Theorem 3. The Mean Value Theorem for Definite Integrals.

If \( f \) is continuous on \([a, b]\), then at some point \( c \) in \([a, b]\),

\[
f(c) = \frac{1}{b - a} \int_a^b f(x) \, dx.
\]
Proof of Theorem 3. By the Max-Min Inequality from Section 5.3, we have
\[
\min f \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \max f.
\]
Since \( f \) is continuous, \( f \) must assume any value between \( \min f \) and \( \max f \), including \( \frac{1}{b - a} \int_a^b f(x) \, dx \) by the Intermediate Value Theorem. \( Q.E.D. \)


If \( f \) is continuous on \([a, b]\) then the function
\[
F(x) = \int_a^x f(t) \, dt
\]
has a derivative at every point \( x \) in \([a, b]\) and
\[
\frac{dF}{dx} = \frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x).
\]
Proof. Notice that

\[ F(x + h) - F(x) = \int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt = \int_x^{x+h} f(t) \, dt. \]

So

\[ \frac{F(x + h) - F(x)}{h} = \frac{1}{h} [F(x + h) - F(x)] = \frac{1}{h} \int_x^{x+h} f(t) \, dt. \]

Since \( f \) is continuous, Theorem 2 implies that for some \( c \in [x, x + h] \) we have

\[ f(c) = \frac{1}{h} \int_x^{x+h} f(t) \, dt. \]

Since \( c \in [x, x + h] \), then \( \lim_{h \to 0} f(c) = f(x) \) (since \( f \) is continuous at \( x \)). Therefore

\[
\frac{dF}{dx} = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} \\
= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt \\
= \lim_{h \to 0} f(c) = f(x)
\]

Q.E.D.

Example. Page 392 numbers 42 and 44.

If \( f \) is continuous at every point of \([a, b]\) and if \( F \) is any antiderivative of \( f \) on \([a, b]\), then

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

**Proof.** We know from the first part of the Fundamental Theorem (Theorem 3a) that

\[
G(x) = \int_a^x f(t) \, dt
\]

defines an antiderivative of \( f \). Therefore if \( F \) is any antiderivative of \( f \), then \( F(x) = G(x) + k \) for some constant \( k \). Therefore

\[
F(b) - F(a) = [G(b) + k] - [G(a) + k] = G(b) - G(a) = \int_a^b f(t) \, dt - \int_a^a f(t) \, dt = \int_a^b f(t) \, dt - 0 = \int_a^b f(t) \, dt.
\]

QED

**Examples.** Page 392 numbers 16 and 60.

**Example.** Find the linearization of \( g(x) = 3 + \int_1^x \sec(t - 1) \, dt \) at \( a = -1 \).