Chapter 10. Infinite Sequences and Series

10.2 Infinite Series

**Definition.** Given a sequence of numbers \( \{a_n\} \), an expression of the form

\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots
\]

is an *infinite series*. The number \( a_n \) is the \( n^{th} \) term of the series. The *partial sums* of the series are the elements of the sequence

\[
s_1 = a_1 \\

s_2 = a_1 + a_2 \\

s_3 = a_1 + a_2 + a_3 \\

\vdots
\\

s_n = \sum_{k=1}^{n} a_k \\

\vdots
\]

If the sequence of partial sums has a limit \( L \), then we say that the *series converges* to the sum \( L \) and write \( \sum_{n=1}^{\infty} a_n = L \). If the sequence of partial sums of the series does not converge, we say that the series *diverges*. 
Definition. A geometric series is a series of the form

\[ a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \]

in which \(a\) and \(r\) are fixed real numbers and \(a \neq 0\). The parameter \(r\) is called the ratio of the series.

Theorem. The geometric series

\[ a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \]

converges to the sum \(a/(1 - r)\) if \(|r| < 1\) and diverges if \(|r| \geq 1\).

Example. Page 569 Number 2. Notice also Example 4 on page 565.

Example. Example 5 page 565. Consider the partial sums of the series

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \]

and find the sums of the series.

Solution. We can apply the partial fractions idea to find that

\[ \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \]

Then for the \(n\)th partial sum, we find that

\[ s_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k \cdot (k+1)} \]

\[ = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{k-1} - \frac{1}{k} \right) + \left( \frac{1}{k} - \frac{1}{k+1} \right) \]
\[ \begin{align*}
&= \frac{1}{1} + \left( \frac{-1}{2} + \frac{1}{2} \right) + \left( \frac{-1}{3} + \frac{1}{3} \right) + \left( \frac{-1}{4} + \frac{1}{4} \right) + \cdots + \left( \frac{-1}{k} + \frac{1}{k} \right) - \frac{1}{k + 1} \\
&= 1 - \frac{1}{k + 1}.
\end{align*} \]

Since \( s_k \to 1 \), then the series sums to 1.

**Example.** Does the series \( 1 - 1 + 1 - 1 + 1 - 1 + \cdots \) converge?

**Theorem 7. Test for Divergence**

If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \). Equivalently, if \( \lim_{n \to \infty} a_n \) does not exist or is not 0, then \( \sum_{n=1}^{\infty} a_n \) diverges.

**Note.** Notice that Theorem 7 is a test for divergence! If \( \lim_{n \to \infty} a_n = 0 \) then it **does not say** that the series converges. As we will see, there are series for which the terms approach 0, but the series still diverges.

**Example.** Page 569 Number 28.
**Theorem 8.** If \( \sum_{n=1}^{\infty} a_n = A \) and \( \sum_{n=1}^{\infty} b_n = B \) are convergent series, then

1. **Sum Rule:** \( \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B. \)

2. **Difference Rule:** \( \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = A - B. \)

3. **Constant Multiple Rule:** \( \sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n = kA \) for any number \( k. \)

**Proof.** The proof of each follows in a way similar to the proof of Theorem 1 of section 10.1 for sequences. See page 567. \( Q.E.D. \)

**Example.** Page 569 Number 12.

**Note.** Similar to Theorem 7 for convergent series, we have the following for divergent series:

**Theorem.** Every nonzero constant multiple of a divergent series diverges.

If \( \sum_{n=1}^{\infty} a_n \) converges and \( \sum_{n=1}^{\infty} b_n \) diverges, then \( \sum_{n=1}^{\infty} (a_n + b_n) \) and \( \sum_{n=1}^{\infty} (a_n - b_n) \) both diverge.

**Example.** Page 570 numbers 88 and 90.