Chapter 5. Integration

5.2 Sigma Notation and Limits of Finite Sums

Note. We use the sigma notation to denote sums:

\[ \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n. \]

Examples. Page 369 number 2, page 370 number 18.

Note. We can verify (by mathematical induction):

\[ \sum_{k=1}^{n} k = \frac{n(n + 1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}, \quad \sum_{k=1}^{n} k^3 = \left( \frac{n(n + 1)}{2} \right)^2. \]


Definition. A partition of the interval \([a, b]\) is a set

\[ P = \{x_0, x_1, \ldots, x_n\} \text{ where } a = x_0 < x_1 < \cdots < x_n = b. \]

partition \(P\) determines \(n\) closed subintervals

\[ [x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]. \]

The length of the \(k\)th subinterval is \(\Delta x_k = x_k - x_{k-1}\).
Note. We now estimate the area bounded between a function $y = f(x)$ and the $x$-axis. We make the convention that the area bounded above the $x$-axis and below the function is positive, and the area bounded below the $x$-axis and above the curve is negative. We estimate this “area” by choosing a $c_k \in [x_{k-1}, x_k]$ and we use $f(c_k)$ as the “height” of a rectangle with base $[x_{k-1}, x_k]$. Then a partition $P$ of $[a, b]$ can be used to estimate this “area” by adding up the “area” of these rectangles.

Figure 5.9, page 368
Definition. With the above notation, a Riemann sum of $f$ on the interval $[a, b]$ is a sum of the form

$$s_n = \sum_{k=1}^{n} f(c_k) \Delta x_k.$$ 

Example. Page 370 number 30.

Definition. The norm of a partition $P = \{x_0, x_1, \ldots, x_n\}$ of interval $[a, b]$, denoted $\|P\|$, is largest subinterval:

$$\|P\| = \max_{1 \leq k \leq n} \Delta x_k = \max_{1 \leq k \leq n} (x_k - x_{k-1}).$$

Note. If $\|P\|$ is “small,” then a Riemann sum is a “good” approximation of the “area” described above.