \cot y = x \) and \( y \in (0, \pi) \).

For all appropriate \( x \) values:

\[
\begin{align*}
\sec^{-1} x &= \cos^{-1}(1/x) \\
\csc^{-1} x &= \sin^{-1}(1/x) \\
\cot^{-1} x &= \pi/2 - \tan^{-1} x.
\end{align*}
\]
Note. The graphs of the six inverse trig functions are:
Example. Page 230 numbers 4, 14, 28 and 38.

Theorem. We differentiate $\sin^{-1}$ as follows:

$$\frac{d}{dx} \left[ \sin^{-1} u \right] = \frac{1}{\sqrt{1 - u^2}} \cdot \frac{du}{dx}$$

where $|u| < 1$.

Proof. We know that if $y = \sin^{-1} x$ then (for appropriate domain and range values) $\sin y = x$ and so by implicit differentiation

$$\frac{d}{dx} [\sin y] = \frac{d}{dx} [x]$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

Since we have restricted $y$ to the interval $[-\pi/2, \pi/2]$, we know that $\cos y \geq 0$ and so $\cos y = +\sqrt{1 - (\sin y)^2} = \sqrt{1 - x^2}$. Making this substitution we get

$$\frac{d}{dx} [\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}.$$

The theorem then follows from the Chain Rule. \textit{Q.E.D.}

Example. Page 231 number 58.
Theorem. We differentiate $\tan^{-1}$ as follows:

$$
\frac{d}{dx} [\tan^{-1} u] = \frac{1}{1 + u^2} \left[ \frac{du}{dx} \right].
$$

Proof. We know that if $y = \tan^{-1} x$ then (for appropriate domain and range values) $\tan y = x$ and so by implicit differentiation

$$
\frac{d}{dx} [\tan y] = \frac{d}{dx} [x]
$$

$$
\sec^2 y \left[ \frac{dy}{dx} \right] = 1
$$

$$
\frac{dy}{dx} = \frac{1}{\sec^2 y}
$$

$$
= \frac{1}{1 + (\tan y)^2}
$$

$$
= \frac{1}{1 + x^2}.
$$

The theorem then follows from the Chain Rule. Q.E.D.

Example. Page 231 number 62.
Theorem. We differentiate $\sec^{-1}$ as follows:

$$\frac{d}{dx} \left[ \sec^{-1} u \right] = \frac{1}{|u| \sqrt{u^2 - 1}} \frac{du}{dx}$$

where $|u| > 1$.

Proof. We know that if $y = \sec^{-1} x$ then (for appropriate domain and range values) $\sec y = x$ and so by implicit differentiation

$$\frac{d}{dx} [\sec y] = \frac{d}{dx} [x]$$

$$\sec y \tan y \left[ \frac{dy}{dx} \right] = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

We now need to express this last expression in terms of $x$. First, $\sec y = x$ and $\tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}$. Therefore we have

$$\frac{d}{dx} [\sec^{-1}] = \pm \frac{1}{x \sqrt{x^2 - 1}}.$$

Notice from the graph of $y = \sec^{-1} x$ above, that the slope of this function is positive where ever it is defined. So

$$\frac{d}{dx} [\sec^{-1} x] = \begin{cases} 
+ \frac{1}{x \sqrt{x^2 - 1}} & \text{if } x > 1 \\
- \frac{1}{x \sqrt{x^2 - 1}} & \text{if } x < -1.
\end{cases}$$

Notice that if $x > 1$ then $x = |x|$ and if $x < -1$ then $-x = |x|$. Therefore

$$\frac{d}{dx} [\sec^{-1} x] = \frac{1}{|x| \sqrt{x^2 - 1}}.$$
The Theorem then follows from the Chain Rule. \textit{Q.E.D.}\bigskip

\textbf{Note.} We can use the following identities to differentiate the other three inverse trig functions:

\[
\cos^{-1} x = \pi/2 - \sin^{-1} x
\]
\[
\cot^{-1} x = \pi/2 - \tan^{-1} x
\]
\[
\csc^{-1} x = \pi/2 - \sec^{-1} x
\]

We then see that the only difference in the derivative of an inverse trig function and the derivative of the inverse of its cofunction is a negative sign. In summary, that is (Table 3.1 page 229):

1. \[
\frac{d}{dx} \left[ \sin^{-1} u \right] = \frac{du/dx}{\sqrt{1-u^2}}, |u| < 1
\]
2. \[
\frac{d}{dx} \left[ \cos^{-1} u \right] = -\frac{du/dx}{\sqrt{1-u^2}}, |u| < 1
\]
3. \[
\frac{d}{dx} \left[ \tan^{-1} u \right] = \frac{du/dx}{1+u^2}
\]
4. \[
\frac{d}{dx} \left[ \cot^{-1} u \right] = -\frac{du/dx}{1+u^2}
\]
5. \[
\frac{d}{dx} \left[ \sec^{-1} u \right] = \frac{du/dx}{|u|\sqrt{u^2-1}}, |u| > 1
\]
6. \[
\frac{d}{dx} \left[ \csc^{-1} u \right] = \frac{-du/dx}{|u|\sqrt{u^2-1}}, |u| < 1
\]

\textbf{Example.} Page 231 numbers 60 and 90.